PROBLEMS IN COMBINATORIAL SET THEORY

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In this lecture I shall mention a number of results and problems from combinatorial set theory.

The starting point for all these investigations is the well known theorem of F.P. Ramsey (1930) which states that

$$\aleph_0 \rightarrow (\aleph_0, \aleph_0)^r \qquad (1)$$

holds for finite r. Here we are using the partition symbol

$$a \rightarrow (b_0, b_1)^r \qquad (2)$$

which means that the following is true: *If* $|S| = a$ *and* $[S]^r = \{X \subset S: |X| = r\} = K_0 \cup K_1$, *then there are* $T \subset S$ *and* $i < 2$ *such that* $|T| = b_i$ *and* $[T]^r \subset K_i$. The symbol (2) was first used in a paper by Rado and myself (1952) but a number of papers dealing with generalizations of Ramsey's theorem appeared before then. The first of these was a paper by W. Sierpinski who proved that $2^{\aleph_0} \nrightarrow (\aleph_1, \aleph_1)^2$. References to the earlier literature will be found in [1] and [2]. Finally in [2] the truth value of (2) is determined for arbitrary cardinals a, b_0, b_1 provided the generalized continuum hypothesis is assured and that a is not larger than the first inaccessible cardinal. Here I want to state just one problem. Without assuming the continuum hypothesis, is it possible to split the pairs of real numbers into three classes so that in any uncountable subset of the reals there is a pair in each of the three classes? With the continuum hypothesis this and much more is true (see [2]).

The symbol (2) defined for cardinal numbers above, has an obvious interpretation when the symbols are replaced by ordinal numbers or order types. Rado and I conjectured that $\omega^2 \to (3,\omega^2)^2$ and more generally that

$$\omega^2 \to (m,\ \omega^2)^2 \qquad\qquad (m < \omega).$$

This conjecture was settled by E. Specker [3] and he also showed, much to our surprise, that $\omega^k \nrightarrow (3,\omega^k)^2$ $(3 \le k < \omega)$. This suggested the problem whether

$$(?) \quad \omega^\omega \to (3,\omega^\omega)^2.$$

This seems to be very difficult and I offered \$250 for either a proof or disproof. It is now rumoured that C.C. Chang has settled the problem affirmatively, but his method does not seem to give

$$(?) \quad \omega^\omega \to (4,\omega^\omega)^2.$$

Other results of this kind have been found. E.C. Milner [4] proved $\omega^4 \to (3,\omega^3)^2$. I proved $\omega^{2n+1} \to (4,\omega^{n+1})^2$ and Milner [4] showed $\omega^{3k+1} \nrightarrow (3,\omega^{2k+1})^2$. Independently of each other A. Hajnal and F. Galvin (unpublished) have reduced the question of deciding the truth status of

$$\omega^l \to (m,\omega^n)^2 \qquad\qquad (l,m,n < \omega)$$

to a finite but difficult combinatorial problem. In particular their method gives $\omega^4 \to (4,\omega^3)^2$, $\omega^4 \nrightarrow (s,\omega^3)^2$. Also, for every integer n, there is an integer $f(n)$ such that $\omega^n \nrightarrow (f(n),\ \omega^3)^2$.

If λ is the order type of the real numbers (or more generally the order type of any uncountable subset of the reals), Rado and I [1] proved $\lambda \to (\omega+n,\omega+n)^2$ $(n<\omega)$. In fact, if the pairs of reals are split into three classes, we have $\lambda \to (\omega+n,\ \omega+n,\ \omega+n)^2$. We also showed that $\omega_1 \to (\omega+n,\omega+n)^2$. Hajnal [5] proved $\lambda \to (\omega n,\alpha)^2$ for $\alpha < \omega_1$. We conjectured that $\lambda \to (\alpha,\alpha)^2$ holds for every $\alpha < \omega$, and I offered \$100 for a proof of this. Recently F. Galvin deservedly collected the prize although his method does not seem to give our further conjectures

$$(?) \quad \lambda \to (\alpha,\ \alpha,\dots,\alpha)^2,$$
$$(?) \quad \omega_1 \to (\alpha,\ \alpha,\dots,\alpha)^2$$

when the splitting involves an arbitrary finite number of classes. The simplest unsolved problems in this connection are

$$(?) \quad \omega_1 \to (\omega 3,\omega 3)^2$$

(Hajnal has shown $\omega_1 \to (\omega 2+n,\ \omega 2+n)^2$),

$$(?) \quad \omega_1 \to (\omega+n,\ \omega+n,\ \omega+n)^2,$$

$$(?) \quad \lambda \to (\omega+n,\ \omega+n,\ \omega+n,\ \omega+n)^2.$$

Not much work has been done about partitioning of triples rather than pairs. Rado and I [1] proved that $\lambda \to (\omega+n,\ 4)^3$ and $\lambda \to (\omega+2,\omega)^3$. But the relations

$$(?) \quad \lambda \to (\omega+n,5)^3\ ,\quad \lambda \to (\omega.2,5)^3\ ,\quad \lambda \to (\omega+1,\omega)^3$$

are all undecided.

Rado and I [1] proved that for given m,n there is an integer $f(m,n)$ such that

$$\omega_\alpha\ f(m,n) \to (m,\ \omega_\alpha n)^2$$

holds for every α. This left open the question whether $\omega_1\omega \to (3,\omega_1\omega)^2$. Hajnal and I showed that this was false and more generally $\omega_1\alpha \to (3,\omega_1\omega)^2$ for every $\alpha < \omega_1$. We also proved that $\omega_1{}^2 \to (3,\ \omega_1\omega)^2$ but we do not know if

$$(?) \quad \omega_1{}^2 \to (3,\ \omega_1{}^2)^2.$$

A set mapping is a function from a set S into the subsets of S such that $x \notin f(x)$. Specker and I [6] showed that if $\overline{S} = \omega_1$ and f is a set mapping of order $\alpha(<\omega_1)$ on S (i.e. $\overline{f(n)} < \alpha$ for all $x \in S$), then there is a free set S_1 of type ω_1, i.e. $S_1 \cap f(S_1) = \emptyset$. Recently, Hajnal, Milner and I have extended this and shown that if $\overline{S} = \omega_1^{\rho+1}$ $(\rho \leq \omega)$ and f is a set mapping of type $\alpha < \omega$, then there is a free set of type $\omega_1^{\rho+1}$. The corresponding result is false if $\overline{S} = \omega_1{}^\omega$ unless $\alpha \leq \omega$. If $\overline{S} = \omega_1^\rho \geq \omega_1{}^{\omega+2}$ the result is false even for $\alpha = \omega$, i.e. we can construct a finite set mapping on $\omega_1{}^\rho$ $(\geq \omega_1{}^{\omega+2})$ so that there is no free set of type $\omega_1{}^\rho$. Using these results we were able to show that if $\gamma = \omega\beta < \omega_1{}^{\omega+2}$, then

$$\gamma \to (\gamma,\ \text{infinite path})^2. \tag{3}$$

This means that if G is any graph on a set of type γ, then either there is an independent set of type γ or there is an infinite path in the graph. Our method using the set mapping result completely breaks down if $\gamma \geq \omega_1{}^{\omega+2}$ but (3) may very well hold for every $\gamma = \omega\beta$. We were able to prove that

$$\gamma \to (\omega,\square)^2 \tag{4}$$

holds for $\gamma = \omega\beta$, i.e. either there is an independent set of type γ or there is a circuit in the graph of length 4. We conjecture that (4) holds for any order type γ which has no fixed point (x is a fixed point of the ordered set S if $\overline{S - \{x\}} \not\cong \overline{S}$).

Finally I mention some problems concerning the order type η_α of all 0,1 sequences of length ω_α having a final 1 and ordered lexicographically. η_0 is the order type of the rationals and it is an easy result of Rado and myself that $\eta_0 \to (\eta_0, \aleph_0)^2$. Milner, Rado and I generalized this and we showed $\eta_\alpha \to (\eta_\alpha, \aleph_\beta)^2$ holds provided $\aleph_\alpha$ is regular and $\aleph_\gamma^k < \aleph_\alpha$ ($\gamma < \alpha$; $k < \aleph_\beta$). With the generalized continuum hypothesis Hajnal, Milner and I showed that $\eta_\alpha \to (\eta_\alpha, [m, \eta_\alpha])^2$ if $m^+ < \aleph_\alpha = \aleph_{cf\alpha}$. Here $[m, \eta_\alpha]$ indicates a complete bipartite graph on sets A, B with $|A| = m$ and $\overline{B} = \eta_\alpha$. We cannot prove any of the following

$$(?) \quad \eta_\omega \to (\eta_\omega, 3)^2, \qquad (?) \quad \eta_\omega \to (\eta_\omega, \aleph_0)^2$$

$$(?) \quad \eta_\omega \to (\eta_\omega, \text{infinite path})^2.$$

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