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ON SOME PROBLEMS OF A STATISTICAL GROUP THEORY, VI

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(To the memory of S. Minakshisundaram)

1. In the second paper of this series we proved the following two theorems. Let S_n stand for the symmetric group with n letters, P a generic element of it and $O(P)$ its order. Then we have

THEOREM A. *For almost all P's in S_n, i.e. with the exception of $o(n!)$ P's at most, $O(P)$ is divisible by all prime powers not exceeding*

$$\frac{\log n}{\log \log n}\left\{1+3\frac{\log\log\log n}{\log\log n}-\frac{\omega(n)}{\log\log n}\right\}$$

if only $\omega(n) \nearrow +\infty$ arbitrarily slowly.

The other theorem shows that the theorem is best possible in the following strong sense.

THEOREM B. *If $\omega(n) \nearrow +\infty$ arbitrarily slowly, then almost no P's (i.e. only $o(n!)$ of it) have the property that $O(P)$ is divisible by all primes not exceeding*

$$\frac{\log n}{\log \log n}\left\{1+3\frac{\log\log\log n}{\log\log n}+\frac{\omega(n)}{\log\log n}\right\}.$$

Since the P's in a conjugacy class H of S_n have the same order, we may denote by $O(H)$ the common order of its elements and it is natural to ask the corresponding statistical theorem for $O(H)$. The total number of conjugacy classes in S_n is, as well known, $p(n)$, the number of partitions of n. As announced in the fifth paper of this series (to appear in *Acta Math. Hung.*) we prove the following two theorems in the above mentioned direction.

THEOREM I. *For almost all classes H, i.e. with exception of $o(p(n))$ classes, $O(H)$ is divisible by all prime powers not exceeding*

$$\frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n} \left\{ 1 + 5 \frac{\log \log n}{\log n} - \frac{\omega(n)}{\log n} \right\}$$

if only $\omega(n) \nearrow +\infty$ arbitrarily slowly.

This is again best possible in the following strong sense.

THEOREM II. *If $\omega(n) \nearrow +\infty$ arbitrarily slowly, then almost no classes H (i.e. only $o(p(n))$ of it) have the property that $O(H)$ is divisible by all primes not exceeding*

$$\frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n} \left\{ 1 + 5 \frac{\log \log n}{\log n} + \frac{\omega(n)}{\log n} \right\}.$$

The quantity in Theorems I and II is much bigger than in Theorems A and B. The interest of Theorems I and II is perhaps enhanced by the theorems proved in the fifth paper according to which the *maximal* prime factor of $O(H)$ is for almost all classes

$$\leqslant \frac{\sqrt{6}}{2\pi} \sqrt{n} \log n \left\{ 1 - 2 \frac{\log \log n}{\log n} + \frac{\omega(n)}{\log n} \right\} \tag{1.1}$$

and this is again best possible in the above sense.

It seems to be possible and would be of interest to prove that for any real x's the number $K(n, x)$ of classes H in S_n for which $O(H)$ is divisible by all prime powers

$$\leqslant \frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n} \left\{ 1 + 5 \frac{\log \log n}{\log n} - \frac{x}{\log n} \right\}$$

divided by $p(n)$ tends to a distribution function $\psi(x)$.

2. Now we turn to the proof of our Theorem I. Let, for $y > 0$,

$$f(y) = \prod_{v=1}^{\infty} \frac{1}{1 - e^{-vy}} = \sum_{n=0}^{\infty} p(n) e^{-ny}. \tag{2.1}$$

For this we have the classical functional equation

$$f(y) = \frac{1}{\sqrt{2\pi}} \sqrt{y} f\left(\frac{4\pi^2}{y}\right) \exp\left(-\frac{y}{24} + \frac{\pi^2}{6y}\right) \tag{2.2}$$

and hence for $y \to +0$

$$f(y) = (1 + o(1)) \sqrt{\frac{y}{2\pi}} \exp\left(\frac{\pi^2}{6y}\right). \tag{2.3}$$

Let $Y = Y(n) \to \infty$ with n to be determined later and let q run through all prime powers with

$$q \leqslant Y(n). \tag{2.4}$$

Let further $p_q(n)$ be the number of all partitions of n with the property that no summand is divisible by q. Then we have for $y > 0$

$$\sum_{n=0}^{\infty} p_q(n)\, e^{-ny} = \prod_{q \nmid n} \frac{1}{1 - e^{-ny}} = \frac{f(y)}{f(qy)} \tag{2.5}$$

Putting

$$\sum_{q \leqslant Y} p_q(n) \overset{\text{def.}}{=} h_Y(n)$$

we get

$$\sum_{n=0}^{\infty} h_Y(n) e^{-ny} = \sum_{q \leqslant Y} \frac{f(y)}{f(qy)}. \tag{2.6}$$

Using (2.3) we get for all q's in (2.6)

$$\frac{f(y)}{f(qy)} = \frac{1 + o(1)}{\sqrt{q}} \exp\left\{\frac{\pi^2}{6}\left(1 - \frac{1}{q}\right)\frac{1}{y}\right\} \tag{2.7}$$

if only

$$qy \to 0. \tag{2.8}$$

Hence, if y and $\frac{1}{Y}$ are sufficiently small, we have

$$\sum_{n=0}^{\infty} h_Y(n)\, e^{-ny} < 2 \exp\left\{\frac{\pi^2}{6}\left(1 - \frac{1}{q}\right)\frac{1}{y}\right\} \sum_{q \leqslant Y} \frac{1}{\sqrt{q}}$$

$$< 5 \frac{\sqrt{Y}}{\log Y} \exp\left\{\frac{\pi^2}{6}\left(1 - \frac{1}{Y}\right)\frac{1}{y}\right\}.$$

Putting

$$y = \frac{\pi}{\sqrt{6}} \frac{\sqrt{1-\frac{1}{Y}}}{\sqrt{n}} \stackrel{\text{def.}}{=} \frac{\lambda}{\sqrt{n}},$$

we get

$$h_Y(n)\, e^{-\lambda\sqrt{n}} = h_Y(n)\, e^{-ny} < \sum_{m=0}^{\infty} h_Y(m)\, e^{-my}$$

$$< 5\frac{\sqrt{Y}}{\log Y} \exp\left\{\frac{\pi}{\sqrt{6}}\sqrt{n} - \frac{1}{Y}\sqrt{n}\right\}$$

and hence

$$h_Y(n) < 5\frac{\sqrt{Y}}{\log Y} \exp\left\{\frac{2\pi}{\sqrt{6}}\sqrt{n} - \frac{1}{Y}\sqrt{n}\right\} \tag{2.9}$$

$$< 5\frac{\sqrt{Y}}{\log Y} \exp\left\{\frac{2\pi}{\sqrt{6}}\left(1-\frac{1}{2Y}\right)\sqrt{n}\right\}.$$

Using the classical formula of Hardy-Ramanujan, we have

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\frac{2\pi}{\sqrt{6}}\cdot\sqrt{n}\right) \tag{2.10}$$

which gives for all sufficiently large n,

$$h_Y(n) < 40\frac{\sqrt{Y}}{\log Y}\, p(n)\, n \exp\left\{-\frac{\pi}{\sqrt{6}}\frac{\sqrt{n}}{Y}\right\}. \tag{2.11}$$

Now choosing

$$Y = \frac{4}{5}\cdot\frac{\pi}{\sqrt{6}}\cdot\frac{\sqrt{n}}{\log n}, \tag{2.12}$$

the restriction (2.8) is satisfied and hence (2.11) gives

$$\frac{h_Y(n)}{p(n)} \to 0 \text{ for } n \to \infty. \tag{2.13}$$

3. Now, as is well known, there is a one-to-one correspondence between the conjugacy classes H of S_n and partitions

$$n = m_1 n_1 + m_2 n_2 + \dots + m_k n_k$$
$$1 \leq n_1 < n_2 < \dots < n_k \tag{3.1}$$

of n; moreover

$$O(H) = [n_1, n_2, \dots, n_k]\ V. \tag{3.2}$$

Hence $O(H)$ is divisible by a prime power q if and only if q is the divisor of some summand n_j and $h_Y(n)$ is an upper bound for the number of conjugacy classes H of S_n whose order is *not* divisible by *some* prime power $q \leqslant Y$. Hence (2.13) means that for almost all classes H the quantity $O(H)$ is divisible by all prime powers not exceeding

$$\frac{4}{5} \cdot \frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}. \tag{3.3}$$

4. Next we consider the divisibility of $O(H)$ by the prime powers q satisfying

$$\frac{4}{5} \cdot \frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n} \leqslant q \leqslant \frac{10\,\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}. \tag{4.1}$$

For this sake, we need a more delicate treatment of $p_q(n)$. Taking into account the Euler-Legendre "Pentagonalsatz" according to which for $\mathrm{Re}\, z > 0$ the relation

$$\left(\frac{1}{f(z)} =\right) \prod_{v=1}^{\infty} (1 - e^{-vz}) = \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(-\frac{3k^2 + k}{2} z\right) \tag{4.2}$$

holds, equation (2.5) gives the representation

$$p_q(n) = \sum\nolimits'_{(k)} (-1)^k p\left(n - \frac{3k^2 + k}{2} q\right), \tag{4.3}$$

where the summation is to be extended over the k's with

$$\frac{3k^2 + k}{2} \leqslant \frac{n}{q}. \tag{4.4}$$

5. First we shall estimate the contribution of the k's with

$$|k| > 10 \frac{\sqrt{n}}{q} \tag{5.1}$$

to the sum in (4.3). Then we have

$$\frac{3k^2 + k}{2} \geqslant k^2 \geqslant 10 \frac{\sqrt{n}}{q} k$$

and thus

$$n - \frac{3k^2+k}{2} q \leqslant n - 10\sqrt{n}k < (\sqrt{n} - 5k)^2;$$

since from (2.10)†

$$p(n) < c \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right), \tag{5.2}$$

we have for the k's in (5.1)

$$p\left(n - \frac{3k^2+k}{2} q\right) < c \exp\left(\frac{2\pi}{\sqrt{6}}\left(n - \frac{3k^2+k}{2}\right) q\right)$$
$$< \exp\left\{\frac{2\pi}{\sqrt{6}}(\sqrt{n} - 5k)\right\}.$$

Hence

$$\left|\sum_{|k|>10\sqrt{n/q}} (-1)^k p\left(n - \frac{3k^2+k}{2} q\right)\right|$$
$$< c \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) \sum_{k>10\sqrt{n/q}} \exp\left(-\frac{10\pi}{\sqrt{6}} k\right)$$
$$< cn^{-6} \exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right) < cn^{-5} p(n)$$

by (2.10). Hence, from (4.3),

$$p_q(n) = \sum_{|k|\leqslant 10\sqrt{n/q}} (-1)^k p\left(n - \frac{3k^2+k}{2} q\right) + O(n^{-5})\, p(n). \tag{5.3}$$

6. Next we use Hardy-Ramanujan's stronger formula (see [2]) in the form

$$p(m) = \frac{\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{(m-1/24)}\right)}{4\left(m - \frac{1}{24}\right)\sqrt{3}} \left\{1 - \frac{1}{\pi}\sqrt{\frac{3}{2}} \frac{1}{\sqrt{(m-1/24)}}\right\} +$$
$$+ O(1) \exp\left\{-0,49 \frac{2\pi}{\sqrt{6}}\sqrt{m}\right\}. \tag{6.1}$$

Noticing the elementary relation

† c means throughout this paper an unspecified (explicitly calculable) positive constant.

$$\exp\{c_1(\sqrt{(x-y)}-\sqrt{x})\}\frac{x}{x-y}\cdot\frac{1-\dfrac{c_2}{\sqrt{(x-y)}}+O(1)\exp(-c_3\sqrt{(x-y)})}{1-\dfrac{c_2}{\sqrt{x}}+O(1)\exp(-c_3\sqrt{x})}$$

$$=\exp\left(-\frac{c_1 y}{2\sqrt{x}}\right)\left\{1+c_4\frac{y^2}{x^{3/2}}+c_5\frac{y^3}{x^{5/2}}+c_6\frac{y^4}{x^3}+O(x^{-1,46})\right\} \quad (6.2)$$

where the c_ν's are positive constants and

$$0<y\leqslant x^{0.51}, \quad (6.3)$$

we obtain using (6.1) for the k's in (5.3) and q's in (4.1) from (6.2) with

$$c_1=\frac{2\pi}{\sqrt{6}},\ x=n-\frac{1}{24},\ y=\frac{3k^2+k}{2}q \quad (6.4)$$

that

$$\frac{p\left(n-\dfrac{3k^2+k}{2}q\right)}{p(n)}=\exp\left(-\frac{3k^2+k}{2}\cdot\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{(m-1/24)}}\right)\times$$

$$\times\left\{1+c_4\left(\frac{3k^2+k}{2}\right)^2\cdot\frac{q^2}{\left(n-\dfrac{1}{24}\right)^{3/2}}+c_5\left(\frac{3k^2+k}{2}\right)^3\frac{q^3}{\left(n-\dfrac{1}{24}\right)^{5/2}}+\right.$$

$$\left.+c_6\left(\frac{3k^2+k}{2}\right)^4\frac{q^4}{\left(n-\dfrac{1}{24}\right)^3}+O(n^{-1,46})\right\}. \quad (6.5)$$

Putting this into (5.3), we get at once

$$\frac{p_q(n)}{p(n)}=\sum_{|k|\leqslant 10\sqrt{n/q}}(-1)^k\exp\left(-\frac{3k^2+k}{2}\cdot\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{(n-1/24)}}\right)+$$

$$+c_4\frac{q^2}{(n-1/24)^{3/2}}\sum_{|k|\leqslant 10\sqrt{n/q}}(-1)^k\left(\frac{3k^2+k}{2}\right)^2\times$$

$$\times\exp\left(-\frac{3k^2+k}{2}\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{(n-1/24)}}\right)+$$

$$+c_5\frac{q^3}{(n-1/24)^{5/2}}\sum_{|k|\leqslant 10\sqrt{n/q}}(-1^k)\left(\frac{3k^2+k}{2}\right)^3\times$$

$$\times \exp\left(-\frac{3k^2+k}{2}\frac{\pi}{\sqrt{6}}\frac{q}{\sqrt{(n-1/24)}}\right)+$$

$$+c_6\frac{q^4}{(n-1/24)^3}\sum_{|k|\leq 10\sqrt{n}/q}(-1)^k\left(\frac{3k^2+k}{2}\right)^4\times$$

$$\times \exp\left(-\frac{3k^2+k}{2}\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{(n-1/24)}}\right)+O(n^{-1.46}\log n). \quad (6.6)$$

Obviously the same error term holds completing the sum in (6.6) to $-\infty<k<+\infty$; putting

$$\sum\nolimits_{(k)}(-1)^k\left(\frac{3k^2+k}{2}\right)^\nu \exp\left(-\frac{3k^2+k}{2}\cdot\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{n-1/24)}}\right) \quad (6.7)$$

equal to $S_\nu(n, q)$, we get

$$\frac{p_q(n)}{p(n)}=S_0(n,q)+c_4\frac{q^2}{\left(n-\frac{1}{24}\right)^{3/2}}S_2(n,q)+$$

$$+c_5\frac{q^3}{\left(n-\frac{1}{24}\right)^{5/2}}S_3(n,q)+c_6\frac{q^4}{\left(n-\frac{1}{24}\right)^3}S_4(n,q)+O\ (n^{-1.45}). \quad (6.8)$$

7. In order to investigate $S_\nu(n, q)$ we take the reciprocal of (2.2) and apply the functional equation (4.2). This gives for $y>0$

$$\sum_{k=-\infty}^{\infty}(-1)^k\exp\left(-\frac{3k^2+k}{2}y\right)=\sqrt{\frac{2\pi}{y}}\exp\left(\frac{y}{24}-\frac{\pi^2}{6y}\right)\times$$

$$\times\sum_{k=-\infty}^{\infty}(-1)^k\exp\left(-\frac{3\,k^2+k}{2}\cdot\frac{4\pi^2}{y}\right) \quad (7.1)$$

and hence

$$S_0(n,q)=\sqrt{(2\sqrt{6})}\,\frac{(n-1/24)^{1/4}}{\sqrt{q}}\exp\left\{\frac{\pi}{\sqrt{6}}\cdot\frac{q}{\sqrt{(n-1/24)}}-\right.$$

$$\left.-\frac{\pi}{\sqrt{6}}\frac{\sqrt{(n-1/24)}}{q}\right\}\left\{1+O(1)\exp\left(-4\pi\sqrt{6}\frac{\sqrt{(n-1/24)}}{q}\right)\right\}. \quad (7.2)$$

For our present aims it is enough to write

$$S_0(n, q) = (1 + o(1)) \sqrt{(2\sqrt{6})}\, \frac{n^{1/4}}{\sqrt{q}} \exp\left(-\frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{q}\right). \tag{7.3}$$

Differentiation in (7.1) leads easily to

$$S_\nu(n, q) = O(\log^{10} n) \exp\left(-\frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{q}\right) \tag{7.4}$$

and thus (6.8) together with (4.1) gives

$$p_q(n) = (1 + o(1)) \sqrt{(2\sqrt{6})}\, \frac{n^{1/4}}{\sqrt{q}} \exp\left(-\frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{q}\right) p(n). \tag{7.5}$$

For further aims we shall need a more exact formula for $S_\nu(n, q)$. Let us differentiate the identity (7.1) ν times ($1 \leqslant \nu \leqslant 4$). This is the sum of $(\nu + 1)$ terms each of the form

$$p_j(y) \exp\left(\frac{y}{24} - \frac{\pi^2}{6y}\right) \sum_{(k)} (-1)^k \left(\frac{3k^2 + k}{2}\right)^j \exp\left(-\frac{3k^2 + k}{2} \cdot \frac{4\pi^2}{y}\right), \tag{7.6}$$

$$j = 0, 1, \dots, \nu,$$

where the $p_j(y)$'s are polynomials in $\frac{1}{\sqrt{y}}$ of degree $\leqslant 20$ with bounded coefficients. In particular, for $j = 0$, we have

$$\left\{ \sqrt{\frac{2\pi}{y}} \exp\left(\frac{y}{24} - \frac{\pi^2}{6y}\right)\right\}^{(\nu)} \sum_{(k)} (-1)^k \exp\left(-\frac{3k^2 + k}{2} \cdot \frac{4\pi^2}{y}\right)$$

whereas for the terms with $j \geqslant 1$, since the term with $k = 0$ is missing from the sum, we have an upper bound

$$O(\log^{10} n) \exp\left\{-\left(\frac{\pi}{\sqrt{6}} + 4\pi\sqrt{6}\right)\frac{\sqrt{n}}{q}\right\}.$$

Hence, for $1 \leqslant \nu \leqslant 4$ we have

$$S_\nu(n, q) = \left\{ \sqrt{\frac{2\pi}{y}} \exp\left(\frac{y}{24} - \frac{\pi^2}{6y}\right)\right\}^{(\nu)}_{y = \frac{\pi}{\sqrt{6}} \cdot \frac{q}{\sqrt{(n - 1/24)}}} +$$

$$+ O(\log^{10} n) \exp\left\{-\left(\frac{\pi}{\sqrt{6}} + 4\pi\sqrt{6}\right)\frac{\sqrt{n}}{q}\right\}. \tag{7.7}$$

8. Now we may complete the proof of Theorem I. Let

$$Y_1 = \frac{4}{5}\frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}, \tag{8.1}$$

$$Y_2 = \lambda \frac{\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n},$$

where λ will be determined later. Putting

$$h^*(n) \overset{\text{def}}{=} \sum_{Y_1 \leq q \leq Y_2} p_q(n) \tag{8.2}$$

gives (7.5) for all sufficiently large n's,

$$h^*(n) < 3\, p(n)\, n^{1/4} \int_{Y_1}^{Y_2} \frac{1}{\sqrt{x}} \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{x}\right) d\Theta(x) \tag{8.3}$$

where $\Theta(x)$ stands for the number of prime powers not exceeding x. Using the prime number theorem in the form

$$\Theta(x) = \mathrm{Li}\, x + O(x) \exp(-\sqrt{\log x}),$$

the factor of $p(n)$ in (8.3) is

$$(1 + o(1)) n^{1/4} \int_{Y_1}^{Y_2} \frac{1}{\sqrt{x} \log x} \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{x}\right) dx$$

$$= o\left(\frac{1}{\sqrt{\log n}}\right) \int_{Y_1}^{Y_2} \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{x}\right) dx.$$

Since the last integral

$$= \frac{\pi}{\sqrt{6}} \cdot \sqrt{n} \int_{(1/\lambda)\log n}^{(5/4)\log n} \frac{1}{y^2} e^{-y}\, dy = o\left(\frac{n^{\frac{1}{2}-1/\lambda}}{\log^2 n}\right)$$

we have

$$\frac{h^*(n)}{p(n)} = O\left(\frac{n^{\frac{1}{2}-1/\lambda}}{\log^{5/2} n}\right) = o\,(1)$$

choosing

$$\lambda = 2\left(1 + 5\frac{\log \log n}{\log n} - \frac{\omega(n)}{\log n}\right) \tag{8.4}$$

if only

$$\omega(n) \nearrow +\infty$$

arbitrarily slowly. Repeating the reasoning of **3**, the proof of Theorem I is finished.

9. Next we turn to show that the theorem is best possible, i.e. to Theorem II. Let again $\omega(n) \nearrow \infty$ arbitrarily slowly; further

$$X_1 = \frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}\left(1 + 5\,\frac{\log\log n}{\log n} - \frac{\omega(n)}{\log n}\right),$$

$$X_2 = \frac{2\pi}{\sqrt{6}} \cdot \frac{\sqrt{n}}{\log n}\left(1 + 5\,\frac{\log\log n}{\log n} + \frac{\omega(n)}{\log n}\right) \tag{9.1}$$

and

$$X_1 \leqslant q_1 < q_2 < \dots < q_l \leqslant X_2 \tag{9.2}$$

all primes of this interval. We define the class-function $k(H)$ by

$$k(H) = \sum_{q_\nu | O(H)}^{(\nu)} 1. \tag{9.3}$$

First we investigate

$$S_1 = \sum_{(H)} k(H). \tag{9.4}$$

Obviously

$$S_1 = \sum_{\nu=1}^{l} \sum_{q_\nu | O(H)}^{(H)} 1 = \sum_{\nu=1}^{l} p_{q_\nu}(n).$$

Using the representation (7.5) (which can be used owing to (4.1),

$$S_1 = (1 - o(1))\sqrt{(2\sqrt{6})}\,p(n)n^{\frac{1}{4}} \sum_{\nu=1}^{l} \frac{1}{\sqrt{q_\nu}} \exp\left(-\frac{\pi}{\sqrt{6}}\,\frac{\sqrt{n}}{q_\nu}\right)$$

$$= (1 + o(1))\sqrt{(2\sqrt{6})}\,p(n)\,n^{\frac{1}{4}} \int_{X_1}^{X_2} \frac{\exp\left(-\frac{\pi}{\sqrt{6}}\cdot\frac{\sqrt{n}}{x}\right)}{\sqrt{x}\log x}.$$

Since this time we need asymptotic formula for S_1 we have to proceed a bit more carefully than in **8**. Now

$$S_1 = (1 + o(1))\,2\sqrt{\frac{6}{\pi}}\,\frac{p(n)}{\sqrt{\log n}} \int_{X_1}^{X_2} \exp\left(-\frac{\pi}{\sqrt{6}}\,\frac{\sqrt{n}}{x}\right) dx$$

$$= (1 + o(1))\,8\sqrt{\pi}\,p(n)\exp\left(\frac{\omega}{2}\right)(\to +\infty). \tag{9.5}$$

10. Next let

$$S_2 = \sum_{(H)} k(H)^2. \tag{10.1}$$

Then

$$S_2 = \sum_{(H)} \sum_{q_\mu / O(H)}^{(\mu)} \sum_{q_\nu / O(H)}^{(\nu)} 1$$

$$= S_1 + \sum_{1 \leq \mu \neq \nu \leq l} \sum_{\substack{q_\mu / O(H) \\ q_\nu / O(H)}}^{(H)} 1. \tag{10.2}$$

Fixing μ and ν the inner sum is the number of such partitions of n in which no summand is divisible either by q_μ or by q_ν. With the notation of (4.2) this quantity is as easy to see

$$\text{the coefficient } e^{-nz} \text{ in } \frac{f(z)\, f(q_\mu q_\nu z)}{f(q_\mu z)\, f(q_\nu z)}. \tag{10.3}$$

Hence

$$S_2 = \text{the coefficient } e^{-nz} \text{ in } f(z) \left\{ \sum_{\mu=1}^{l} \frac{1}{f(q_\mu z)} + \right.$$

$$\left. + \sum_{1 \leq \mu \neq \nu \leq l} \frac{f(q_\mu q_\nu z)}{f(q_\mu z)\, f(q_\nu z)} \right\}. \tag{10.4}$$

The function in the curly bracket is

$$\left(\sum_{\mu=1}^{l} \frac{1}{f(q_\mu z)} \right) + \left(\sum_{\mu=1}^{l} \frac{1}{f(q_\mu z)} \right)^2 - \left(\sum_{\mu=1}^{l} \frac{1}{f(q_\mu z)^2} \right) +$$

$$+ \sum_{1 \leq \mu \neq \nu \leq l} \frac{f(q_\mu q_\nu z) - 1}{f(q_\mu z)\, f(q_\nu z)} \tag{10.5}$$

and accordingly we split S_2 into the parts

$$S_1,\ S_2^{(1)}\ S_2^{(2)} \text{ and } S_2^{(3)}. \tag{10.6}$$

11. Since from (2.1) and (4.2)

$$\frac{f(z)\,\{f(q_\mu q_\nu z) - 1\}}{f(q_\mu z)\, f(q_\nu z)} = \left\{ \sum_{k_1=0}^{\infty} p(k_1)\, e^{-k_1 z} \right\}. \tag{11.1}$$

$$\left\{\sum_{k_2=1}^{\infty} p(k_2)\exp(-k_2 q_\mu q_\nu z)\right\}\left\{\sum_{k_3,k_4=-\infty}^{\infty}(-1)^{k_3+k_4}\times\right.$$
$$\left.\times\exp\left(-\frac{3k_3^2+k_3}{2}q_\mu+\frac{3k_4^2+k_4}{2}q_\nu\right)z\right\}$$

we have

$$S_2^{(3)} = \sum_{k_2,k_3,k_4}^{1}(-1)^{k_3+k_4}p(k_2)\times$$
$$\times\sum_{1\leqslant\mu\neq\nu\leqslant l}\left(n-k_2 q_\mu q_v-\frac{3k_3^2+k_3}{2}q_\mu-\frac{3k_4^2+k_4}{2}q_\nu\right) \qquad (11.2)$$

where the outer summation is to be extended to all (k_2, k_3, k_4) systems with

$$k_2 \geqslant 1$$
$$q_\mu q_\nu k_2+\frac{3k_3^2+k_3}{2}q_\mu+\frac{3k_4^2+k_4}{2}q_\nu \leqslant n. \qquad (11.3)$$

Using (5.2) and (9.1) – (9.2), the inner sum in (11.2) is quite roughly

$$< c\sum_{1\leqslant\mu,\nu\leqslant l}\exp\left(\frac{2\pi}{\sqrt{6}}\left\{n-\frac{2\pi^2}{3}\frac{n}{\log^2 n}\right\}^{1/2}\right)$$
$$< c\exp\left(\frac{2\pi}{\sqrt{6}}\sqrt{n}\right)\cdot\frac{n\,\omega(n)^2}{\log^2 n}\exp\left(-\frac{2\pi^3}{3\sqrt{6}}\frac{\sqrt{n}}{\log^2 n}\right)$$
$$< c\,p(n)n^2\exp\left(-\frac{2\pi^3}{3\sqrt{6}}\cdot\frac{\sqrt{n}}{\log^2 n}\right). \qquad (11.4)$$

Since roughly k_2 takes at most $O(\log^2 n)$-values, further k^3 and k^4 each at most $O(n^{\frac{1}{4}}\log n)$-values, we get from (11.4) at once

$$S_2^{(3)} = o(p(n)). \qquad (11.5)$$

12. Next we consider $S_2^{(2)}$. Since from (10.5) and (4.2), we have

$$-\frac{f(z)}{f(q_\mu z)^2} = -\left(\frac{f(z)}{f(q_\mu z)}\right)\frac{1}{f(q_\mu z)}$$
$$= \left(\sum_{m=0}^{\infty}p_{q_\mu}(m)\,e^{-mz}\right)\left(\sum\nolimits_{(k)}(-1)^{k+1}\exp\left\{-\frac{3k^2+k}{2}q_\mu z\right\}\right)$$

we get

$$S_2^{(2)} = \sum_{\mu=1}^{l} \sum\nolimits_{(k)} (-1)^{k+1} p_{q_\mu}\left(n - \frac{3k^2+k}{2} q_\mu\right). \tag{12.1}$$

The contribution of terms with $|k| > 10 \log n$ is absolutely

$$< \sum_{\mu=1}^{l} \sum_{10 \log n \leq |k| \leq \sqrt{n}/q_\mu} p\left(n - \frac{3k^2+k}{2} q_\mu\right) = O(p(n))$$

as in **5**. For the remaining terms in (12.1) we can apply the representation (6.8) — (7.2) — (7.4) in the form

$$p_q(n) = \sqrt{(2\sqrt{6})} \frac{n^{\frac{1}{4}}}{\sqrt{q}} p(n) \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{q}\right) \left\{1 + O\left(\frac{1}{\log n}\right)\right\}. \tag{12.2}$$

The contribution of the error term to (12.1) is absolutely

$$O(1)\, p(n) \sum_{\mu=1}^{l} \frac{n^{\frac{1}{4}}}{\sqrt{q_\mu}} \sum_{|k| \leq 10\sqrt{n}/q_\mu} \exp\left(\frac{\pi}{\sqrt{6}} \cdot \frac{\left\{n - \frac{3k^2+k}{2} q_\mu\right\}^{1/2}}{q_\mu}\right) = o(p(n))$$

using (9.1). Hence, from (12.1) and (12.2), we have

$$S_2^{(2)} = o(p(n)) + \sqrt{(2\sqrt{6})} \sum_{\mu=1} \frac{1}{\sqrt{q_\mu}} \sum_{k \leq 10 \log n}^{(k)} (-1)^{k+1} \times$$

$$\times \left(n - \frac{3k^2+k}{2} q_\mu\right)^{\frac{1}{4}} p\left(n - \frac{3k^2+k}{2} q_\mu\right) \times$$

$$\times \exp\left(-\frac{\pi}{\sqrt{6}} \cdot \frac{\left\{n - \frac{3k^2+k}{2} q_\mu\right\}^{1/2}}{q_\mu}\right). \tag{12.3}$$

Rough estimations show that replacing

$$\left(n - \frac{3k^2+k}{2} q_\mu\right)^{1/4} \text{ by } n^{1/4}$$

and

$$\exp\left(-\frac{\pi}{\sqrt{6}} \frac{\left\{n - \frac{3k^2+k}{2} q_\mu\right\}^{1/2}}{q_\mu}\right) \text{ by } \exp\left(-\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{q_\mu}\right)$$

the error is again $o(p(n))$ and hence

$$S_2^{(2)} = o(p(n)) + \sqrt{(2\sqrt{6})}\, n^{\frac{1}{4}} \sum_{\mu=1}^{l} \frac{1}{\sqrt{q_\mu}} . \exp\left\{ -\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{q_\mu} \right\} \times$$

$$\times \left(\sum_{|k| \leqslant 10 \log n} (-1)^{k+1} p(n - \frac{3k^2+k}{2} q_\mu) \right). \quad (12.4)$$

Completing the inner sum means again an error of $o(p(n))$ and using (4.3) we get

$$S_2^{(2)} = o(p(n)) - \sqrt{(2\sqrt{6})}\, n^{\frac{1}{4}} \sum_{\mu=1}^{l} \frac{p_{q_\mu}(n)}{\sqrt{q_\mu}} \times$$

$$\times \exp\left\{ -\frac{\pi}{\sqrt{6}} \frac{\sqrt{n}}{q_\mu} \right\} < o(p(n)).^\dagger \quad (12.5)$$

13. Next we consider $S_2^{(1)}$. Using (4.2) and (2.1)

$$f(z) \left(\sum_{\mu=1}^{l} \frac{1}{f(q_\mu z)} \right)^2 = \left\{ \sum_{m=0}^{\infty} p(m)\, e^{-mz} \right\} \times$$

$$\times \left\{ \sum_{\mu_1=1}^{l} \sum_{\mu_2=1}^{l} \sum_{k_1} \sum_{k_2} (-1)^{k_1+k_2} \exp\left(-\frac{3k_1^2+k_1}{2} q_{\mu_1} - \frac{3k_2^2+k_2}{2} q_{\mu_2} \right) z \right\}$$

and hence the representation

$$S_2^{(1)} = \sum_{\mu_1=1}^{l} \sum_{\mu_2=1}^{l} \sum_{k_1} \sum_{k_2} (-1)^{k_1+k_2} \times$$

$$\times\, p\left(n - \frac{3k_1^2+k_1}{2} q_{\mu_1} - \frac{3k_2^2+k_2}{2} q_{\mu_2} \right). \quad (13.1)$$

One can see easily as in **5**, that the contribution of k_2's with $|k_2|\, m > 10 \log n$ is $o(p(n))$ and hence using also (4.3)

$$S_2^{(1)} = o(p(n)) + \sum_{\mu_1=1}^{l} \sum_{\mu_2=1}^{l} \sum_{|k_2| \leqslant 10 \log n} (-1)^{k_2}\, p_{q_{\mu_1}}\left(n - \frac{3k_2^2+k_2}{2} q_{\mu_2} \right).$$

(13.2)

To go further, we shall need for $p_{q_{\mu_1}}(m)$ an asymptotic representation which is finer than the one in (7.5) (even the one in (12.2)).

†It would be easy to show $\int_2^{(2)} = o(p(n))$ but, for our aims, (12.5) is enough.

Using (6.8) and the formula (7.7) we get

$$\frac{p_q(m)}{p(m)} = S_0(m, q) + \Big\{ c_4 \frac{q^2}{(m-1/24)^{3/2}} \left(\sqrt{\frac{2\pi}{y}} \exp\left(\frac{y}{24} - \frac{\pi^2}{6y}\right)\right)^{''} +$$

$$+ c_5 \frac{q^3}{(m-1/24)^{5/2}} \left(\sqrt{\frac{2\pi}{y}} \exp\left(\frac{y}{24} - \frac{\pi^2}{6y}\right)\right)^{(3)} +$$

$$+ c_6 \frac{q^4}{(m-1/24)^{3}} \left(\sqrt{\frac{2\pi}{y}} \exp\left(\frac{y}{24} - \frac{\pi^2}{6y}\right)\right)^{(4)} \Big\}_{y=\pi/\sqrt{6}\, q/\sqrt{m-1/24}} +$$

$$+ O(m^{-1,45}). \qquad (13.3)$$

The contribution of the error term in (13.3) to $S_1^{(2)}$ in (13.2) is seen to be by (9.1) easily $o(p(n))$. Further we shall discuss in detail the contribution of the $S_0(m, q)$. $p(m)$-term. The others could be dealt with quite analogously; their contribution will be $o(p(n))$ owing to the factors

$$\frac{q^2}{(m-1/24)^{3/2}}, \frac{q^3}{(m-1/24)^{5/2}}, \frac{q^4}{(m-1/24)^{3}}$$

which are by (9.1) of order $1/\sqrt{n}$, if only

$$\omega(n) = o(\log \log n).$$

The contribution U of $p(m) S_0(m, q)$ is by (7.2)

$$o(p(n)) + \sqrt{(2\sqrt{6})} \sum_{\mu_1=1}^{l} \sum_{\mu_2=1}^{l} \sum_{|k_2| \leq 10 \log n} (-1)^{k_2} \frac{\left(n - \frac{3k_2^2+k_2}{2} q_{\mu_2} - \frac{1}{24}\right)}{q_{\mu_2}}$$

$$\times \exp\left[\frac{\pi}{\sqrt{6}} \frac{q_{\mu_2}}{\left\{n - \frac{3k_2^2+k_2}{2} q_{\mu_2} - \frac{1}{24}\right\}^{1/2}} - \right.$$

$$\left. - \frac{\pi}{\sqrt{6}} \frac{\left\{n - \frac{3k_2^2+k_2}{2} q_{\mu_2} - \frac{1}{24}\right\}^{1/2}}{q_{\mu_1}} \right] \times$$

$$\times p\left(n - \frac{3k_2^2+k_2}{2} q_{\mu_2}\right). \qquad (13.4)$$

By the elementary formula (with suitable numerical constants d_ν)

$$(x-y)^{\frac{1}{4}} \exp\left\{c\left(\frac{q}{\sqrt{(x-y)}}-\frac{\sqrt{(x-y)}}{q}\right)\right\}=x^{\frac{1}{4}} \exp\left\{c\left(\frac{q}{\sqrt{x}}-\frac{\sqrt{x}}{q}\right)\right\} \times$$

$$\times\left\{1+d_1 \frac{y\, q}{x^{3/2}}+d_2 \frac{y^2 q^2}{x^3}+d_3 \frac{y^2 q}{x^{5/2}}+d_4 \frac{y}{q\, x^{1/2}}+\right.$$

$$\left.+d_5 \frac{y^2}{q x}+d_6 \frac{y^2}{q^2 x}+O\left(\frac{y^3 q}{x^{9/2}}\right)+O\left(\frac{y^3}{q\, x^{5/2}}\right)\right\}, \tag{13.5}$$

valid for

$$0<y \leqslant x^{0,51}, \quad q<\sqrt{x}.$$

Using it with

$$c=\frac{\pi}{\sqrt{6}}, x=n-\frac{1}{24}, y=\frac{3 k_2^2+k_2}{2} q_{\mu_2}, q=q_{\mu_1},$$

we obtain analogously as in **6** and **7**,

$$U=O(\rho(n))+\sqrt{(2 \sqrt{6})} \sum_{\mu_1=1}^{l} \frac{\left(n-\frac{1}{24}\right)^{\frac{1}{4}}}{q_{\mu_1}} \times$$

$$\times \exp\left(\frac{\pi}{\sqrt{6}} \frac{q_{\mu_1}}{\sqrt{(n-1/24)}}-\frac{\sqrt{(n-1/24)}}{q_{\mu_1}}\right) \times$$

$$\times\left\{\sum_{\mu_2=1}^{l} \sum_{k_2}(-1)^{k_2} p\left(n-\frac{3k_2^2+k_2}{2} q_{\mu_2}\right)\right\}. \tag{13.6}$$

The sum in the curly brackets is by (4.3) (or (5.3))

$$=\sum_{\mu_2=1}^{l} p_{q_{\mu_2}}(n)=S_1$$

and the sum with respect to μ_1 is

$$\frac{1}{p(n)}(1+o(1)) S_1$$

by (6.8), (7.2) and (7.7). Thus using (9.5), we have

$$S_2^{(1)} = (1 + o\,(1))\,\frac{1}{p(n)}\,S_1^2$$

$$= (1 + o\,(1))\,p(n)\left\{8\sqrt{\pi}\exp\left(\frac{\omega}{2}\right)\right\}^2. \qquad (13.7)$$

Collecting (10.2), (10.6), (9.5), (11.5), (12.5) and (13.7) we get for S_2 in (10.1) the inequality

$$S_2 < (1 + o\,(1))\,p(n)\left\{8\sqrt{\pi}\exp\left(\frac{\omega}{2}\right)\right\}^2. \qquad (13.8)$$

By Cebysev's inequality, in order to complete the proof of Theorem II, it is enough to show that

$$Z \overset{\text{def}}{=} \frac{1}{p(n)}\sum_{H}\left\{k(H) - 8\sqrt{\pi}\exp\left(\frac{\omega}{2}\right)\right\}^2 = o\,(1)\exp\omega.$$

But this follows from (9.5) and (13.8) at once.

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