CHILD PRODIGIES

by

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I will talk about the child prodigies which I have known. Perhaps I should start with myself since I was something of a child prodigy when I was three years old. I knew to calculate very well with fairly large numbers. I suppose two of my biggest discoveries were the following ones. When I was four years old I informed my mother that if you take away 250 from 150 you get a number 100 below zero. Also once when I was five years old I computed the distance to the sun when I knew how many years it takes a train to travel there. Incidently my parents were both mathematicians. And it is supposed that I learned to compute in the following way. My father was a prisoner of war in Siberia during the first world war and my mother taught high school and I was left with a German governess. So I was naturally interested when my mother will be at home, and therefore when I was about three years old I ployed with a calendar and knew many months ahead when there will be a holiday. And it is probable that I learned to count that way. Actually I started to do mathematical research fairly early. My first paper was written when I was eighteen years old when I gave a new proof for the theorem of Tchebychev that there always is a prime between N and $2N$. But enough about myself.

Not all mathematicians were child prodigies though many of them were. For example, Gauss certainly was a child prodigy on the other hand Hardy certainly wasn't, von Neumann and Norbert Wiener both were child prodigies. But now I will talk only about the child prodigies which I had personal contact with.

The first two child prodigies whom I will mention I didn't have too much contact with so I will be short.

Peter Lax, who was recognized as a child prodigy in Hungary when he was about twelve, came to the United States when he was sixteen and he had a letter of introduction to me so I met him almost on his arrival, and we met a great deal in New York, and Princeton. I still remember when Halmos said at the Institute for Advanced Study; there is a baby here from Hungary and you can talk mathematics to him. Peter Lax wrote his first paper actually on a problem of myself on polynomials but pretty soon he moved off into a far away field about which I know unfortunately nothing on /differential equations/ and I did not have much mathematical contact with him since then.

The second child prodigy was Peter Ungar who was discovered in Hungary when he was still very young, about fourteen or fifteen years old, and by the time he was fifteen Turán quated some of his results in a paper on the Riemann ζ-Function. One of his problems in the American Mathematical Monthly he composed when he was only fourteen years old. I didn't have too much contact with him so I don't want to talk more about him. Both Peter Ungar and Peter Lax are now Professors at New York University.

I will start to talk about Pósa who is now 22 years old and the author of about eight papers. I met him before he was twelve years old. When I returned from the United States in the summer of 1959 I was told that there is a little boy whose mother is a mathematician and who knows all that there is to be known in high school. I was very interested and next day I had lunch with him and with Róza Péter a Hungarian mathematician who worked with him. While we had lunch and Pósa was eating his soup I told him the following problem. Prove that if you have $n + 1$ integers less than or equal to $2n$ then there are always two of them which are relatively prime. It is quite easy to see that the theorem is not true for n integers because if you take the multiples of 2 there are n of them not exceeding $2n$ and no two of them are relatively prime. Actually I discovered this simple result some years ago but it took me about ten minutes until I found the very simple proof. Pósa ate his soup, and then said, "If you have $n + 1$ integers less then or equal to $2n$ two of them are consecutive and therefore they are relatively prime." It was needless to see that I was very much impressed, and I think that this is on the same level as Gauss when he was seven years old summed the integers ≤ 100. Incidentally the following problem is still unsolved: Denote by $f_k/n/$ the largest value of r so that there is a sequence $1 \leq a_1 < \dots < a_r \leq n$ so that one cannot select $k + 1$ a's which are pairwise relatively prime. I believe $f_k/n/$ equals the number of multiples not exceeding n of the first k primes. Since then I worked systematically with Pósa and I wrote to him many letters with problems during my travels. And before he was about twelve years old, he proved the following theorem which I told him. If you have a graph of $2n$ vertices and $n^2 + 1$ edges it always contains a triangle. This is a special case of a well-known theorem of Turán. Also, I gave him the following problem. Take an infinite series whose n^{th} term is defined as follows. The numerator is 1 and the denominator is the least common multiple of the integers from 1 to n . Prove that the sum of the series is irrational. This is not very difficult to prove but certainly surprising that a twelve year old child could do this. When he was a little over thirteen, I explained to him Ramsay's Theorem for the case k = 2 . The theorem states as follows. Suppose you have an infinite graph then the graph either contains a infinite complete graph or infinite independent set. In other words there is an infinite set so that either any two of these vertices are joined or no two of the vertices are joined. It took about fifteen minutes until Pósa understood it and then he went home, thought about it all evening and before going to sleep he had the proof. By the time Pósa was about fourteen you could talk to him as a grown up mathematician. I called him on the phone and asked him about a problem and if the problem was about elementary mathematics it was very likely that he had some relevant and intelligent comment. It is perhaps interesting to remark that he had some difficulty with calculus and he understood them and could use them but he never was as much at home as with combinatorial analysis elementary number theory. He never liked geometry. I tried to give him some math problems in elementary geometry but he never liked them. He always liked to do only what he was really interested in, but at that he was extremely good. Our first joint paper was written when he was fourteen and one half years old, and I will tell you his contribution. Pósa wrote many significant papers also by himself. Some of which still have a great deal of effect. His best known and most quoted paper is on Hamiltonian lines and he wrote it when he was fifteen. Unfortunately since about 4 or 5 years he has not proved and conjectured much and I often comment sadly that he is dead, but I very much hope that he will come back to life soon. I got first worried about him when he told me when he was 16 that he rather would be Dostojewsky then Einstein.

His first theorem which he conjectured and proved by himself and which think was new states as follows: Every graph of n vertices and $2n - 3$ edges contains a circuit having a diagonal. The result is best possible: There is a graph of n vertices and $2n - 4$ edges which does not contain a circuit with a diagonal.

Denote by $f/n; k/$ the smallest integer so that every graph of n vertices and $f/n; k/$ edges contains k vertex disjoint circuits. In our first joint paper we determine $f/n; k/$ for $n > 24k$. I previously proved $f/n; 2/ = 3n - 5$ for $n \geq 6$ but my proof was complicated and did not generalize for $k > 2$, /later I found out that Dirac also proved $f/n; 2/ = 3n - 5/$. I told Pósa the problem of determining $f/n; 2/$ and in a very few days he found a very simple proof of $f/n; 2/ = 3n - 5$. I could then easily extend his proof for general k. I now give the outlines of his proof. We use induction for n. It is easy to see that $f/n; 2/ = 3n - 5$ holds for $n = 6$. Assume that it holds for every $6 \leq m < n$ and we prove it for n. Our graph must have a vertex of valency ≤ 5

(since $\frac{1}{2} \sum_{i=1}^{n} v(X_i) = 3n - 5$, where $v(X_i)$ is the valency of the vertex X_i). If our graph $\mathcal{G}$ has a vertex of valency ≤ 3, we omit it and by the induction hypothesis the remaining graph has two vertex disjoint circuits. Thus we can assume that our graph has a vertex X_1 of valency 4 or 5. Assume first $v(X_1) = 5$ and let $X_2, \ldots, X_6$ be the vertices joined to X_1. Since our theorem holds for $n = 6$ the subgraph spanned by the vertices $X_1, \ldots, X_6$ has at most 12 edges - hence without loss of generality we can assume that the edges (X_2, X_3) and (X_3, X_4) are missing from our graph. Omit now from $\mathcal{G}$ the vertex X_1 and all edges incident to it but add (X_2, X_3) and (X_3, X_4). The new graph has $n - 1$ vertices and $3n - 8$ edges thus by the induction hypothesis it contains two vertex disjoint circuits, only one of these can contain the new edges (X_2, X_3) or (X_3, X_4). Let us assume that one of the circuits contains (X_2, X_3). We then omit (X_2, X_3) and replace it by (X_2, X_1), (X_1, X_3). Thus our original graph has two vertex disjoint circuits, as stated. Assume next that X_1 has valency four and is joined to X_2, X_3, X_3, X_4 and X_5. All the edges (X_i, X_j), $1 \leq i < j \leq 5$ must be in $\mathcal{G}$, for if say (X_2, X_3) would not be in $\mathcal{G}$, we would add (X_2, X_3), omit X_1 and the previous proof would apply.

Now count the number of edges of $\mathcal{G}$ incident to one of the vertices X_1, X_2 and X_3. X_i, $(i > 5)$ cannot be joined to X_1 and if it is joined to both X_2 and X_3 the two triangles (X_2, X_3, X_i) and (X_1, X_4, X_5) are vertex disjoint. Thus we can assume that each X_i ; $i > 5$ is joined to only one of X_1, X_2, X_3. This gives at most $n - 4$ edges, there are 9 more edges spanned by $X_1, \ldots, X_5$ incident to X_1, X_2 or X_3. Thus there are at most $n + 4$ edges of G incident to X_1, X_2 and X_3. Omit now the vertices X_1, X_2, X_3 and all the edges incident to them. The remaining graph has $n - 3$ vertices and at most $3n - 5 - (n + 4) = 2n - 9$ edges. Since $2n - 9 \geq n - 3$ $(n \geq 6)$ this graph contains a circuit which is vertex disjoint from (X_1, X_2, X_3). This completes the proof. You will agree with me that this is a remarkable proof for a child of 14. It was not difficult to extend this proof for $k > 2$ by induction with respect to k. Pósa also found a very beautiful proof that every graph of n vertices and $n + 4$ edges contains two edge disjoint circuits.

There was another child prodigy in the university town of Szeged called Máté he was a year older than Pósa and in fact the two children first met in my mothers apartment. As many of you know I call children ε-s and Máté when he was once phoning to me (I was abroad) introduced himself to my mother: I am the ε from Szeged. The two ε-s these days were really enthusiastic and always wanted new problems. I remember once I had a bad headache and wanted to go home to rest and hardly could get away from them.

Máté got to the university two years earlier by winning the Kürschák competition, Pósa also could have gone to the university two years ahead of time but he liked high school very much and refused - I will tell more about this high school later.

Máté works mainly on est theory now, has written several papers and is writing a book with the well known set theorist G. Fodor. One of his first results he found when he was a little over 16. I proved some time ago the following theorem: To every real X there corresponds a nowhere dense set of reals A/X/ . Two points X and Y are independent if $X \notin A/Y/$ and $Y \notin A/X/$. A set is independent if every two of its elements are independent. I proved that there always is an uncountable independent set. This question is still open. Máté proved the following result: To every ordinal $\alpha < \omega_1$ there is an independent set of type α . He found this theorem all by himself, the proof uses category arguments very cleverly.

In Hungary children go to elementary school for 8 years and then there are 4 years of high school. A few years ago they started a special high school for children gifted in mathematics. The school was opened just when Pósa was due to go to high school, he liked this high school very much. Very soon he told me there are some boys in my class who are better in elementary mathematics than I. I want to say a few words about two of them Lovász and Pelikán, all of them are now 22 years old. Lovász is perhaps the the most successful of the prodigies up to now, his career showed no breaks like Pósa's. He started scientific work a little later at the ripe old age of nearly 17, but has done outstandingly well mainly in combinatorial mathematics, he was the first to give a construction for a graph of arbitrarily large chromatic number and arbitrarily large girth, he did that while he was still in high school, the construction is very ingenious and difficult.

Pelikán also worked mainly in graph theory and is the author of several papers, those who were at the graph theory meeting in Tihany in 1966 will remember him, he was just out of high school and gave a talk about his recent results in very good English it was a masterfully clear lecture nobody could have believed that this is his first lecture. I had much less contanct with Lovász and Pelikán then with Pósa.

Before I continue I would like to make a few conjectures about the reasons that there are so many child prodigies in Hungary. First of all there was for at least 80 years, a mathematical periodical for high school students, then they have many mathematical competitions, the Eötvös-Kurshak competition goes back for 75 years /see the Hungarian problem book/ after the first world war a new competition was started for high school students who are just finishing high school and after the second world war several new competitions were started, the most interesting is the Schweitzer competition, there about ten problems are given and the competitors have more than a weeks time to send in the solutions, collaboration is not permitted but books can be used. Recently a book has been published with these problems. In the last few years I heard many reports about child prodigies in this country too I met some of them /e.g. Grost from Michigan State/ but do not know enough about them. I want to mention only one namely Turansky. I met him at the University of Pennsylvania when he was 17 and he was extremely talented but, due to unknown causes he never got a Ph.D. and was killed in a traffic accident when he was 35.

A few years ago a new kind of competition was started in Hungary for high school students it is held on television. The competitors are given questions which they have to answer in 2 or 3 minutes. The questions are usually very ingenious and the competitors are judged by a panel of the leading mathematicians /e.g. Alexits, Hajós and Turán/. It seems many people watch these competitions with great interest even if they do not understand the problems.

Graph theory and combinatorial analysis is a very suitable field for young mathematicians to do original work - there are still many unsolved problems whose solution requires only ingenuity and not much knowledge. There is also some danger in this since there is a temptation for young geniuses just to prove and conjecture and not to learn other branches of mathematics.

Finally, I want to speak about a child prodigy who is still in high school, I. Ruzsa. Pósa introduced him to me 2 1/2 years ago he was not yet 15 at that time. His special interest was number theory, he was especially good at raising new and interesting problems and he has several papers which will soon be published. Ruzsa, Sárközi and I have a long forthcoming paper on additive number theoretic functions. Let me mention some of his problems and results. Let $f/n/$ be an integer valued function which satisfies $f/a/ \equiv f/b/$ $/\text{mod } m/$ if $a \equiv b$ $/\text{mod } m/$ for every a, b and m. Polynomials satisfy this condition. Ruzsa proved that if $f(n)$ satisfies this condition and is not a polynomial then

$$/1/ \qquad \lim_{n \to \infty} |f(n)| / n^k = \infty$$

for every k, he also proves that /1/ is best possible. He further shows that for infinitely many n

$$|f(n)| > (e - 1)^{n(1+o(1))}$$

and he conjectures that $f(n) > e^{n(1+o(1))}$ infinitely often.

Let $f(n)$ be a multiplicative function whose value domain is an abelian group of order N. Is it true that for every element g of our group the integers n for which $f(n) = g$ have a density $1/N$. If the abelian group has only the elements ± 1 this is an old conjecture of mine proved first by Wirsing.

Some time ago I asked the following question: Let $1 \leq a_1 < \dots < a_k \leq X$ be a sequence of integers so that for every n the number of solutions of $n = p\, a_i$, /p prime/ is at most 2. Is it true that $\max k = o(X)$? It is easy to see that if we require that the products $p\, a_i$ are all distinct /i.e. the number of solutions of $n = p\, a_i$ is at most 1/ then $k = o(X)$ this easily follows from the fact that the numbers $\frac{a_i}{P(a_i)}$ must all be distinct where $P(a_i)$ is the greatest prime factor of a_i.

I told this problem to Ruzsa who very soon showed that

$k > c X$ is indeed possible. He argued as follows: Let $b_1 < b_2 < \dots$ be the sequence of those integers which do not have two prime factors p and q satisfying $p < q < 2p$. It is not too difficult to show that the density of the b's exists and is positive. Thus if this density is α there are $(1 + o(1))\ \alpha X/2$ b's in the interval $\left(\frac{X}{2}, X\right)$. A simple argument shows that for these b's the number of solutions of $p\, b_i$ is indeed < 3. Thus $\max k \geq \left(\frac{\alpha}{2} + o(1)\right) X$ which answers my question.

You will agree with me that Ruzsa shows ability of the highest degree and one can hope and expect that he will become a great mathematician. Ruzsa incidentally did outstandingly well this year at the mathematical olimpiad.

Before I finish my lecture let me tell you a few anecdotes about the child prodigies. Lovász and Pósa when they still went to high school once asked me why are there so few girl mathematicians. I told them: Suppose the slave children /boys/ would be brought up with the idea that if they are very clever the bosses /girls/ will not like them - would there be then many boys who do mathematics? Both said: well perhaps not so many.

In Hungary many mathematicians drink strong coffee, in fact Rényi once said: a mathematician is a machine which turns coffee into theorems, at the mathematical institute they make particularly good coffee, when Pósa was not quite 14 I offered him a little strong coffee which he drank with an infinite amount of sugar. My mother was very angry that I gave the little boy strong coffee. I answered that Pósa could have said:

madam I do a mathematicians work and drink a mathematicians drink. /I saw a movie many years ago where a lady sees a boy of 16 drink whicky with the older man and is shocked. The boy says: madam I do a mans work and drink a mans drink./

When Lovász was still an ε in the first year in high school he and a friend a fellow mathematician courted the same boss-child also a mathematician /not a bad one as bosses go/. The two slave-children asked her to choose. She chose Lovász, in fact they married last year. Milner improved on this story by changing her answer as follows: "I will choose the one who proves the Riemann hypothesis". If as I hope Lovász will become a great mathematician whose name will be remembered the story will perhaps survive in this form.