SOME PROBLEMS IN ADDITIVE NUMBER THEORY

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1. Let $1 \leqq a_1 < \cdots < a_k \leqq x$ be a sequence of integers for which the sums

$$\sum_{i=1}^{k} \epsilon_i a_i, \qquad \epsilon_i = 0 \quad \text{or} \quad 1 \tag{1}$$

are all distinct. Put max $k = f(x)$ where the maximum is taken over all sequences satisfying (1). It is easy to see that

$$f(x) < \frac{\log x}{\log 2} + \frac{\log \log x}{\log 2} + O(1),$$

and Moser and I proved

$$f(x) < \frac{\log x}{\log 2} + \frac{\log \log x}{2 \log 2} + O(1). \tag{2}$$

This is the best-known upper bound for $f(x)$.

Is it true that

$$f(x) = (\log x / \log 2) + O(1)? \tag{3}$$

Moser and I asked: Is it true that $f(2^k) \geqq k+2$ for sufficiently large k? Conway and Guy showed that the answer is affirmative (unpublished).

P. Erdös, Problems and results in additive number theory, *Colloque, Théorie des Nombres*, Bruxelles 1955, p. 137.

2. Let $1 \leqq a_1 < \cdots < a_k \leqq x$ be a sequence of integers so that all the sums

$$a_{i_1} + \cdots + a_{i_s}, \quad i_1 \leqq i_2 \leqq \cdots \leqq i_s, \qquad 1 \leqq s \leqq r$$

are distinct. Put max $k = g_r(x)$. Turán and I proved

$$g_2(x) < x^{1/2} + O(x^{1/4}). \tag{4}$$

This was recently improved by Lindström to $g_2(x) \leqq x^{1/2} + x^{1/4} + 1$. The lower bound $g_2(x) \geqq (1+o(1))x^{1/2}$ easily follows from a classical result of Singer on difference sets. Turán and I conjectured

$$g_2(x) = x^{1/2} + O(1). \tag{5}$$

Bose and Chowla proved that $g_r(x) \geqq (1+o(1))x^{1/r}$ for each $r \geqq 2$, and they conjectured

$$g_r(x) = (1 + o(1))x^{1/r}. \tag{6}$$

This is known only for $r = 2$.

R. C. Bose and S. Chowla, *Theorems in the additive theory of numbers*, Comment. Math. Helv., 37 (1962–63) 141–147.

H. Halberstam and K. Roth, *Sequences*, Oxford Univ. Press, Oxford 1966.

P. Erdös and P. Turán, *On the Problem of Sidon in additive number theory and on some related problems*, Journal London Math. Soc., 16 (1941) 212–215 and 19 (1944) 208.

J. Singer, *A theorem in finite projective geometry and some applications to number theory*, Trans. Amer. Math. Soc., 43 (1938) 377–385.

B. Lindström, *An inequality for B_2 sequences*, Journal Combinatorial Theory, 6 (1969) 211–212.

3. Let $1 \leqq a_1 < \cdots$ be an infinite sequence of integers. Denote by $h(n)$ the number of solutions of $n = a_i + a_j$. Turán and I conjectured that if $h(n) > 0$ for $n > n_0$ then

$$\limsup_{n\to\infty} h(n) = \infty. \tag{7}$$

Another stronger conjecture states that if $a_k < ck^2$ for every k then (7) holds.

These conjectures seem very difficult. It is a curious fact that the multiplicative analogues of these conjectures are not intractable.

Let $1 < b_1 < \cdots$ be an infinite sequence of integers. Denote by $H(n)$ the number of solutions of $n = b_i b_j$. Assume $H(n) > 0$ for $n > n_0$. I proved

$$\limsup_{n\to\infty} H(n) = \infty.$$

Turán and I further conjectured that the number of solutions of $a_i + a_j \leqq x$ cannot be of the form $cx + O(1)$, where $c < \infty$. In other words

$$\sum_{n=1}^{x} h(n) = cx + O(1) \tag{8}$$

can hold only if $c = 0$ and the sequence, $a_1 < \cdots$, is finite. Fuchs and I proved a very much stronger result than (8). We in fact showed that if $c > 0$ then

$$\sum_{n=1}^{x} h(n) = cx + o\left(\frac{x^{1/4}}{(\log x)^{1/2}}\right) \tag{9}$$

is impossible. Jurkat showed (unpublished) that (9) is impossible even with $o(x^{1/4})$. Perhaps Jurkat's result is best possible and

$$\sum_{n=1}^{x} h(n) = cx + O(x^{1/4})$$

can hold for a suitable sequence $a_1 < \cdots$.

Is it true that the number of solutions of $a_i + a_j + a_r \leqq x$ cannot be of the form $cx + O(1)$? Our method used with Fuchs does not apply here.

I have several times offered \$250 for the proof or disproof of any of the conjectures (3), (5), or (7).

Paul Erdös, *On extremal problems of graphs and generalised graphs*, Israel Journal of Math., 2 (1965) 183–190, and *On the multiplicative representation of* *integers*, *ibid.*, 2 (1965) 251–261.

Paul Erdös and W. H. J. Fuchs, *On a problem of additive number theory*, Journal London Math. Soc., 31 (1956) 67–73. See also the well-known book: H. Halberstam and K. Roth, *Sequences*, Oxford Univ. Press, Oxford, 1966.

A. Stohr, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe*, I and II, Journal reine u. angew. Math., 194 (1955) 40–65 and 111–140.

CLASSROOM NOTES

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AN EXISTENCE THEOREM FOR NON-NOETHERIAN RINGS

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In developing the theory of Noetherian rings, it is desirable to have at hand some examples of non-Noetherian rings. One such example is the ring of polynomials in infinitely many indeterminates over any nonzero ring. A second example is $R[X]$, where R is any nonzero commutative ring without identity [1], but in this case, $R[X]$ is also a ring without identity. We present here a theorem which provides a method for constructing, as subrings of a polynomial ring in finitely many indeterminates, a wide class of commutative rings with identity which are not Noetherian. It should be noted that the proof of the theorem given uses only one result (Lemma 1) outside the basic theory of Noetherian rings, namely, that a finitely generated idempotent ideal of a commutative ring is principal and is generated by an idempotent element. An examination of the proof of this result reveals that even it is obtained as a direct application of Cramer's Rule for determinants over a commutative ring with identity.

THEOREM 1. *Suppose that R is a nonzero commutative ring, that A is a nonzero ideal of R distinct from R, and that $\{X_\lambda\}_{\lambda\in\Lambda}$ is a set of indeterminates over R. The subring $S=R+A[\{X_\lambda\}]$ of $R[\{X_\lambda\}]$, consisting of those polynomials over R having each of their nonconstant coefficients in A, is Noetherian if and only if these three conditions hold*: (1) Λ *is finite*, (2) R *is Noetherian, and* (3) *the ideal A is idempotent.*

Proof. Suppose that S is Noetherian. It is clear that Λ must be finite, for if not, the ideal $A[\{X_\lambda\}]$ of S would not be finitely generated. The mapping on S which sends each polynomial $f\in S$ onto its constant term is a homomorphism from S onto R, so that R is Noetherian. If σ is a fixed element of Λ and if $a\in A$, the ideal $(aX_\sigma, aX_\sigma^2, \cdots, aX_\sigma^m, \cdots)$ of S is finitely generated, so that $aX_\sigma^{m+1} \in(aX_\sigma, \cdots, aX_\sigma^m)$ for some positive integer m: