

## Polarized Partition Relations for Ordinal Numbers

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### 1. Introduction

There have been several instances where some particularly well-chosen symbol has enhanced the development of a branch of mathematics, and the partition symbol

$$\alpha \rightarrow (\alpha_0, \alpha_1)^r \quad (1.1)$$

invented by Richard Rado is a case in point. By definition, (1.1) means that the following relation between the ordinal (or cardinal) numbers  $\alpha, \alpha_0, \alpha_1$  holds: *If  $A$  is an ordered set of order type  $\alpha$  (we shall write  $\text{tp } A = \alpha$ ) and if  $[A]^r = \{X \subset A : |X| = r\}$  is partitioned in any way into two sets  $K_0$  and  $K_1$ , then there are  $\rho < 2$  and  $A' \subset A$  such that  $\text{tp } A' = \alpha_\rho$  and  $[A']^r \subset K_\rho$ .* Erdős and Rado were the first to realize that a large number of seemingly unrelated problems in set theory could be reduced to a question of deciding whether or not some partition relation like (1.1) holds. In (5) and (6) they began a systematic study of these relations and laid the foundations of what they called a *partition calculus* to serve as a kind of unifying principle in set theory. Since these two pioneer papers several others have been written on the subject. In particular we refer to the long paper by Erdős, Hajnal, and Rado (3) which contains an almost complete analysis for partition relations involving infinite cardinal numbers. Rado's compact symbol (1.1), which reveals at a glance the whole content of a fairly complicated combinatorial statement, proved to be particularly convenient and flexible for the development of this calculus. Apart from the merit of compactness, the symbol enjoys other advantages. The negation of any statement (1.1) is conveniently expressed by replacing the arrow  $\rightarrow$  by a non-arrow  $\nrightarrow$ . The symbol has the following obvious monotonicity properties, if  $\alpha' \geq \alpha$ ,  $\beta' \leq \beta$  and  $\gamma' \leq \gamma$ , then (1.1) implies that

$$\alpha' \rightarrow (\beta', \gamma')^r.$$

The arrow in (1.1) separates the two kinds of monotonicity involved and this is

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helpful in recognizing which relations are *best possible*. Finally, the symbol readily lends itself to a number of interesting generalizations (see (6) and (3)).

In this paper we investigate one of these generalizations, the so-called polarized partition symbol. We consider only the simplest of such relations, namely those of the form

$$\left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{array} \right\}^{1,1}. \quad (1.2)$$

By definition, this means that: *If  $A$  and  $B$  are ordered sets,  $\text{tp } A = \alpha$ ,  $\text{tp } B = \beta$ , and if the cartesian product  $A \times B$  is partitioned in any way into two sets  $K_0$  and  $K_1$ , then there are  $\rho < 2$  and sets  $A' \subset A$ ,  $B' \subset B$  such that  $\text{tp } A' = \alpha_\rho$ ,  $\text{tp } B' = \beta_\rho$  and  $A' \times B' \subset K_\rho$ .* If one considers instead, partitions of  $[A]^r \times [B]^s$  for arbitrary integers  $r, s$ , the corresponding relation is represented by replacing the exponents 1,1 in (1.2) by  $r, s$ ; these more general polarized relations clearly include the ordinary partition relations (1.1). Since we only consider relations with the exponents 1,1, for the remainder of this paper we shall omit these from (1.2) and simply write

$$\left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{array} \right\}.$$

Note that, as for the ordinary partition symbol, the negation of (1.2) is expressed by replacing  $\rightarrow$  by  $\nrightarrow$ . Also, we have the same monotonicity properties: if  $\alpha' \geq \alpha$ ,  $\beta' \geq \beta$ ,  $\alpha_\rho' \leq \alpha_\rho$ ,  $\beta_\rho' \leq \beta_\rho$  ( $\rho < 2$ ), then (1.2) implies that

$$\left\{ \begin{array}{c} \alpha' \\ \beta' \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} \alpha'_0 & \alpha'_1 \\ \beta'_0 & \beta'_1 \end{array} \right\}.$$

Polarized partition relations were first introduced in (6), and in (3) a number of these relations involving cardinal numbers were established. As we already remarked, the theory for the ordinary partition relations involving cardinal numbers is fairly complete, but for polarized relations the situation is very different. There remain unsolved problems involving only the smallest transfinite cardinal numbers. For example, it is not known if the relation

$$\left\{ \begin{array}{c} \aleph_1 \\ \aleph_2 \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} \aleph_1 & \aleph_1 \\ \aleph_0 & \aleph_0 \end{array} \right\}$$

is true or false. In this paper we shall establish relations of the form (1.2) which involve ordinal numbers.

As a starting point for our investigation we mention the simple, but slightly surprising, negative relation

$$\lambda \nrightarrow (\omega_\alpha^n)_{n < \omega} \quad (\lambda < \omega_{\alpha+1}), \quad (1.3)$$

proved by Milner and Rado (9). This asserts that, if  $\text{tp } S = \lambda < \omega_{\alpha+1}$ , then  $S$

is the union of  $\aleph_0$  'small' sets  $A_n$  ( $n < \omega$ ), i.e.

$$S = \bigcup_{n < \omega} A_n$$

and  $\text{tp } A_n < \omega_\alpha^n$  ( $n < \omega$ ). If we put  $B_n = A_0 \cup \dots \cup A_n$  ( $n < \omega$ ), then the sets  $B_n$  are also 'small' ( $\text{tp } B_n < \omega_\alpha^n$ ) and the union of any  $\aleph_0$  of these is the whole set  $S$ . This fact may be expressed by means of a negative polarized relation

$$\left\{ \begin{array}{c} \omega \\ \lambda \end{array} \right\} \nrightarrow \left\{ \begin{array}{cc} 1 & \omega \\ \omega_\alpha^\omega & 1 \end{array} \right\} \quad (\lambda < \omega_{\alpha+1}).$$

From (1.3), Hajnal (see (2)) deduced the following seemingly paradoxical theorem: *If  $\text{tp } S = \lambda < \omega_2$ , then there are  $\aleph_1$  subsets  $F_\mu$  ( $\mu < \omega_1$ ) of  $S$  such that  $\text{tp } F_\mu < \omega_1^{\omega+2}$  (i.e. the sets are 'small') and the union of any  $\aleph_0$  of these  $F_\mu$  is the whole set  $S$ .* This is equivalent to the relation

$$\left\{ \begin{array}{c} \omega_1 \\ \lambda \end{array} \right\} \nrightarrow \left\{ \begin{array}{cc} 1 & \omega \\ \omega_1^{\omega+2} & 1 \end{array} \right\} \quad (\lambda < \omega_2). \quad (1.4)$$

In this paper we establish some analogous relations. We only consider relations (1.2) involving ordinal numbers of cardinal  $\aleph_1$  and for the special case in which  $\alpha = \omega_1$ ,  $\beta = \omega_1^\gamma$  and  $\alpha_0 = 1$  our discussion is complete. Some of our results do generalize to ordinals of higher cardinality, but new difficulties are encountered even in the case of  $\aleph_2$  and a discussion of these results must be left to a later paper.

In contrast to (1.4) we show (Theorem 1) that

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} 1 & \alpha \\ \xi & \beta \end{array} \right\}, \quad (1.5)$$

where  $\xi < \omega_1^{\omega+2} \leq \omega_1^\gamma < \omega_2$ ,  $\alpha < \omega_1$  and  $\beta < \omega_1^\gamma$ . It follows from (1.4) that the condition  $\xi < \omega_1^{\omega+2}$  is necessary in (1.5). Also, the trivial cardinal relation (6, 21.2)

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1 \end{array} \right\} \nrightarrow \left\{ \begin{array}{cc} 1 & \omega_1 \\ \omega_1 & 1 \end{array} \right\} \quad (1.6)$$

shows that (1.5) is false when  $\alpha = \omega_1$ . In general, the condition  $\beta < \omega_1^\gamma$  is also necessary for (1.5) since (Theorem 6)

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \nrightarrow \left\{ \begin{array}{cc} 1 & \omega \\ \omega_1^{\omega+1} + 1 & \omega_1^\gamma \end{array} \right\} \quad (1.7)$$

if  $\dagger \text{co}(\omega_1^\gamma) = \omega_1$  and  $\gamma < \omega_2$ . On the other hand, if  $\text{co}(\omega_1^\gamma) = \omega$ , then it is

$\dagger$  We write  $\text{co}(\lambda)$  to denote the least ordinal number which is cofinal with  $\lambda$ . Thus  $\text{co}(\lambda)$  is either 1 or an initial ordinal.

possible to strengthen (1.5), i.e. (Theorem 2)

$$\begin{pmatrix} \omega_1 \\ \omega_1^\gamma \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \alpha \\ \xi & \omega_1^\gamma \end{pmatrix} \quad (1.8)$$

if  $\xi < \omega_1^{\omega+2} < \omega_1^\gamma$ ,  $\alpha < \omega_1$  and  $co\gamma = \omega$ . Note that (1.7) is best possible since (Theorem 3)

$$\begin{pmatrix} \omega_1 \\ \omega_1^\gamma \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \alpha \\ \omega_1^{\omega+1} & \omega_1^\gamma \end{pmatrix} \quad (1.9)$$

holds if  $\omega + 1 \leq \gamma < \omega_2$ .

We shall prove (Theorem 4) that

$$\begin{pmatrix} \omega_1 \\ \omega_1^\gamma \end{pmatrix} \rightarrow \begin{pmatrix} k & \alpha \\ \omega_1^\gamma & \omega_1^\gamma \end{pmatrix} \quad (1.10)$$

holds for  $k < \omega$ ,  $\alpha < \omega_1$  and  $\gamma \leq \omega + 1$ . This is stronger than (1.9) when  $\gamma = \omega + 1$ . It is not possible to replace  $k$  by  $\omega$  in (1.10) since it is known (3, Theorem 32) that

$$\begin{pmatrix} \omega_1 \\ \omega_1 \end{pmatrix} \nrightarrow \begin{pmatrix} \omega & \omega \\ \omega_1 & \omega_1 \end{pmatrix}. \quad (1.11)$$

However, this raises the question whether one can replace 1 by any integer  $k$  in (1.5), (1.8) and (1.9). This is not possible in the case of (1.9) since we can prove (see (1.16)) with the continuum hypothesis<sup>†</sup> that

$$(*) \quad \begin{pmatrix} \omega_1 \\ \omega_1^\gamma \end{pmatrix} \nrightarrow \begin{pmatrix} 2 & \omega \\ \omega_1^\omega & \omega_1^\gamma \end{pmatrix} \quad (1.12)$$

if  $\omega + 2 \leq \gamma < \omega_2$  and  $co(\omega_1^\gamma) = \omega_1$ . We do not know the status of (1.5) and (1.8) in this connection.

**PROBLEM 1.** Is the relation

$$\begin{pmatrix} \omega_1 \\ \omega_1^{\omega^2} \end{pmatrix} \rightarrow \begin{pmatrix} 2 & \omega \\ \omega_1^\omega & \omega_1^{\omega^2} \end{pmatrix}$$

true or false?

There is another problem of this kind (see §4) which we cannot settle.

<sup>†</sup> Where we use the continuum hypothesis to prove a result, we prefix the statement by (\*) for easier recognition.

PROBLEM 2. Does the relation

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^{\omega+2} \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} 3 & \alpha \\ \omega_1^{\omega+1} & \xi \end{array} \right\}$$

hold for  $\alpha < \omega_1$  and  $\xi < \omega_1^{\omega+2}$ ?

It follows as a special case of a result proved in (2, Theorem 1) that

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} 1 & \omega_1 \\ \alpha & \omega_1^\gamma \end{array} \right\} \quad (1.13)$$

if  $\alpha < \omega_1, \gamma < \omega_2$  and  $co(\omega_1^\gamma) = \omega_1$ . We also showed (2, Theorem 2) that

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} 1 & \omega_1 \\ \omega + 1 & \omega_1^\gamma \end{array} \right\} \quad (1.14)$$

if  $co(\omega_1^\gamma) = \omega$ . The method we used to prove (1.13) is very different from the methods used in this paper. We shall not give the details, but with the same method used in (2) one can also show that

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} 1 & \omega_1 \\ \omega & \omega_1^\gamma \end{array} \right\} \quad (\gamma < \omega_2). \quad (1.15)$$

We mention these results because these three relations (1.13)–(1.15), together with (1.4), (1.5), (1.7), (1.8), (1.9) and (1.10) give a complete analysis of the symbol (1.2) for the case  $\alpha = \omega_1, \beta = \omega_1^\gamma$  and  $\alpha_0 = 1$ .

In §6 we establish some strong negative results. Using the continuum hypothesis we prove (Theorem 7) that, for  $\gamma < \omega_2$  and  $co(\omega_1^\gamma) = \omega_1$ ,

$$(*) \quad \left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{cccc} 1 & 2 & \omega & \omega \\ \omega_1^{\omega+1} + 1 & \vee & \omega_1^\omega & \vee & 1 & \omega_1^\gamma \end{array} \right\}. \quad (1.16)$$

Here we are using the partition symbol with *alternatives* (for the definition see §2). An equivalent formulation of (1.16) is the following. If  $tp S = \omega_1^\gamma < \omega_2$  and the continuum hypothesis is assumed, then there is a family of  $\aleph_1$  sets  $F_\mu \subset S (\mu < \omega_1)$  such that (i)  $tp F_\mu \leq \omega_1^{\omega+1}$  (i.e. 'small' sets), (ii)  $tp F_\mu \cap F_\lambda < \omega_1^\omega$  if  $\lambda \neq \mu$  (i.e. the intersection of any pair is 'very small'), (iii) each point of  $S$  belongs to only finitely many of the sets  $F_\mu$  and (iv) the union of any  $\aleph_0$  of these sets is nearly all of  $S$ , i.e.

$$tp \left( S - \bigcup_{\mu \in N} F_\mu \right) < \omega_1^\gamma$$

for any infinite set of indices  $N$ . Using a different kind of notation, we proved (1, Theorem 10.14) that

$$\left\{ \begin{array}{c} \omega \\ \lambda \end{array} \right\} \rightarrow \left\{ \begin{array}{ccc} 1 & \omega & \omega \\ \omega_1^\omega & \vee & 1 & \omega_1^{\omega+2} \end{array} \right\} \quad (\lambda < \omega_2).$$

In Theorem 8 we establish the analogous result

$$\left\langle \begin{matrix} \omega_1 \\ \lambda \end{matrix} \right\rangle \leftrightarrow \left\langle \begin{matrix} 1 & \omega & \omega \\ \omega_1^{\omega+2} & \vee & 1 \\ & & \omega_1^{\omega+2} \end{matrix} \right\rangle \quad (\lambda < \omega_2).$$

This last relation is also best possible since (see §7)

$$\left\langle \begin{matrix} \omega_1 \\ \omega_1^\gamma \end{matrix} \right\rangle \rightarrow \left\langle \begin{matrix} 1 & \omega & \omega \\ \omega_1^\gamma & \vee & 1 \\ & & \xi \end{matrix} \right\rangle$$

holds if  $\xi < \omega_1^{\omega+2} < \omega_1^\gamma < \omega_2$ .

## 2. Notation and preliminary results

Unless stated otherwise small Latin and Greek letters denote ordinal numbers. Capital letters denote sets and, in particular,

$$W_1 = \{v : v < \omega_1\}$$

is the set of countable ordinals. The obliterater sign  $\wedge$  above any symbol means that that symbol is to be disregarded, e.g.  $\{x_0, \dots, \hat{x}_\lambda\} = \{x_v : v < \lambda\}$ . We write  $\{x_0, \dots, \hat{x}_\lambda\}_<$  to indicate that the set  $\{x_0, \dots, \hat{x}_\lambda\}$  is ordered so that  $x_\mu < x_\nu$  for  $\mu < \nu < \lambda$ . Similarly,  $\{x_0, \dots, \hat{x}_\lambda\}_\neq$  means that  $x_\mu \neq x_\nu$  for  $\mu < \nu < \lambda$ . The order type of an ordered set  $S$  is denoted by  $\text{tp } S$ . If  $X, Y \subset S$ , then  $X < Y$  means that  $x < y$  holds for all  $x \in X$  and  $y \in Y$ . If  $S$  is the disjoint union of sets  $S_v$  ( $v < \lambda$ ) and  $S_\mu < S_\nu$  holds for  $\mu < \nu < \lambda$ , then we write

$$S = S_0 \cup \dots \cup \hat{S}_\lambda(<) \text{ or } S = \bigcup_{v < \lambda} S_v(<).$$

A subset  $X$  of  $S$  is *cofinal* with  $S$  if  $X < \{\alpha\}$  is false for every  $\alpha \in S$ . We define  $\text{co}(\lambda)$  to be the least ordinal  $\mu$  such that  $\text{tp } T = \mu$  for some cofinal subset  $T$  of  $\{v : v < \lambda\}$ .

The cardinal of  $S$  is  $|S|$ , and  $[S]^r = \{X \subset S : |X| = r\}$  for any cardinal number  $r$ . The partition symbol

$$\alpha \rightarrow (\alpha_v)_{v < \lambda}^r \quad (2.1)$$

means: if  $\text{tp } S = \alpha$  and  $[S]^r = \bigcup_{v < \lambda} K_v$ , then there are  $\mu < \lambda$  and  $A \subset S$  such that  $\text{tp } A = \alpha_\mu$  and  $[A]^r \subset K_\mu$ . If  $\alpha_v = \beta$  for all  $v < \lambda$  we write (2.1) as

$$\alpha \rightarrow (\beta)_\lambda^r.$$

We need the following simple relations of the form (2.1) with  $r = 1$  (see (9)):

$$\omega_1^n \rightarrow (\omega_1^n)_\lambda^1 \quad \text{if } \lambda < \omega_1 \text{ and } n < \omega, \quad (2.2)$$

$$\omega_1^l \rightarrow (\omega_1^l)_1^1 \quad \text{if } l < \omega, \quad (2.3)$$

$$\xi \mapsto (\omega_1^n)_{n < \omega}^1 \quad \text{if } \xi < \omega_2. \quad (2.4)$$

These results generalize to ordinal numbers of arbitrary cardinality, but we do not use this fact.

A set mapping is a function  $f: S \rightarrow \{X: X \subset S\}$  such that  $x \notin f(x)$  ( $x \in S$ ). A *free set* in this mapping is a subset  $S'$  of  $S$  such that  $x \notin f(y)$  for all  $x, y \in S'$ . It was shown by Erdős and Specker (7) that, if  $S = \{v: v < \omega_\lambda\}$  and  $f$  is a set mapping on  $S$  such that  $\text{tp}(f(x)) < \alpha$  ( $< \omega_\lambda$ ), then there is a free set  $S' \subset S$  of type  $\omega_\lambda$ . Their proof required the generalized continuum hypothesis but Hajnal (8) showed how to eliminate this hypothesis. In (2) we pointed out a general connection between the theory of set mappings and the polarized partition relations. We need the above theorem only for the case  $\lambda = 1$  and this may be expressed by the relation

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1 \end{array} \right\} \rightarrow \left\{ \begin{array}{c} 1 \ \omega_1 \\ \alpha \ \omega_1 \end{array} \right\} \quad (\alpha < \omega_1). \quad (2.5)$$

The polarized partition symbol (1.2) has already been defined. In §§6,7 we use a slight extension of this by allowing alternative entries. Formally,

$$\left\{ \begin{array}{c} \alpha \\ \beta \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \alpha_{v_1} \quad \alpha_{v_2} \quad \dots \quad \alpha_{v_{k_v}} \\ \beta_{v_1} \quad \beta_{v_2} \quad \dots \quad \beta_{v_{k_v}} \end{array} \right\}_{v < \lambda}$$

means: if  $\text{tp } A = \alpha$ ,  $\text{tp } B = \beta$  and

$$A \times B = \bigcup_{v < \lambda} K_v$$

is any partition, then there are  $\mu < \lambda$  and  $l < k_\mu$  and sets  $A' \subset A$ ,  $B' \subset B$  such that  $\text{tp } A' = \alpha_\mu$ ,  $\text{tp } B' = \beta_\mu$  and  $A' \times B' \subset K_\mu$ .

If  $K_0 \cup K_1$  is a partition of  $A \times B$  then we define

$$K_\rho(a) = \{b \in B: \{a, b\} \in K_\rho\} \quad (a \in A),$$

$$K_\rho(b) = \{a \in A: \{a, b\} \in K_\rho\} \quad (b \in B).$$

Also, if  $X \subset A \cup B$ , then we write

$$K_\rho(X) = \bigcup_{x \in X} K_\rho(x) \quad (\rho = 0 \text{ or } 1).$$

A *graph* is an ordered pair  $G = (S, E)$  with  $E \subset [S]^2$ . The elements of  $E$  are the *edges* of  $G$ .  $G = (S, E)$  is a *complete graph* if  $E = [S]^2$ . A *circuit* of  $G$  of

length  $k (> 2)$  is a sequence  $x_1, x_2, \dots, x_k$  of  $k$  distinct elements of  $S$  such that  $\{x_i, x_{i+1}\} \in E$  ( $1 \leq i \leq k$ ), where  $x_{k+1} = x_1$ . A graph without circuits is called a *forest*.

### 3. Some lemmas

We establish here some simple lemmas which will be used in the next section.

LEMMA 1. *Let  $\alpha < \omega_1$  and let  $M_n \in [W_1]^{\aleph_1}$  ( $n < \omega$ ). Then there are  $\mu_n \in M_n$  ( $n < \omega$ ) such that  $\text{tp} \{\mu_n : n < \omega\} \geq \alpha$ .*

*Proof.* It is enough to prove this in the case  $\alpha = \omega^\beta$ . For  $\beta = 0$ , the result is obvious. Now assume that  $\beta > 0$  and use induction. We may write  $\omega^\beta = \alpha_0 + \alpha_1 + \dots + \hat{\alpha}_\omega$ , where  $\alpha_n = \omega^{\beta_n} < \omega^\beta$  ( $n < \omega$ ). Let  $\{n : n < \omega\}$  be partitioned into  $\aleph_0$  disjoint infinite sets  $N_i$  ( $i < \omega$ ). Let  $l < \omega$  and suppose we have already chosen  $\mu_n \in M_n$  for  $n \in N_0 \cup \dots \cup \hat{N}_l$ . For  $k \in N_l$ , let

$$M'_k = \{\mu \in M_k : \mu > \mu_n \text{ for all } n \in N_0 \cup \dots \cup \hat{N}_l\}.$$

Then  $M'_k$  is a cofinal subset of  $W_1(k \in N_l)$ . Since  $N_l$  is infinite, it follows from the induction hypothesis that there are elements  $\mu_n \in M'_n$  ( $n \in N_l$ ) such that

$$\text{tp} \{\mu_n : n \in N_l\} \geq \alpha_l.$$

This defines  $\mu_n \in M_n$  for all  $n < \omega$ . By the construction,

$$\text{tp} \{\mu_n : n < \omega\} = \sum_{l < \omega} \text{tp} \{\mu_n : n \in N_l\} \geq \sum_{l < \omega} \alpha_l = \omega^\beta.$$

We frequently use the following result.

LEMMA 2. *If  $T \subset S_0 \cup S_1 \cup \dots \cup \hat{S}_{\omega_1}(<)$  and  $\text{tp} T < \omega_1^{\omega+1}$ , then there are  $v < \omega_1$  and  $n < \omega$  such that  $\text{tp}(T \cap S_\rho) < \omega_1^n$  ( $v < \rho < \omega_1$ ).*

*Proof.* Suppose the lemma is false. For  $\rho < \omega_1$  we may write  $\rho = \omega^\xi + n$ , where  $\xi = \xi(\rho) < \omega_1$  and  $n = n(\rho) < \omega$ . Suppose  $\rho < \omega_1$  and that  $v_\sigma < \omega_1$  has been defined for  $\sigma < \rho$ . By our assumption, there is  $v_\rho < \omega_1$  such that  $v_\sigma < v_\rho$  ( $\sigma < \rho$ ) and  $\text{tp}(T \cap S_{v_\rho}) \geq \omega_1^{n(\rho)}$ . This defines  $v_\rho < \omega_1$  for all  $\rho < \omega_1$ . From the definition we have

$$\text{tp} T \geq \sum_{\rho < \omega_1} \text{tp}(T \cap S_{v_\rho}) \geq \sum_{\rho < \omega_1} \omega_1^{n(\rho)} = \omega_1^{\omega+1},$$

a contradiction.

LEMMA 3. Let  $S_n$  ( $n < \omega$ ) be  $\aleph_0$  disjoint sets and suppose that  $\text{tp } S_n = \omega_1^{\gamma_n}$ , where  $\omega < \gamma_n < \omega_1$  and  $\text{co}(\omega_1^{\gamma_n}) = \omega_1$ . Let  $\alpha < \omega_1$  and let  $F_\mu$  ( $\mu < \omega_1$ ) be  $\aleph_1$  sets such that

$$\text{tp}(S_n \cap F_\mu) < \omega_1^{\omega+1} \quad (n < \omega; \mu < \omega_1). \quad (3.1)$$

Then there is  $A \subset W_1$  such that  $\text{tp } A \geq \alpha$  and

$$\text{tp}\left(S_n - \bigcup_{\mu \in A} F_\mu\right) = \text{tp } S_n \quad (n < \omega). \quad (3.2)$$

*Proof.* By the hypothesis, there are  $\gamma_{nv}$  ( $n < \omega; v < \omega_1$ ) such that

$$\omega \leq \gamma_{n0} \leq \gamma_{n1} \leq \dots \leq \hat{\gamma}_{n\omega_1} < \gamma_n$$

and

$$\sum_{v < \omega_1} \omega_1^{\gamma_{nv}} = \omega_1^{\gamma_n}.$$

We may write

$$S_n = \bigcup_{v < \omega_1} S_{nv}(<),$$

where  $\text{tp } S_{nv} = \omega_1^{\gamma_{nv}}$ . By (3.1) and Lemma 2, there are  $v(n, \mu) < \omega_1$  and  $i(n, \mu) < \omega$  ( $n < \omega; \mu < \omega_1$ ) such that

$$\text{tp}(S_{nv} \cap F_\mu) < \omega_1^{i(n, \mu)} \quad (v(n, \mu) < v < \omega_1).$$

There are sets  $M_n \in [W_1]^{\aleph_1}$  ( $n < \omega$ ) such that  $M_0 \supset M_1 \supset \dots$  and

$$i(n, \mu) = i(n) \quad (\mu \in M_n).$$

By Lemma 1, there are  $\mu_n \in M_n$  ( $n < \omega$ ) such that  $A = \{\mu_n : n < \omega\}$  has order type  $\geq \alpha$ . Choose  $\lambda < \omega_1$  such that  $v(n, \mu_m) < \lambda$  ( $m, n < \omega$ ) and let

$$k(n) = \max \{i(n), i(n, \mu_0), \dots, i(n, \mu_n)\} \quad (n < \omega).$$

Then

$$\text{tp}(F_{\mu_m} \cap S_{nv}) < \omega_1^{k(n)} \quad (m, n < \omega; \lambda < v < \omega_1).$$

Since  $k(n) < \omega$ , it follows from (2.2) and (2.3) that

$$\text{tp}\left(S_{nv} - \bigcup_{\mu \in A} F_\mu\right) = \omega_1^{\gamma_{nv}} \quad (n < \omega; \lambda < v < \omega_1)$$

and (3.2) follows.

LEMMA 4. Let  $\omega + 1 \leq \gamma_0 \leq \gamma_1 \leq \dots \leq \hat{\gamma}_{\omega_1} < \omega_2$  and let

$$\beta < \sum_{v < \omega_1} \omega_1^{\gamma_v} = \omega_1^\gamma.$$

Then there are a countable set  $N \subset W_1$  and ordinals  $\beta_v$  ( $v \in N$ ) such that

$$\omega + 1 \leq \beta_v \leq \gamma_v, \quad \text{co}(\omega_1^{\beta_v}) = \omega_1$$

and

$$\beta < \sum_{v \in N} \omega_1^{\beta_v}. \quad (3.3)$$

*Proof.* By the hypothesis, there is  $\theta$  such that  $\omega + 1 \leq \theta < \gamma$  and  $\beta < \omega_1^{\theta+1}$ . If  $\theta < \gamma_v$  for some  $v < \omega_1$ , then (3.3) holds with  $N = \{v\}$  and  $\beta_v = \theta + 1$ . We may therefore assume that there is  $v_0 < \omega_1$  such that  $\gamma_v = \theta$  ( $v_0 \leq v < \omega_1$ ). Therefore,  $\gamma = \theta + 1$  and there is  $\xi < \omega_1$  such that  $\beta < \omega_1^\theta \xi$ . If  $\text{co}(\omega_1^\theta) = \omega_1$ , then (3.3) holds with  $N = \{v : v_0 \leq v < v_0 + \xi\}$  and  $\beta_v = \theta$ . Suppose, on the other hand, that  $\text{co}(\omega_1^\theta) = \omega$ . Then there are  $\theta_n$  ( $n < \omega$ ) such that  $\omega + 1 \leq \theta_0 < \theta_1 < \dots < \hat{\theta}_\omega < \theta = \lim_{n < \omega} \theta_n$ . In this case, (3.3) holds with

$$N = \{v : v_0 \leq v < v_0 + \omega_\xi^{\rho}\} \quad \text{and} \quad \beta_{v_0 + \omega_\rho + n} = \omega_1^{\theta_n + 1} \quad (\rho < \xi; n < \omega).$$

#### 4. Positive results

In contrast to the negative relation

$$\left\{ \begin{array}{c} \omega_1 \\ \lambda \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} 1 & \omega \\ \omega_1^{\omega+2} & 1 \end{array} \right\} \quad (\lambda < \omega_2) \quad (4.1)$$

proved in (2), we shall establish the following theorem.

**THEOREM 1.** *If  $\alpha < \omega_1$ ,  $\xi < \omega_1^{\omega+2}$ ,  $\beta < \omega_1^\gamma$  and  $\omega + 2 \leq \gamma < \omega_1$ , then*

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} 1 & \alpha \\ \xi & \beta \end{array} \right\}. \quad (4.2)$$

*Proof.* We first prove the result for the case  $\text{co}(\omega_1^\gamma) = \omega_1$ .

Let  $\text{tp } C = \omega_1^\gamma$  and let  $W_1 \times C = K_0 \cup K_1$  be a partition such that  $\text{tp}(K_0(\mu)) < \xi$  for all  $\mu \in W_1$ . We have to show that there are sets  $A \subset W_1$  and  $B \subset C$  such that  $\text{tp } A \geq \alpha$ ,  $\text{tp } B \geq \beta$  and  $A \times B \subset K_1$ .

Since  $\text{co}(\omega_1^\gamma) = \omega_1$ , we may write

$$C = \bigcup_{v < \omega_1} C_v(<),$$

where  $\text{tp } C_v = \omega_1^{\gamma_v}$  ( $v < \omega_1$ ) and

$$\omega + 1 \leq \gamma_0 \leq \gamma_1 \leq \dots \leq \hat{\gamma}_{\omega_1} < \gamma.$$

There is  $\eta < \omega_1$  such that  $\xi < \omega_1^{\omega+1}\eta$  and the sets

$$F(\mu) = \{v < \omega_1 : \text{tp}(K_0(\mu) \cap C_v) \geq \omega_1^{\omega+1}\} \quad (\mu < \omega_1)$$

have order type less than  $\eta$ . Therefore, by (2.5), there are  $M, N \in [W_1]^{\aleph_1}$  such that

$$\text{tp}(K_0(\mu) \cap C_v) < \omega_1^{\omega+1} \quad (\mu \in M; v \in N). \quad (4.3)$$

By Lemma 4 there are a countable set  $N_0 \subset N$  and ordinals  $\beta_v$  ( $v \in N_0$ ) such that  $\omega + 1 \leq \beta_v \leq \gamma$ , and  $\text{co}(\omega_1^{\beta_v}) = \omega_1$  ( $v \in N_0$ ) and such that

$$\beta < \sum_{v \in N_0} \omega_1^{\beta_v}.$$

Let  $S_v$  be a subset of  $C_v$  of type  $\omega_1^{\beta_v}$  ( $v \in N_0$ ). Then, by (4.3) and Lemma 3, there is a set  $A \subset M$  such that  $\text{tp} A \geq \alpha$  and

$$\text{tp} \left( S_v - \bigcup_{\mu \in A} K_0(\mu) \right) = \text{tp} S_v \quad (v \in N_0).$$

This implies that the set

$$B = \bigcup_{v \in N_0} S_v - \bigcup_{\mu \in A} K_0(\mu)$$

has type  $\geq \beta$  and  $A \times B \subset K_1$ . This proves (4.2) for the case  $\text{co}(\omega_1^\gamma) = \omega_1$ .

The case  $\text{co}(\omega_1^\gamma) = \omega$  follows immediately from this. For, if  $\text{co} \gamma = \omega$  and  $\beta < \omega_1^\gamma$ , then  $\beta < \omega_1^{\delta+1}$  for some  $\delta + 1 < \gamma$  and (4.2) is implied by the relation

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^{\delta+1} \end{array} \right\} \rightarrow \left\{ \begin{array}{c} 1 \quad \alpha \\ \xi \quad \beta \end{array} \right\}.$$

The condition  $\xi < \omega_1^{\omega+2}$  in Theorem 1 is necessary because of (4.1). Also, in view of the trivial relation (1.6), the condition  $\alpha < \omega_1$  is necessary. The relation (4.2) is best possible in a third sense since (Theorem 6)

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \nrightarrow \left\{ \begin{array}{cc} 1 & \omega \\ \omega_1^{\omega+1} + 1 & \omega_1^\gamma \end{array} \right\} \quad \text{if } \text{co}(\omega_1^\gamma) = \omega_1. \quad (4.4)$$

This shows that we cannot replace  $\beta$  by  $\omega_1^\gamma$  in (4.2) when  $\text{co}(\omega_1^\gamma) = \omega_1$ . In the next theorem we show that (4.2) can be strengthened if  $\text{co}(\omega_1^\gamma) = \omega$ .

**THEOREM 2.** *If  $\alpha < \omega_1$ ,  $\xi < \omega_1^{\omega+2} < \omega_1^\gamma < \omega_2$  and  $\text{co}(\gamma) = \omega$ , then*

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{c} 1 \quad \alpha \\ \xi \quad \omega_1^\gamma \end{array} \right\}. \quad (4.5)$$

*Proof.* We shall prove the result by induction on  $\alpha$ . For  $\alpha = 0$  or  $1$  the result is obvious. Now assume that  $1 < \alpha' < \omega_1$  and that (4.5) holds for all  $\alpha < \alpha'$ .

Since  $1 < \alpha' < \omega_1$ , there are ordinals  $\alpha_n < \alpha'$  ( $n < \omega$ ) such that

$$\alpha' = \alpha_0 + \alpha_1 + \dots$$

Also, by the hypothesis of the theorem, there are  $\gamma_n$  ( $n < \omega$ ) such that

$$\omega < \gamma_0 < \gamma_1 < \dots < \gamma = \lim_{n < \omega} \gamma_n.$$

Let  $\text{tp } C_0 = \omega_1^\gamma$ ,  $M_0 = W_1$ , and let  $K_0 \cup K_1$  be any partition of  $M_0 \times C_0$  such that  $\text{tp } (K_0(\mu)) < \xi$  ( $\mu \in M_0$ ). We will show that there are sets  $A \subset M_0$  and  $B \subset C_0$  such that  $\text{tp } A \geq \alpha'$ ,  $\text{tp } B = \omega_1^\gamma$  and  $A \times B \subset K_1$ .

By the induction hypothesis, (4.5) holds for  $\alpha = \alpha_0$  and hence there are sets  $A_0 \subset M_0$  and  $D_0 \subset C_0$  such that  $\text{tp } A_0 = \alpha_0$ ,  $\text{tp } D_0 = \omega_1^\gamma$  and  $A_0 \times D_0 \subset K_1$ . Let

$$D_0 = E_0 \cup C_1 (<),$$

where  $E_0$  is the initial section of  $D_0$  of type  $\omega_1^{\gamma_0+2}$ . We may write

$$E_0 = \bigcup_{\rho < \omega_1} E_{0\rho}(<), \quad E_{0\rho} = \bigcup_{\sigma < \omega_1} E_{0\rho\sigma}(<),$$

where  $\text{tp } E_{0\rho\sigma} = \omega_1^{\gamma_0}(\rho, \sigma < \omega_1)$ . There is  $\eta < \omega_1$  such that  $\xi < \omega_1^{\omega+1}\eta$ . Therefore, since  $\text{tp } K_0(\mu) < \xi$  ( $\mu \in M_0$ ) and

$$\left\{ \begin{array}{c} \omega_1 \\ \omega_1 \end{array} \right\} \rightarrow \left\{ \begin{array}{cc} 1 & \omega_1 \\ \eta & 1 \end{array} \right\} \quad (4.6)$$

by (2.5), it follows that there are  $M_0' \in [M_0]^{\aleph_1}$  and  $\rho_0 < \omega_1$  such that

$$\text{tp } (K_0(\mu) \cap E_{0\rho_0}) < \omega_1^{\omega+1} \quad (\mu \in M_0').$$

Therefore, by Lemma 2, for each  $\mu \in M_0'$  there are  $p_0(\mu) < \omega$  and  $\sigma_0(\mu) < \omega_1$  such that

$$\text{tp } (K_0(\mu) \cap E_{0\rho_0\sigma}) < \omega_1^{p_0(\mu)} \text{ for } \sigma_0(\mu) < \sigma < \omega_1.$$

There is a set  $M_1 \in [M_0']^{\aleph_1}$  such that  $A_0 < M_1$  and such that

$$p_0(\mu) = p_0(\mu \in M_1).$$

More generally, suppose that  $n < \omega$  and  $C_n \subset C_0$ ,  $M_n \subset M_0$  have been defined so that  $\text{tp } C_n = \omega_1^\gamma$  and  $\text{tp } M_n = \omega_1$ . By the induction hypothesis, (4.5) holds with  $\alpha = \alpha_n$  and so there are sets  $A_n \subset M_n$  and  $D_n \subset C_n$  such that  $\text{tp } A_n = \alpha_n$ ,  $\text{tp } D_n = \omega_1^\gamma$  and  $A_n \times D_n \subset K_1$ . Let  $E_n$  be the initial section of type  $\omega_1^{\gamma_n+2}$  and let

$$D_n = E_n \cup C_{n+1} (<).$$

We may write

$$E_n = \bigcup_{\rho < \omega_1} E_{n\rho}(<) \text{ and } E_{n\rho} = \bigcup_{\sigma < \omega_1} E_{n\rho\sigma}(<), \quad (4.7)$$

where  $\text{tp}(E_{n\rho\sigma}) = \omega_1^{\gamma n}$ . It follows from (4.6) that there are  $p_n < \omega_1$  and  $M_n' \in [M_n]^{\aleph_1}$  such that

$$\text{tp}(K_0(\mu) \cap E_{n\rho_n}) < \omega_1^{\omega+1} \quad (\mu \in M_n').$$

Therefore, by Lemma 2, for each  $\mu \in M_n'$  there are  $p_n(\mu) < \omega$  and  $\sigma_n(\mu) < \omega_1$  such that

$$\text{tp}(K_0(\mu) \cap E_{n\rho_n\sigma}) < \omega_1^{p_n(\mu)} \text{ for } \sigma_n(\mu) < \sigma < \omega_1.$$

Now choose  $M_{n+1} \in [M_n']^{\aleph_1}$  such that  $p_n(\mu) = p_n(\mu \in M_{n+1})$  and such that  $A_n < M_{n+1}$ .

Proceeding inductively in the above manner, we define sets  $A_n \subset M_0$  and  $E_n \subset C_0$  ( $n < \omega$ ) such that (4.7) holds,  $\text{tp} A_n = \alpha_n$ ,  $\text{tp} E_{n\rho\sigma} = \omega_1^{\gamma n}$ ,

$$\begin{aligned} A_0 &< A_1 < \dots, \\ E_0 &< E_1 < \dots, \\ A_m \times E_n &\subset K_1 \quad (m \leq n < \omega). \end{aligned} \quad (4.8)$$

Also, there are

$$p_n < \omega, \rho_n < \omega_1 \text{ and } \sigma_n(\mu) < \omega_1 \text{ for } \mu \in A_n' = A_{n+1} \cup A_{n+2} \cup \dots,$$

such that

$$\text{tp}(K_0(\mu) \cap E_{n\rho_n\sigma}) < \omega_1^{p_n} \text{ for } \sigma_n(\mu) < \sigma < \omega_1 \text{ and } \mu \in A_n'. \quad (4.9)$$

Since  $A_n'$  is countable, there is  $\sigma_n < \omega_1$  such that  $\sigma_n(\mu) < \sigma_n$  for all  $\mu \in A_n'$ . Put

$$B_n = E_{n\rho_n\sigma_n} - \bigcup_{\mu \in A_n'} K_0(\mu). \quad (4.10)$$

Since  $\text{tp}(E_{n\rho_n\sigma_n}) = \omega_1^{\gamma n} > \omega_1^{\omega}$ , it follows from (4.9) and (2.2) that  $\text{tp} B_n = \omega_1^{\gamma n}$ . Therefore,  $B = B_0 \cup B_1 \cup \dots (<)$  has type  $\omega_1^{\gamma}$ . From (4.8) and (4.10) it follows that  $A \times B \subset K_1$ , where  $A = A_0 \cup A_1 \cup \dots (<)$ . This completes the proof of the theorem, since  $\text{tp} A = \alpha_0 + \alpha_1 + \dots = \alpha'$ .

As we have already noted, the negative relation (4.4) shows that the condition placed on the cofinality type of  $\gamma$  in Theorem 2 is essential. The next theorem shows that we can drop this condition if we strengthen the restriction on  $\xi$  to  $\xi \leq \omega_1^{\omega+1}$ . Theorem 3 shows that (4.4) is a best possible relation.

**THEOREM 3.** *If  $\alpha < \omega_1$  and  $\omega + 1 \leq \gamma < \omega_2$ , then*

$$\left\{ \begin{array}{l} \omega_1 \\ \omega_1^{\gamma} \end{array} \right\} \rightarrow \left\{ \begin{array}{ll} 1 & \alpha \\ \omega_1^{\omega+1} & \omega_1^{\gamma} \end{array} \right\}. \quad (4.11)$$

*Proof.* In view of Theorem 2 we may assume that  $\text{co}(\omega_1^{\gamma}) = \omega_1$ . Let  $\text{tp} C = \omega_1^{\gamma}$  and let  $K_0 \cup K_1$  be any partition of  $W_1 \times C$  such that  $\text{tp}(K_0(\mu)) < \omega_1^{\omega+1}$

( $\mu < \omega_1$ ). Since  $\text{co}(\omega_1^\gamma) = \omega_1$ , we may write

$$C = \bigcup_{\rho < \omega_1} C_\rho(<),$$

where  $\text{tp } C_\rho = \omega_1^{\gamma\rho}$  ( $\rho < \omega_1$ ) and  $\omega \leq \gamma_0 \leq \gamma_1 \leq \dots \leq \hat{\gamma}_{\omega_1} < \gamma$ . By Lemma 2, there are  $\rho(\mu) < \omega_1$  and  $n(\mu) < \omega$  for  $\mu < \omega_1$  such that

$$\text{tp}(F_\mu \cap C_\rho) < \omega_1^{n(\mu)} \quad (\rho(\mu) < \rho < \omega_1).$$

There is  $M \in [W_1]^{\aleph_1}$  such that  $n(\mu) = n(\mu \in M)$ . Let  $A$  be any subset of  $M$  of order type  $\alpha$  and chose  $\rho_0 < \omega_1$  such that  $\rho(\mu) < \rho_0$  for all  $\mu \in A$ . By (2.2) and (2.3), we have

$$\text{tp}\left(C_{\rho_0} - \bigcup_{\mu \in A} K_0(\mu)\right) = \omega_1^{\gamma\rho_0} \quad (\rho_0 < \rho < \omega_1)$$

and (4.11) follows.

It is easy to prove that, for  $1 \leq \gamma \leq \omega + 1$  and  $\alpha < \omega_1$ ,

$$\left\{ \begin{array}{l} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{ll} 1 & \alpha \\ \omega_1^\gamma & \omega_1^\gamma \end{array} \right\}.$$

In Theorem 4 we establish a stronger result.

**THEOREM 4.** *If  $k < \omega$ ,  $\alpha < \omega_1$  and  $1 \leq \gamma \leq \omega + 1$ , then*

$$\left\{ \begin{array}{l} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{ll} k & \alpha \\ \omega_1^\gamma & \omega_1^\gamma \end{array} \right\}. \quad (4.12)$$

*Proof.* We shall first prove (4.12) for the case  $k = 1$ . If  $\gamma = \omega + 1$ , then this is a special case of (4.11). Suppose  $\gamma \leq \omega$ . Let  $\text{tp } C = \omega_1^\gamma$  and let  $W_1 \times C = K_0 \cup K_1$  be a partition such that  $\text{tp}(K_0(\mu)) < \omega_1^\gamma$  ( $\mu < \omega_1$ ). Since  $\gamma \leq \omega$ , it follows that there are  $n < \omega$  and  $M \in [W_1]^{\aleph_1}$  such that  $\text{tp}(K_0(\mu)) < \omega_1^n$  ( $\mu \in M$ ). Let  $A$  be any subset of  $M$  of order type  $\alpha$ . It follows from (2.2) and (2.3) that

$$\text{tp}\left(C - \bigcup_{\mu \in A} K_0(\mu)\right) = \omega_1^\gamma$$

and so (4.12) holds with  $k = 1$ .

Now assume  $k > 1$  and use induction. Let  $W_1 \times C = K_0 \cup K_1$ , where  $\text{tp } C = \omega_1^\gamma$ . Suppose that

$$\text{tp}(K_0(X)) < \omega_1^\gamma$$

holds for any  $X \in [W_1]^k$ . If  $\text{tp}(K_0(\mu)) < \omega_1^\gamma$  for all  $\mu \in W_1$ , then, since (4.12) holds with  $k = 1$ , there are  $A \subset W_1$  and  $B \subset C$  such that  $\text{tp } A = \alpha$ ,



PROBLEM 2. Does the relation

$$\left\{ \begin{array}{l} \omega_1 \\ \omega_1^{\omega+2} \end{array} \right\} \rightarrow \left\{ \begin{array}{ll} 3 & \alpha \\ \omega_1^{\omega+1} & \beta \end{array} \right\}$$

hold for  $\alpha < \omega_1$  and  $\beta < \omega_1^{\omega+2}$ ?

### 5. Lemmas

We need the following three lemmas in order to prove Theorem 7 in the next section. Lemma 5 is essentially the same as Lemma 10.5 of (1) and Lemma 6 is a known result due to Erdős, Kakutani and Tukey (4). We repeat the short proofs of these for the convenience of the reader.

LEMMA 5. Let  $0 < |I| \leq \aleph_0$ ,  $|M_i| = \aleph_0$  ( $i \in I$ ) and let

$$M = \bigcup_{i \in I} M_i = \{\mu_0, \mu_1, \dots, \hat{\mu}_\omega\} \neq \emptyset.$$

If  $\text{tp } S < \omega_2$ , then there are sets  $A_\mu \subset S$  ( $\mu \in M$ ) such that

- (i)  $\text{tp } A_{\mu_n} < \omega_1^n$  ( $n < \omega$ ),
- (ii)  $|\{\mu \in M : x \in A_\mu\}| < \aleph_0$  for  $x \in S$ ,
- (iii)  $\text{tp} \left( S - \bigcup_{\mu \in M_i} A_\mu \right) < \omega_1^\omega$  ( $i \in I$ ).

*Proof.* Since the sets  $M_i$  ( $i \in I$ ) are infinite, there are mutually disjoint infinite sets  $M'_i \subset M_i$  ( $i \in I$ ). Let  $M'_i = \{\mu_{n_{ij}} : j < \omega\} \neq \emptyset$ , where  $n_{i0} < n_{i1} < \dots$  ( $i \in I$ ).

By (2.4) there is a partition of  $S$  into disjoint sets  $C_n$  ( $n < \omega$ ) such that  $\text{tp } C_n < \omega_1^n$  ( $n < \omega$ ). Let  $r < \omega$ . If  $r = n_{ij}$  for some pair  $i, j$  with  $i \in I$  and  $0 < j < \omega$ , then we put

$$A_{\mu_r} = \bigcup_{n_{i,j-1} \leq n \leq n_{i,j}} C_n.$$

Otherwise, if

$$\mu_r \notin \bigcup_{i \in I} (M'_i - \{\mu_{i0}\}),$$

then we put  $A_{\mu_r} = \emptyset$ . In either case,  $\text{tp } A_{\mu_r} < \omega_1^r$  ( $r < \omega$ ) by (2.2), and (i) holds. If  $x \in S$ , then there is a unique  $n < \omega$  such that  $x \in C_n$ . But there are only finitely many pairs  $i, j$  with  $i \in I$  and  $0 < j < \omega$  such that  $n_{ij-1} \leq n \leq n_{ij}$  and so (ii) holds. Finally, (iii) holds since

$$S - \bigcup_{\mu \in M_i} A_\mu \subset \bigcup_{n < n_{i0}} C_n = D$$

and the order type of  $D$  is less than  $\omega_1^{n_{i0}}$  by (2.2).

LEMMA 6. *The complete graph on  $W_1$  is the union of  $\aleph_0$  forests, i.e. there is a partition  $[W_1]^2 = E_0 \cup E_1 \cup \dots \cup E_\omega$  such that the graph  $T_i = (W_1, E_i)$  ( $i < \omega$ ) contains no circuit.*

*Proof.* For  $\mu < \omega_1$  there is  $t = t(\mu) \leq \omega$  such that  $\{v : v < \mu\} = \{v_{\mu 0}, v_{\mu 1}, \dots, v_{\mu t}\} \neq \emptyset$ . Thus, for  $v < \mu$ , there is a unique  $i < t(\mu)$  such that  $v = v_{\mu i}$ . Put

$$E_i = \bigcup_{\mu < \omega_1} \{v_{\mu i}, \mu\}_<.$$

Then

$$[W_1]^2 = \bigcup_{i < \omega} E_i.$$

Suppose that  $T_i = (W_1, E_i)$  contains a circuit. Then there are  $\{v, v', \mu\}_< \subset W_1$  such that  $\{v, \mu\}_< \in E_i$  and  $\{v', \mu\}_< \in E_i$ . But this implies the contradiction  $v = v' = v_{\mu i}$ .

LEMMA 7. *Let  $|N_i| = \aleph_0$  ( $i < \omega$ ) and let  $G_i = (S, E_i)$  ( $i < \omega$ ) be a graph without circuits of length 4. Then there are disjoint sets  $K_i$  ( $i < \omega$ ) such that*

- (i)  $K_i \in [N_0 \cup N_1 \cup \dots \cup N_i]^{\leq i+1}$ ,
- (ii)  $K_i \cap N_j \neq \emptyset \quad (j \leq i)$ ,
- (iii)  $[K_i]^2 \cap E_j = \emptyset \quad (j \leq i)$ .

*Proof.* We will say that a set  $K$  has property  $P_{ni}$  ( $i \leq n < \omega$ ) if the conditions (i) and (ii) of the lemma are satisfied with  $K_i = K$  and if

$$(iii)' [K]^2 \cap E_j = \emptyset \quad (j \leq n).$$

This last condition is stronger than (iii) since  $i \leq n$ . We will prove that for fixed  $i$  and  $n$  ( $i \leq n < \omega$ ), there are infinitely many mutually disjoint sets having property  $P_{ni}$ .

If  $i = 0$ , this is obvious since each one-element subset of  $N_0$  has property  $P_{n0}$ . Now assume that  $0 < i \leq n < \omega$  and use induction on  $i$ . By assumption there are infinitely many mutually disjoint sets with property  $P_{n, i-1}$  and we choose any  $n+2$  of these, say  $L_0, L_1, \dots, L_{n+1}$ . We claim that, if  $F \in [N_i]^t$  and  $t > \frac{1}{2}(n+2)(n+1)^2 i^2$ , then there are  $x \in F$  and  $\rho \leq n+1$  such that  $L_\rho \cup \{x\}$  has property  $P_{ni}$ . Suppose this is false. Then, for each  $x \in F$  and  $\rho \leq n+1$  there are  $j(\rho, x) \leq n$  and  $y(\rho, x) \in L_\rho$  such that

$$\{y(\rho, x), x\} \neq \emptyset \in E_{j(\rho, x)}.$$

This follows since (i) and (ii) hold for  $L_\rho \cup \{x\}$  and (iii)' holds for  $L_\rho$ . There

are  $\rho_1(x), \rho_2(x) \leq n+1$  such that  $\rho_1(x) \neq \rho_2(x)$  and  $j(\rho_1(x), x) = j(\rho_2(x), x) = j(x)$ . There are at most  $\frac{1}{2}(n+2)(n+1)^2 i^2$  different vectors

$$v(x) = (\rho_1(x), \rho_2(x), j(x), y(\rho_1(x), x), y(\rho_2(x), x)))$$

and hence, there are  $x_1, x_2 \in F$  such that  $x_1 \neq x_2$  and  $v(x_1) = v(x_2) = (\rho_1, \rho_2, j, y_1, y_2)$ . Note that  $\rho_1 \neq \rho_2$  and hence  $y_1 \neq y_2$  since  $y_1 \in L_{\rho_1}, y_2 \in L_{\rho_2}$  and  $L_{\rho_1}, L_{\rho_2}$  are disjoint. Therefore,  $\{x_1, y_1, x_2, y_2\} \neq$  is a circuit in  $G_j$  of length 4. This contradiction proves our claim, i.e. there are  $x \in F$  and  $\rho \leq n+1$  such that  $J_0 = L_{\rho} \cup \{x\}$  has property  $P_{ni}$ . This argument may be repeated choosing another set  $L_0', \dots, L_{n+1}'$  of  $n+2$  sets with property  $P_{n, i-1}$  so that these are mutually disjoint and disjoint from  $J_0$ . As before, there are  $x' \in N_i - J_0$  and  $p' \leq n+1$  such that  $J_1 = L_{p'} \cup \{x'\}$  has property  $P_{ni}$ . In this way we construct infinitely many mutually disjoint sets with property  $P_{ni}$ . The assertion of the previous paragraph now follows by induction.

In particular, there are infinitely many mutually disjoint sets having property  $P_{ii}$  ( $i < \omega$ ). Therefore, we can choose the finite sets  $K_i$  ( $i < \omega$ ) so that (i), (ii) and (iii) hold and so that these are mutually disjoint.

## 6. Negative relations

In Theorem 6 we establish the negative polarized partition relation (4.4) discussed in §4. The condition placed upon the cofinality type of  $\omega_1^\gamma$  is necessary by Theorem 2.

**THEOREM 6.** *If  $\gamma < \omega_2$  and  $co(\omega_1^\gamma) = \omega_1$ , then*

$$\left\{ \begin{array}{l} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{ll} 1 & \omega \\ \omega_1^{\omega+1} + 1 & \omega_1^\gamma \end{array} \right\}. \quad (6.1)$$

*Proof.* Let  $S$  be an ordered set of type  $\omega_1^\gamma$ . Then we may write

$$S = S_0 \cup S_1 \cup \dots \cup \hat{S}_{\omega_1}(<),$$

where  $tp S_v = \omega_1^{\gamma_v}$  and  $\gamma_0 \leq \gamma_1 \leq \dots \leq \hat{\gamma}_{\omega_1} \leq \gamma$ . By (2.4) there is a partition of  $S_v$ ,

$$S_v = \bigcup_{v < \omega} A_{vn},$$

with  $tp A_{vn} < \omega_1^n$  ( $v < \omega_1; n < \omega$ ). For  $v < \omega_1$ , there is a mapping  $f_v$  of  $\{n : n < \omega\}$  onto  $\{\mu : \mu \leq v\}$ , and for each  $\mu \leq v$  there is an integer  $n = n(\mu, v)$  such that  $\mu = f_v(n)$ . Now consider the partition  $W_1 \times S = K_0 \cup K_1$  in which

$$K_0(\mu) = \bigcup_{\mu \leq v < \omega_1} \bigcup_{n < n(\mu, v)} A_{vn} \quad (6.2)$$

for  $\mu < \omega_1$ .

By (2.2),  $\text{tp}(K_0(\mu) \cap S_\nu) < \omega_1^{n(\mu, \nu)} < \omega_1^\omega$  if  $\mu \leq \nu < \omega_1$ . Therefore, since  $K_0(\mu) \cap S_\nu = \emptyset$  ( $\nu < \mu < \omega_1$ ), we have

$$\text{tp } K_0(\mu) \leq \omega_1^{\omega+1}. \quad (6.3)$$

Let  $N \in [W_1]^{\aleph_0}$ . Then  $N$  contains an increasing sequence of ordinals  $\mu_i$  ( $i < \omega$ ). Let

$$\lambda = \lim_{i < \omega} \mu_i.$$

For  $\nu \geq \lambda$ , the integers  $n(\mu_i, \nu)$  ( $i < \omega$ ) are all distinct and therefore, by (6.2),

$$S_\nu \subset \bigcup_{i < \omega} K_0(\mu_i) \quad (\lambda \leq \nu < \omega_1).$$

Therefore

$$\bigcap_{\mu \in N} K_1(\mu) \subset S_0 \cup S_1 \cup \dots \cup \hat{S}_\lambda,$$

and

$$\text{tp} \left\{ \bigcap_{\mu \in N} K_1(\mu) \right\} < \omega_1^\gamma. \quad (6.4)$$

The theorem follows from (6.3) and (6.4).

We now establish a much stronger result than (6.1) by using the continuum hypothesis. Note that (6.5) implies (4.13).

(\*) THEOREM 7. *If  $\omega + 1 < \gamma < \omega_2$  and  $co(\omega_1^\gamma) = \omega_1$ , then*

$$\left\{ \begin{array}{l} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{cccc} 1 & 2 & \omega & \omega \\ & \vee & \vee & \\ \omega_1^{\omega+1} + 1 & \omega_1^\omega & 1 & \omega_1^\gamma \end{array} \right\}. \quad (6.5)$$

*Proof.* Let  $\text{tp } S = \omega_1^\gamma$ . In order to prove (6.5) it is enough to construct sets  $F_\mu \subset S$  ( $\mu < \omega_1$ ) such that

- (i)  $\text{tp } F_\mu \leq \omega_1^{\omega+1}$  ( $\mu < \omega_1$ ),
- (ii)  $\text{tp}(F_\mu \cap F_\lambda) < \omega_1^\omega$  ( $\mu < \lambda < \omega_1$ ),
- (iii)  $\{\mu < \omega_1 : x \in F_\mu\}$  is finite for each  $x \in S$ , and
- (iv)  $\text{tp} \left( S - \bigcup_{\mu \in N} F_\mu \right) < \omega_1^\gamma$  whenever  $N \in [W_1]^{\aleph_0}$ .

Let  $S = S_0 \cup S_1 \cup \dots \cup \hat{S}_{\omega_1}(<)$ , where  $\text{tp } S_\nu = \omega_1^{\gamma_\nu} < \omega_1^\gamma$ . By Lemma 5, for each  $\nu < \omega_1$  there are sets  $A_{\nu n} \subset S_\nu$  ( $n < \omega$ ) such that

$$\text{tp } A_{vn} < \omega_1^n \quad (n < \omega), \quad (6.6)$$

$$\{n < \omega : x \in A_{vn}\} \text{ is finite for } x \in S_v, \quad (6.7)$$

$$\text{tp} \left( S_v - \bigcup_{n_0 < n < \omega} A_{vn} \right) < \omega_1^{\omega}. \quad (6.8)$$

By the continuum hypothesis,  $[W_1]^{\aleph_0} = \{M_0, M_1, \dots, M_{\omega_1}\} \neq \emptyset$ . For  $v < \omega_1$ , let  $\{M_0, M_1, \dots, M_v\} \neq \{N_{v0}, N_{v1}, \dots, N_{v\omega}\}$  (the sets  $N_{vn}$  ( $n < \omega$ ) are not necessarily different). By Lemma 6, there is a partition of  $[W_1]^2$  into  $\aleph_0$  sets,

$$[W_1]^2 = E_0 \cup E_1 \cup \dots \cup \hat{E}_\omega,$$

such that each graph  $T_i = (W_1, E_i)$  ( $i < \omega$ ) is a forest. By Lemma 7, there are disjoint sets  $K_{vn}$  ( $n < \omega$ ) such that

$$K_{vn} \subset [N_{v0} \cup \dots \cup N_{vn}]^{\leq n+1}, \quad (6.9)$$

$$K_{vn} \cap N_{vj} \neq \emptyset \quad (j \leq n), \quad (6.10)$$

$$[K_{vn}]^2 \cap E_j = \emptyset \quad (j \leq n). \quad (6.11)$$

We shall define the sets  $F_\mu$  ( $\mu < \omega_1$ ) by describing the intersections  $F_\mu \cap S_v$  ( $\mu, v < \omega_1$ ). Let  $\mu, v < \omega_1$  be fixed. Since the sets  $K_{vn}$  ( $n < \omega$ ) are mutually disjoint, there is at most one integer  $n$  such that  $\mu \in K_{vn}$ . If  $\mu \in K_{vn}$ , then we define  $F_\mu \cap S_v = A_{vn}$ . If, on the other hand,

$$\mu \notin \bigcup_{n < \omega} K_{vn},$$

then we put  $F_\mu \cap S_v = \emptyset$ . This defines the sets  $F_\mu$  ( $\mu < \omega_1$ ) and we have to verify that (i)–(iv) hold.

Clearly (i) holds since  $\text{tp}(F_\mu \cap S_v) < \omega_1^\omega$  ( $\mu, v < \omega_1$ ). Let  $\mu < \lambda < \omega_1$ . There is a unique integer  $l$  such that  $\{\mu, \lambda\} \subset$  is an edge of  $T_l$ , i.e. such that  $\{\mu, \lambda\} \in E_l$ . Therefore, by (6.11), if  $v < \omega_1$  and  $l \leq n < \omega$ , then  $\mu$  and  $\lambda$  are not both elements of  $K_{vn}$ . It follows from this and the way  $F_\mu \cap S_v$  and  $F_\lambda \cap S_v$  are defined, that

$$F_\mu \cap F_\lambda \cap S_v \subset A_{v0} \cup \dots \cup \hat{A}_{vl}$$

for all  $v < \omega_1$ . Therefore,  $\text{tp}(F_\mu \cap F_\lambda \cap S_v) < \omega_1^l$  by (6.6) and (2.2). Thus,  $\text{tp}(F_\mu \cap F_\lambda) \leq \omega_1^{l+1} < \omega_1^\omega$  and (ii) holds.

If  $x \in S$ , then there is a unique  $v < \omega_1$  such that  $x \in S_v$ . By (6.7), there is a finite set of integers  $N(x)$  such that  $x \notin A_{vn}$  if  $n \notin N(x)$ . From the definition of the sets  $F_\mu \cap S_v$ , it follows that  $x \notin F_\mu$  unless

$$\mu \in \bigcup_{n \in N(x)} K_{vn}.$$

This proves that (iii) holds since the sets  $K_{vn}$  are finite.

Finally, let  $N \in [W_1]^{\aleph_0}$ . Then  $N = M_\sigma$  for some  $\sigma < \omega_1$ . Let  $\sigma \leq \nu < \omega_1$ . Then there is an integer  $j(\sigma, \nu)$  such that  $N = N_{\nu, j(\sigma, \nu)}$ . By (6.10) there is an element

$$\mu_n \in K_{\nu n} \cap N_{\nu, j(\sigma, \nu)}$$

for  $j(\sigma, \nu) \leq n < \omega$ . Therefore, by the definition of  $F_{\mu_n}$ ,

$$A_{\nu n} \subset F_{\mu_n} \subset \bigcup_{\mu \in N} F_\mu \quad (j(\sigma, \nu) \leq n < \omega).$$

Therefore, by (6.8),

$$\text{tp} \left( S_\nu - \bigcup_{\mu \in N} F_\mu \right) < \omega_1^\omega \quad (\sigma \leq \nu < \omega_1).$$

It follows that

$$\text{tp} \left( S - \bigcup_{\mu \in N} F_\mu \right) \leq \sum_{\nu < \sigma} \omega_1^{\gamma_\sigma} + \omega_1^{\omega+1} < \omega_1^\gamma.$$

This proves (iv) and completes the proof of Theorem 7.

We proved in (1, Theorem 10.14) with the continuum hypothesis that

$$(*) \quad \left\{ \begin{array}{c} \omega \\ \lambda \end{array} \right\} \rightarrow \left\{ \begin{array}{ccc} 1 & \omega & \omega \\ \omega_1^\omega & \vee & 1 \\ & & \omega_1^{\omega+2} \end{array} \right\} \quad (6.12)$$

holds for all  $\lambda < \omega_2$ . This result is best possible in the sense that none of the entries on the right side of (6.12) can be decreased (see §7). We shall use (6.12) to establish an analogous result for  $(\omega_1, \lambda)$ -systems.

(\*) THEOREM 8. If  $\lambda < \omega_2$ , then

$$\left\{ \begin{array}{c} \omega_1 \\ \lambda \end{array} \right\} \rightarrow \left\{ \begin{array}{ccc} 1 & \omega & \omega \\ \omega_1^{\omega+2} & \vee & 1 \\ & & \omega_1^{\omega+2} \end{array} \right\}. \quad (6.13)$$

*Proof.* It is enough to prove (6.13) for the case when  $\lambda = \omega_1^\gamma$ ,  $\gamma < \omega_2$ .

For  $\gamma < \omega + 2$ , the result is immediately obvious. We shall, therefore, assume that  $\gamma \geq \omega + 2$  and use induction on  $\gamma$ .

Let  $\text{tp } S = \omega_1^\gamma$ . We shall construct sets  $F_\mu \subset S$  ( $\mu < \omega_1$ ) such that

$$(i) \text{ tp } F_\mu < \omega_1^{\omega+2} \quad (\mu < \omega_1),$$

$$(ii) |\{\mu < \omega_1 : x \in F_\mu\}| < \aleph_0 \quad (x \in S),$$

and

$$(iii) \text{ tp} \left( S - \bigcup_{\mu \in N} F_\mu \right) < \omega_1^{\omega+2} \text{ whenever } N \in [W_1]^{\aleph_0}.$$

Case 1.  $co(\gamma) = \omega$ . In this case  $S = S_0 \cup \dots \hat{S}_\omega(<)$ , where  $tp S_n = \omega_1^{\gamma_n} < \omega_1^\gamma$ . By the induction hypothesis, there are sets  $F_{n\mu} \subset S_n$  ( $\mu < \omega_1; n < \omega$ ) such that, for  $n < \omega$ ,

$$tp F_{n\mu} < \omega_1^{\omega+2} \quad (\mu < \omega_1),$$

$$|\{\mu < \omega_1 : x \in F_{n\mu}\}| < \aleph_0 \quad (x \in S_n),$$

$$tp \left( S_n - \bigcup_{\mu \in N} F_{n\mu} \right) < \omega_1^{\omega+2} \text{ for } N \in [W_1]^{\aleph_0}.$$

It is easy to verify that the sets  $F_\mu = \bigcup_{n < \omega} F_{n\mu}$  satisfy all the conditions (i), (ii) and (iii).

The next case is less trivial.

Case 2.  $co(\omega_1^\gamma) = \omega_1$ .

In this case,

$$S = \bigcup_{v < \omega_1} S_v(<),$$

where  $tp S_v = \omega_1^{\gamma_v} < \omega_1^\gamma$  ( $v < \omega_1$ ). Therefore, by the induction hypothesis, there are sets  $F_{v\mu}^1 \subset S_v$  ( $\mu < \omega_1; v < \omega_1$ ) such that

$$tp F_{v\mu}^1 < \omega_1^{\omega+2} \quad (\mu, v < \omega_1), \quad (6.14)$$

$$|\{\mu < \omega_1 : x \in F_{v\mu}^1\}| < \aleph_0 \quad (x \in S_v; v < \omega_1), \quad (6.15)$$

$$tp \left( S_v - \bigcup_{\mu \in N} F_{v\mu}^1 \right) < \omega_1^{\omega+2} \quad (N \in [W_1]^{\aleph_0}; v < \omega_1). \quad (6.16)$$

By the continuum hypothesis,  $[W_1]^{\aleph_0} = \{N_0, N_1, \dots, \hat{N}_{\omega_1}\} \neq \emptyset$ . From Lemma 5 it follows that for each  $v < \omega_1$ , there are  $\aleph_0$  sets  $F_{v\mu}^2 \subset S_v$  ( $\mu \in N_0 \cup \dots \cup N_v$ ) such that

$$tp F_{v\mu}^2 < \omega_1^\omega \quad (\mu \in N_0 \cup \dots \cup N_v; v < \omega_1), \quad (6.17)$$

$$|\{\mu \in N_0 \cup \dots \cup N_v : x \in F_{v\mu}^2\}| < \aleph_0 \quad (x \in S_v; v < \omega_1), \quad (6.18)$$

$$tp \left( S_v - \bigcup_{\mu \in N_\rho} F_{v\mu}^2 \right) < \omega_1^\omega \quad (\rho \leq v < \omega_1). \quad (6.19)$$

By (6.12)†, there are sets  $F_{v\mu}^3 \subset S_v$  ( $\mu \leq v < \omega_1$ ) such that

$$tp F_{v\mu}^3 < \omega_1^\omega \quad (\mu \leq v < \omega_1), \quad (6.20)$$

$$|\{\mu \leq v : x \in F_{v\mu}^3\}| < \aleph_0 \quad (x \in S_v; v < \omega_1), \quad (6.21)$$

$$tp \left( S_v - \bigcup_{\mu \in N} F_{v\mu}^3 \right) < \omega_1^{\omega+2} \quad (N \in [\{0, 1, \dots, v\}]^{\aleph_0}; v < \omega_1). \quad (6.22)$$

† Note that we can only properly apply (6.12) when  $\omega \leq v < \omega_1$ . However, for  $v < \omega$ , if we put  $F_{v\mu}^3 = \emptyset$  ( $\mu \leq v$ ), then (6.20), (6.21) and (6.22) are all satisfied, the last vacuously.

Now define the sets  $F_\mu$  ( $\mu < \omega_1$ ) by putting

$$F_\mu = \bigcup_{\nu < \mu} F_{\nu\mu}^1 \cup \left( \bigcup_{\nu < \omega_1} \bigcup_{\mu \in N_0 \cup \dots \cup N_\nu} F_{\nu\mu}^2 \right) \cup \bigcup_{\mu \leq \nu < \omega_1} F_{\nu\mu}^3.$$

We shall verify that (i), (ii) and (iii) hold.

By (6.14) and (6.17) we have that

$$\text{tp}(F_\mu \cap S_\nu) < \omega_1^{\omega+2} \quad (\nu < \mu),$$

and by (6.17) and (6.20)

$$\text{tp}(F_\mu \cap S_\nu) < \omega_1^\omega \quad (\mu \leq \nu).$$

This implies that  $\text{tp} F_\mu < \omega_1^{\omega+2}$  ( $\mu < \omega_1$ ), i.e. (i) holds.

Let  $x \in S$ . Then there is a unique  $\nu < \omega_1$  such that  $x \in S_\nu$ . By (6.15), (6.18) and (6.21) it follows that  $x$  is a member of only finitely many of the sets  $F_{\nu\mu}^\rho$  ( $\rho = 1, 2$  or  $3$ ) and hence (ii) holds.

Let  $N \in [W_1]^{N_0}$ . Then there is  $\rho < \omega_1$  such that  $N_\rho = \{\mu_0, \mu_1, \dots, \hat{\mu}_\omega\} \subset N$ . Let  $\lambda = \lim_{n < \omega} \mu_n$ . If  $\nu < \lambda$ , then  $N' = \{\mu_n : n < \omega; \nu < \mu_n\}$  is infinite and therefore, by (6.16),

$$\text{tp} \left( S_\nu - \bigcup_{\mu \in N} F_\mu \right) \leq \text{tp} \left( S_\nu - \bigcup_{\mu \in N'} F_{\nu\mu}^1 \right) < \omega_1^{\omega+2}. \quad (6.23)$$

If  $\rho \leq \nu < \omega_1$ , then by (6.19),

$$\text{tp} \left( S_\nu - \bigcup_{\mu \in N} F_\mu \right) \leq \text{tp} \left( S_\nu - \bigcup_{\mu \in N_\rho} F_{\nu\mu}^2 \right) < \omega_1^\omega. \quad (6.24)$$

Also, if  $\lambda \leq \nu < \rho$ , then  $N_\rho \in [\{0, 1, 2, \dots, \nu\}]^{N_0}$  and

$$\text{tp} \left( S_\nu - \bigcup_{\mu \in N} F_\mu \right) \leq \text{tp} \left( S_\nu - \bigcup_{\mu \in N_\rho} F_{\nu\mu}^3 \right) < \omega_1^{\omega+2} \quad (6.25)$$

by (6.22). Let  $\pi = \max\{\lambda, \rho\}$ . Then (6.23), (6.24) and (6.25) show that

$$\text{tp} \left( S_\nu - \bigcup_{\mu \in N} F_\mu \right) < \omega_1^{\omega+2} \quad (\mu < \pi)$$

and

$$\text{tp} \left( S_\nu - \bigcup_{\mu \in N} F_\mu \right) < \omega_1^\omega \quad (\pi \leq \nu < \omega_1).$$

Thus (iii) holds.

Theorem 8 now follows by induction on  $\gamma$ .

## 7. Conclusion

We conclude by showing that  $\omega_1^{\omega+2}$  in (6.13) cannot be replaced by anything smaller, i.e.

$$\left\{ \begin{array}{l} \omega_1 \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{ccc} 1 & \omega & \omega \\ \omega_1^\gamma & 1 & \xi \end{array} \right\}, \quad (7.1)$$

holds if  $\xi < \omega_1^{\omega+2} \leq \omega_1^\gamma < \omega_2$ . We remark that the situation is rather different with regard to the relation (6.12). For we have proved (1, Theorem 10.13) that

$$\left\{ \begin{array}{l} \omega \\ \omega_1^\gamma \end{array} \right\} \rightarrow \left\{ \begin{array}{ccc} 1 & \omega & \omega \\ \omega_1^\gamma & 1 & \xi \end{array} \right\} \quad (7.2)$$

holds for  $\xi < \omega_1^{\omega+2} \leq \omega_1^\gamma < \omega_2$  provided that

$$co(\gamma) \neq \omega \text{ and } co(\gamma - 1) \neq \omega. \quad (7.3)$$

Here  $\gamma - 1 = \gamma'$  if  $\gamma = \gamma' + 1$  and  $\gamma - 1 = \gamma$  if  $\gamma$  is a limit ordinal. If (7.3) holds, then (7.2) is stronger than (7.1). However, if (7.3) is false, then (7.2) is also false (see Theorems 10.11 and 10.12 of (1)).

*Proof of (7.1).* In view of the remarks above, we can assume that (7.3) is false, i.e. either  $co(\gamma) = \omega$  or  $co(\gamma - 1) = \omega$ .

Let  $tp S = \omega_1^\gamma$  and let  $W_1 \times S = K_0 \cup K_1$  be a partition such that (i)  $tp(K_0(\mu)) < \omega_1^\gamma (\mu < \omega_1)$  and (ii)  $tp(K_0(x)) < \omega$  for  $x \in S$ . We have to show that there are sets  $A \subset W_1$  and  $B \subset S$  such that  $tp A \geq \omega$ ,  $tp B \geq \xi$  and  $A \times B \subset K_1$ .

*Case 1.*  $co(\gamma) = \omega$ . In this case we may write  $S = S_0 \cup S_1 \cup \dots \cup \hat{S}_\omega(<)$ , where  $tp S_n = \omega_1^{\gamma_n+2}$  and  $\omega \leq \gamma_0 < \gamma_1 < \dots < \hat{\gamma}_\omega < \gamma$ . For each  $\mu < \omega_1$  there is an integer  $n(\mu)$  such that

$$tp(K_0(\mu) \cap S_{n(\mu)}) < \omega_1^{\gamma_{n(\mu)+2}.$$

There is  $M \subset W_1$  such that  $tp M = \omega$  and  $n(\mu) = n (\mu \in M)$ . Since (7.2) holds with  $\gamma$  replaced by  $\gamma_n + 2$ , it follows that there are  $A \subset M$  and  $B \subset S_{n(\mu)}$  such that  $tp A = \omega$ ,  $tp B = \xi$  and  $A \times B \subset K_1$ .

*Case 2.*  $\gamma = \gamma' + 1$  and  $co(\gamma') = \omega$ . We may write  $S = S_0 \cup \dots \cup \hat{S}_{\omega_1}(<)$ , where  $tp S_\nu = \omega_1^{\gamma_\nu} (\nu < \omega_1)$ . Let  $\gamma_n (n < \omega)$  be a sequence of ordinals such that

$$\omega \leq \gamma_0 < \gamma_1 < \dots < \hat{\gamma}_\omega < \gamma' = \lim_{n < \omega} \gamma_n.$$

For  $\mu < \omega_1$  there are  $n(\mu) < \omega$  and  $v(\mu) < \omega_1$  such that

$$\text{tp}(F_\mu \cap S_v) < \omega_1^{v n(\mu)} \quad (v(\mu) < v < \omega_1).$$

There is  $M \subset W_1$  such that  $\text{tp } M = \omega$  and  $n(\mu) = n(\mu \in M)$ . Choose  $v_0 < \omega_1$  so that  $v(\mu) < v_0$  ( $\mu \in M$ ) and let  $S'$  be a subset of  $S_{v_0}$  of order type  $\omega_1^{v_0 n+2}$ . As in Case 1, there are  $A \subset M$  and  $B \subset S'$  such that  $\text{tp } A = \omega$ ,  $\text{tp } B = \xi$  and  $A \times B \subset K_1$ .

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