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ON THE SUM $\sum_{d|2^n-1} d^{-1}$

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To the memory of my friend, colleague and collaborator, Eri Jabotinsky

ABSTRACT

Let $\sigma(n)$ be the sum of divisors of n. In this paper we prove $\sigma(2^n - 1) < c(2^n - 1)\log\log n$.

Denote by $\sigma(n)$ the sum of divisors of n. Clearly

$$\sigma(n)/n = \sum_{d|n} \frac{1}{d}.$$

A well known result in number theory states that

$$\limsup_{n=\infty} \sigma(n)/n\log\log n = e^{\gamma} \tag{1}$$

where γ is Euler's constant. In the present note we prove the following

THEOREM.

$$\frac{\sigma(2^n-1)}{2^n-1} = \sum_{d|2^n-1} \frac{1}{d} < c_1 \log\log n. \tag{2}$$

Throughout this paper $c_1, c_2, \cdots$ denote positive absolute constants. The theorem is perhaps somewhat surprising since in view of (1) one might have expected that $\sum_{d|2^n-1} 1/d$ can occasionally become as large as $\log n$.

First of all observe that apart from the value of c_1 our Theorem is best possible. To see this let n_k be the product of the first k odd primes and let u_k $(u_k \leqq \phi(n_k))$ be the smallest integer with $2^{u_k} \equiv 1 \pmod{n_k}$.

We evidently have by well known results in number theory (prime number theorem and the theorem of Mertens, the p_i's run through the first k odd primes)

$$\sum_{d|2^{u_k}-1} \frac{1}{d} \geqq \prod\left(1 + \frac{1}{p_i}\right) > c_2 \log\log n_k > c_2 \log\log u_k.$$

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Before we prove our Theorem we state a few problems and results. Put

$$\varepsilon_n = \Sigma \frac{1}{d}, \; d \mid 2^n - 1, \; d \nmid 2^m - 1 \text{ for } m < n.$$

A well known result of Romanoff [1] states that $\Sigma_{n=1}^{\infty} \varepsilon_n / n$ converges. This follows easily from $\Sigma_{k=1}^{n} \varepsilon_k < c_3 \log n$. Probably

$$\sum_{k=1}^{n} \varepsilon_k = (c_4 + o(1)) \log n$$

and

$$c_5 < \sum_{k=n}^{2n} \varepsilon_k < c_6.$$

It seems likely that

$$\limsup_{n=\infty} n\,\varepsilon_n = \infty, \; \liminf_{n=\infty} n\,\varepsilon_n = 0$$

and that $n\varepsilon_n$ has a distribution function. I can prove only that $\varepsilon_n \to 0$ and in fact I can even prove that

$$\sum_{\substack{d \mid 2^n - 1 \\ d > n}} \frac{1}{d} \to 0. \tag{3}$$

Very likely

$$\varepsilon_n = 0 \left(\frac{1}{n^{1-\delta}} \right)$$

for every $\delta > 0$ but I could not even prove $\varepsilon_n < 1/n^{c_7}$. I could not obtain a satisfactory estimation of the sum (3). I proved that

$$\sum_{d \mid 2^n - 1} \frac{1}{d} = \frac{\sigma(2^n - 1)}{2^n - 1}$$

has a distribution function, but we do not discuss the proof here.

Now we prove our theorem. To prove (2) it will suffice to show that (p prime)

$$\sum_{p \mid 2^n - 1} \frac{1}{p} < \log\log\log n + c_8. \tag{4}$$

(4) implies (2) by $e^x > 1 + x$.
To prove (4) write

$$\sum_{p \mid 2^n - 1} \frac{1}{p} = \sum_{d \mid n} \Sigma_d \frac{1}{p} = \Sigma_1 + \Sigma_2 + \Sigma_3 \tag{5}$$

where in Σ_d p runs through the primes p satisfying $p \mid 2^d-1$, $p \nmid 2^{d'}-1$, $d' < d$, $d' \mid n$, and in Σ_1 $d \leqq (\log n)^{16}$, in Σ_2 $d > (\log n)^{16}$, $p < n$ and in Σ_3 $p \geqq n$. First we estimate Σ_1. Clearly $2^d - 1$ has fewer than d prime factors, hence Σ_1 has fewer than $(\log n)^{32}$ summands. Thus Σ_1 is less than the sum of the reciprocals of the first $[(\log n)^{32}]$ primes. Hence from the prime number theorem (or a more elementary theorem)

$$\Sigma_1 < \sum_{p<(\log n)^{33}} \frac{1}{p} < \log\log\log n + c_9. \tag{6}$$

Next we estimate Σ_2, this will be considerably more difficult than the estimation of Σ_1. First of all put

$$\Sigma_2 = \Sigma_2' + \Sigma_2'', \tag{7}$$

where in

$$\Sigma_2' = \sum_{d>(\log n)^{16}} \Sigma_{2,d}\frac{1}{p} \tag{8}$$

only primes $p > d^3$ occur in the inner sum and in Σ_2'' are the primes $d < p < d^3$ $p \mid n$ $p \equiv 1 (\operatorname{mod} d)$ (all prime factors of $2^d - 1$ which do not divide any $2^{d'} - 1$, $d' \mid d$ are well known to be $\equiv 1$ (mod d)).

$2^d - 1$ has fewer than d prime factors, thus

$$\Sigma_{2,d}\frac{1}{p} < \frac{1}{d^2}. \tag{9}$$

From (8) and (9) we have

$$\Sigma_2' = \sum_{d>(\log n)^{16}}^{\infty} \frac{1}{d^2} = o(1). \tag{10}$$

Hence we only have to estimate Σ_2'' and this will be the only difficult part of our note. Denote by $q_1 < q_2 < \cdots < q_s < n$ the sequence of primes which occur in Σ''. In other words for every q_i there is a d satisfying

(11) $d > (\log n)^{16}, q_i \equiv 1 (\operatorname{mod} d), d^3 > q_i, q_i \mid 2^d - 1, q_i \nmid 2^d - 1$ for $d_1 < d, d_1 \mid n$,

(since $(2^a - 1, 2^b - 1) = 2^{(a,b)} - 1$, $d_1 \mid n$ could be replaced by $d_1 \mid d$).

Thus

$$\Sigma'' = \sum_i \frac{1}{q_i} = \sum_{k=4}^{\infty} \Sigma_k \frac{1}{q_i} \tag{12}$$

where in Σ_k

$$(\log n)^{2^k} < q_i \leqq (\log n)^{2^{k+1}}. \tag{13}$$

Next we estimate $\Sigma_k 1/q_i$. If q_i occurs in Σ_k we have by (11) that there is a $d \mid n$ for which $q_i \equiv 1 (\text{mod } d)$, $d^3 > q_i$, or by (13)

$$(q_i - 1, n) > (\log n)^{2^{k-2}}. \tag{14}$$

Let $(\log n)^{2^k} < x < (\log n)^{2^{k+1}} (k \geqq 4,\ x \leqq n)$. Denote by $Q(x)$ the number of primes $q < x$ which satisfy (14). Let $r_1, \cdots,$ be the prime factors of n. To estimate $Q(x)$ from above, we first estimate from above (p runs through all the primes $\leqq x$)

$$A(n,x) = \prod_{p<x} (p-1,n). \tag{15}$$

We evidently have

$$A(n,x) \leqq \prod_{r_i \mid n} \prod_{l=1}^{\infty} r_i^{\pi(x, r_i^l, 1)} = \Pi_1 \, \Pi_2 \tag{16}$$

where $\pi(x,d,1)$ denotes the number of primes $p \leqq x$ satisfying $p \equiv 1 (\text{mod } d)$ and in Π_1, $r_i^l \leqq (\log n)^{10}$ and in Π_2, $r_i^l > (\log n)^{10}$.

By a theorem of Brun-Titchmarsh [2] we have for $q_i^l < (\log n)^{10}$, $x > (\log n)^{16}$

$$\pi(x, q_i^l, 1) < c_{10} \frac{x}{q_i^l \log x}. \tag{17}$$

From (17) we obtain by the theorem of Mertens, $(\exp z = e^z)$

$$\begin{aligned} \Pi_1 &\leqq \prod_{r_i^l < (\log n)^{10}} r_i^{c_{10} x / r_i^l \log x} \leqq \exp \frac{c_{10} x}{\log x} \sum_{r_i \leqq (\log n)^{10}} \sum_{l=1}^{\infty} \frac{\log r_i}{r_i^l} \\ &\leqq \exp \frac{c_{11} x \log \log n}{\log x}. \end{aligned} \tag{18}$$

Next we estimate Π_2. If $r^l > (\log n)^{10}$ we use the trivial estimate

$$\pi(x, r_i^l, 1) < \frac{x}{r_i^l} < \frac{x}{(\log n)^{10}}. \tag{19}$$

The number of prime factors of n (multiple factors counted multiply) is clearly at most $\log n / \log 2$, thus from (19) $(x \leqq n)$

$$\Pi_2 < \prod_{r_i^l \mid n} r_i^{x/(\log n)^{10}} < x^{2x/(\log n)^9} < x^{2x/(\log x)^9} = \exp \frac{2x}{(\log x)^8}. \tag{20}$$

From (16), (18) and (20) we have

$$A(n,x) < \exp c_{12}(x \log\log n/\log x). \tag{21}$$

From (20) and the definition of $Q(x)$ we have

$$A(n,x) > (\log n)^{2^{k-2}Q(x)}. \tag{22}$$

Thus finally from (21) and (22)

$$Q(x) < \frac{c_{12}x}{2^{k-2}\log x}. \tag{23}$$

From (23) we immediately obtain

$$\Sigma_k \frac{1}{q_i} < c_{13}/2^k \tag{24}$$

and thus from (23)

$$\sum_{k=4}^{\infty} \Sigma_k \frac{1}{q_i} < c_{13}. \tag{25}$$

From (6), (9), (11) and (24) we finally have

$$\Sigma_2 < c_{13} + o(1). \tag{26}$$

The estimation of Σ_3 is very simple. $2^n - 1$ clearly has fewer than n prime factors, thus

$$\Sigma_3 < 1 \tag{27}$$

(6), (26) and (27) proves (3) which completes the proof of our Theorem.

Perhaps the following stronger result holds:

Let $3, 5 \dots p_k \leqq n < 3.5 \dots p_{k+1}$. Then ($p_i$ runs through the consecutive odd primes)

$$\max_{m \leq n} \sum_{p|2^m-1} \frac{1}{p} = \sum_{i=1}^{k} \frac{1}{p_i} + o(1), \tag{28}$$

but the methods used in this note are not strong enough to decide (28).

Clearly our proof gives that for every a

$$\sum_{d|a^n-1} \frac{1}{d} < c_a \log\log n,$$

but I cannot decide whether

$$\sum_{d|2^n-3} \frac{1}{d} < c_{14} \log\log n$$

holds.

References

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2. C. E. Titchmarch, *On a divisor problem*, Rend. Circ. Mat. Palermo **54** (1930), 414–419.

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