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PROBLEMS AND RESULTS IN CHROMATIC GRAPH THEORY*

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1. Introduction

In this note we shall discuss mostly without proofs, some recent results on graph theory. It will be almost entirely restricted to problems on chromatic graphs and to those problems on which I worked myself. Many of the questions which we shall discuss are joint work with Hajnal.

First we introduce some notations. G_n denotes a graph of n vertices, $G(n;\ t)$ denotes a graph of n vertices and t edges, and $\chi(G)$ denotes the chromatic number of G (i.e., the smallest integer [or cardinal number] such that the vertices of G can be colored by $\chi(G)$ colors and such that two vertices which are joined never have the same color). $I(G)$ denotes the maximum number of independent vertices of G (i.e., the largest number of vertices of G no two of which are joined by an edge). $K(G)$ denotes the number of the vertices of the largest complete graph contained in G. $G(x_1, \ldots, x_n)$ denotes the subgraph of G spanned by the vertices $x_1, \ldots, x_n$. $G - R$ is the graph from which the edge R has been omitted. The number of edges of G are denoted by $R(G)$ the number of vertices of G by $v(G)$. K_n will denote the complete graph of n vertices, C_r will denote a circuit of r edges and $K(u_1\ u_2)$ will denote the complete bipartite graph with $u_i (i = 1, 2)$ vertices of each color, where any two vertices of different color are joined.

We evidently have

$$\chi(G) \geq v(G)/I(G) \tag{1}$$

In the second section we will discuss problems and results on chromatic graphs and in the third we will mention a few miscellaneous problems.

* Dedicated to the memory of Jon Folkman.

2. Problems and Results on Chromatic Graphs

G. Dirac [2] calls the k-chromatic graph G critical if for every edge R of G, $\chi(G-R)<\chi(G)$. For $k=3$, the critical graphs are the odd circuits, but for $k=4$ it seems hopeless to characterize all the 4-critical graphs.

Denote by $f_k(n)$ the largest integer for which there exists a $G(n;f_k(n))$ which is k-chromatic and critical. I asked whether $f_k(n)>c_k n^2$. Dirac [2] proved

$$f_6(4n+2)\geq 4n^2+8n+3 \tag{2}$$

To prove (2) consider a graph, $G(4n+2;\ 4n^2+8n+3)$ which consists of two different C_{2n+1}, any two vertices of which are joined by an edge. It is easy to see that our graph is 6-chromatic and critical. Perhaps in (2) the sign of equality holds.

Dirac's construction easily generalizes to give

$$f_{3k}(k(2n+1))\geq \binom{k}{2}(2n+1)^2+k(2n+1). \tag{3}$$

Perhaps the equality sign holds in (3) also. I conjecture

$$\lim_{n=\infty} f_{3k}(n)/n^2=\lim_{n=\infty} f_{3k+1}(n)/n^2=\lim_{n=\infty} f_{3k+2}(n)/n^2=\tfrac{1}{2}(1-1/k),$$

but could not even prove $f_6(n)=(\frac{1}{4}+o(1))n^2$.

A plausible guess cannot even be made about $\lim_{n\to\infty} f_4(n)/n$ and $\lim_{n\to\infty} f_5(n)/n$. The inequality $f_4(n)\geq 2n+O(1)$ is known [19].

A well-known theorem of de Bruijn and myself [1] states that if $k<\aleph_0$ then every k-chromatic graph contains a finite subgraph which is also k-chromatic and from this result it is easy to see that if $l=\aleph_0$ then every l-chromatic graph contains an l-chromatic subgraph which is denumerable. A first guess might be that every graph which has chromatic number $\aleph_1$ contains a subgraph G of power $\aleph_1$ with $\chi(G)=\aleph_1$. Hajnal and I proved [15] that this not true. We show that for every $k<\aleph_0$ there is an $\aleph_1$-chromatic graph each subgraph of which, of power less than $\aleph_k$, has denumerable chromatic number. In [15] several unsolved problems are mentioned. Here we mention only two of the simplest ones. Is it true that there is an $\aleph_2$-chromatic graph of power $\aleph_2$ such that every subgraph of power $\aleph_1$ has chromatic number $\aleph_0$? Is there a graph of power $\aleph_{\omega+1}$ of chromatic number $\aleph_1$ such that every subgraph of power $\aleph_\omega$ has chromatic number $\aleph_0$? For other problems of this nature we refer to [14, 16].

Let G be a graph whose vertices form a well-ordered set and for which $\chi(G)\geq\aleph_0$. Babai proved (oral communication), using the theorem of de

Bruijn and myself [1], that one can find a subsequence $x_1, x_2, \ldots$ of type ω of the vertices of G so that $\chi(G(x_1, \ldots)) = \aleph_0$.

It was first thought that this theorem could be generalized to higher cardinals, but Hajnal and I showed that this is not true. We construct a graph G with $\chi(G) = \aleph_1$ whose vertices are a well-ordered set of type $\omega_1{}^2$, but every subgraph whose vertices are of type less than $\omega_1{}^2$ has chromatic number less than or equal to $\aleph_0$. Let the vertices of G be $\{x_\alpha, y_\beta\}$, $1 \le \alpha < \omega_1$; $1 \le \beta < \omega_1$ ordered lexicographically ($\{x_\alpha, y_\beta\} < \{x_{\alpha_1}, y_{\beta_1}\}$ if $\alpha_1 > \alpha$ or if $\alpha = \alpha_1$ and $\beta_1 > \beta$). This is a set of type $\omega_1{}^2$. Let $\{x_{\alpha_1}, y_{\beta_1}\} < \{x_{\alpha_2}, y_{\beta_2}\}$. These vertices are joined if and only if $\alpha_1 < \alpha_2$ and $\alpha_2 < \beta_1$. It is not difficult to see that $\chi(G) = \aleph_1$ but for every subgraph G_1 whose vertices have order type less than $\omega_1{}^2$ we have $\chi(G_1) \le \aleph_0$.

On the other hand we can not decide the following questions: Does there exist a graph G whose vertices form a set of type $\omega_2{}^2$, $\chi(G) = \aleph_2$, and for every subgraph G' of G whose vertices form a set of lesser type we have $\chi(G') \le \aleph_0$? We can not solve this question with $\chi(G) = \aleph_1$ and $\chi(G') \le \aleph_0$. (The case $\chi(G) = \aleph_2$, $\chi(G') \le \aleph_1$ can be solved easily by an obvious modification of the previous example.)

Tutte [25] and Zykov [27] were the first to show that for every k there is a G with $\chi(G) = k$ which contains no C_3.

Denote by $g_l(n)$ the largest integer for which there exists a graph G with $K(G) < l$ (i.e., not containing a K_l) satisfying $v(G) = n$ and $\chi(G) = g_l(n)$. The term $g^{(l)}(n)$ denotes the largest integer for which there is a G not containing a C_r for $3 \le r \le l$ and for which $v(G) = n$, $\chi(G) = g^{(l)}(n)$. Clearly $g_3(n) = g^{(3)}(n)$.

Graver and Yackel [20] proved that

$$g_l(n) < c_1(n \log \log n/\log n)^{(l-2)(l-1)} \tag{4}$$

In fact Graver and Yackel [20] proved that if $v(G) = n$ and $K(G) < l$ then

$$I(G) > c(n \log n/\log \log n)^{1/(l-1)}\ ; \tag{5}$$

but (4) is an easy consequence of (5).

I proved by probabilistic methods [4] that there is a G_n with $K(G_n) < 3$ for which $I(G_n) < cn^{1/2} \log n$, hence by (1)

$$g_3(n) > \frac{c_2\, n^{1/2}}{\log n}. \tag{6}$$

Very likely the same method will give for $l > 3$

$$g_l(n) > \frac{c_2\, n^{(l-2)/(l-1)}}{(\log n)^{c_3}}, \tag{7}$$

but the details of a proof of (7) seem to be formidable.

The estimation of $g_l(n)$ is clearly connected with the estimation of the Ramsey numbers, i.e., the smallest integer n for which every G_n either contains a K_l or $I(G_n) \geq k$. We do not discuss here these problems but refer to [4, 20].

It is not at all obvious that

$$\lim_{n\to\infty} g^{(l)}(n) = \infty \tag{8}$$

holds for every l. Equation (8) was proved by probabilistic considerations [4] and in fact the stronger result was shown that there is a G_n which does not contain a C_r for $3 \leq r \leq l$ and for which

$$I(G_n) < n^{1-(c/l)} \tag{9}$$

By Eq. (1), Eq. (9) implies

$$g^{(l)}(n) > n^{c/l} \tag{10}$$

It is easy to see that if G_n does not contain a C_r for $3 \leq r \leq 2l+1$ then

$$I(G_n) > c_1 n^{1-1/(l+1)} \tag{11}$$

Equation (11) gives

$$g^{(2l+1)}(n) < c_2 n^{l/(l+1)}. \tag{12}$$

Equation (12) is almost certainly close to being the best possible since the method used in [4] will probably give that there is a G_n which contains no C_r for $3 \leq r \leq 2l+1$ and for which

$$I(G_n) < n^{1-1/(l+1)}(\log n)^{c_3} \quad \text{or} \quad g^{(2l+1)}(n) > c_4 n^{l/(l+1)}/(\log n)^{c_3}. \tag{13}$$

The technical difficulties of a proof of (13) seem to be great.

A good guess for an asymptotic formula for $\log g^{(2l)}(n)$ is not at hand. Undoubtedly

$$\log g^{(2l)}(n) = (c + o(1)) \log n \qquad \text{where} \quad 1/(l+1) \leq c \leq 1/l, \tag{14}$$

but I cannot even prove (14) for $l = 2$, and I have no good guess for the value of c (if it exists).

Hajnal and I generalized (8) for set systems [16]. We also used probabilistic methods. Lovász [21] proved our results by an ingenious but complicated direct construction; thus he showed (8) by a direct construction. His method does not give (9) and does not even seem to give $I(G_n) = o(n)$ instead of (9). I would like to call attention to the following question: I proved by probabilistic methods [3] that there is a G_n satisfying

$$K(G_n) \leq \frac{2\log n}{\log 2}, \qquad I(G_n) \leq \frac{2\log n}{\log 2}. \tag{15}$$

It would be desirable to prove (15) by a direct construction. I cannot even construct a G_n for which

$$\max\,(K(G_n), I(G_n)) < \varepsilon n^{1/2}.$$

It would also be interesting to determine

$$\lim_{n\to\infty} \min_{G_n} \max\,(K(G_n), I(G_n))/\log n. \tag{16}$$

I cannot even prove that the limit in (16) exists. We have trivially $\chi(G) \geq K(G)$. I proved [10] that

$$c_1 n/(\log n)^2 < \max_{G_n} \chi(G_n)/K(G_n) < c_2\, n/(\log n)^2.$$

Very likely

$$\lim_{n\to\infty} \max_{G_n} \frac{\chi(G_n)}{K(G_n)} \frac{(\log n)^2}{n}$$

exists, but this I have not been able to prove.

In [5, 6] it is shown that for every k there is a c_k such that if G_n has no circuit of length less than $c_k \log n$, then its chromatic number is less than k. Also, this is the best possible apart from the value of c_k. Thus we have the result that there is a c such that if G_n has no circuit of length less than $c \log n$ then it is at most three-chromatic. One could have guessed that if G_n does not contain an odd circuit of length less than $c_1 \log n$, then G_n is three-chromatic, but Gallai [19] constructed a G_n which is four-chromatic and the smallest odd circuit of which has length greater than $n^{1/2}$. Gallai and I conjectured that for every k there is a G_n with $\chi(G_n) = k$ the smallest odd circuit of which has length greater than $c_1 n^{1/(k-2)}$; but that there is a c_2 such that if the smallest odd circuit of G_n has length greater than $c_2\, n^{1/(k-2)}$, then $\chi(G_n) < k$. I proved this for $k = 4$ (unpublished).

Rado and I proved [18] that for every $m \geq \aleph_0$ there is a G satisfying $\chi(G) = v(G) = m$ which does not contain a triangle. Hajnal and I proved [16] that for every l there is a graph with $\chi(G) = v(G) = m$, the smallest odd circuit of which has more than $2l + 1$ edges. On the other hand we proved [16] that every graph with $\chi(G) > \aleph_0$ contains a C_4, in fact it contains for every n a $K(n; \aleph_1)$. We also show that every G with $\chi(G) > \aleph_0$ must contain a two-way infinite simple path. On the other hand, we show that there is a G with $\chi(G) > \aleph_0$ which does not contain a $K(\aleph_0, \aleph_0)$. Several further results are proved in [16] and there are many unsolved problems; here we state only two of them. Is it true that, for every G with $\chi(G) = \aleph_1$, there is an n such that, for every $l > n$, G contains a C_l? We can prove this if we assume $\chi(G) \geq \aleph_2$. Is it true that every graph with $\chi(G) = \aleph_0$ satisfies $\sum 1/n_r = \infty$ where $n_1 < n_2 < \cdots$ is the sequence of integers n for which G contains a C_n (perhaps it even follows that the n_i have positive upper density)? A finite form of the above problem may be stated as follows. Put

$$u(k) = \min_G \sum 1/n_i$$

where the minimum is extended over all graphs for which $\chi(G) = k$. Is it true that $u(k) \to \infty$ as $k \to \infty$?

As an application of one of the results given [16] Hajnal and I observed: Let G be a graph whose vertices are the points in n-dimensional Euclidean space. Let S be any countable set of real numbers. We join two points in G if their distance is in S. We show $\chi(G) \le \aleph_0$.

For simplicity's sake we assume $n = 2$. We show that G does not contain a $K(3, \aleph_1)$ and thus by our theorem with Hajnal [16] we obtain $\chi(G) \le \aleph_0$.

Let us assume that there are three points in the plane x_1, x_2, x_3 and a set $\{y_\alpha\}$ so that $d(x_1, y_\alpha)$, is always in S [$d(x, y)$ is the distance between x and y]. Since S is denumerable, only denumerably many y's can lie on the three lines joining the x's. Let y_α be a y not on these lines. There are only countably many choices for $d(y_\alpha, x_1) - d(y_\alpha, x_2)$ and for $d(y_\alpha, x_1) - d(y_\alpha, x_3)$. Thus the y's are the points of intersection of countably many pairs of hyperbolas, or $\{y_\alpha\}$ is countable, which proves our assertion.

A similar but slightly more complicated argument shows that in the case of the n-dimensional space, G does not contain a $K(n + 1, \aleph_1)$ and thus as shown by Erdös and Hajnal [16] $\chi(G) \le \aleph_0$.

Hajnal and I [17] proved that for every $c < \frac{1}{2}$ there is a G with $\chi(G) = \aleph_0$ such that for any set $x_1, \ldots, x_n$ of vertices of G we have

$$I(G(x_1, \ldots, x_n)) > cn. \tag{17}$$

We can show that there is a G satisfying (17) for $c < \frac{1}{4}$ with $\chi(G) = \aleph_1$, but we could not show the same for $c < \frac{1}{2}$.

If for every choice of the vertices $I(G(x_1, \ldots, x_n)) \ge n/2$, then trivially $\chi(G) \le 2$. We conjectured that if for every n and every choice of $x_1, \ldots, x_n$

$$I(G(x_1, \ldots, x_n)) \ge (n - k)/2,$$

then $\chi(G) \le k + 2$. We did not even prove this for $k = 1$.

An interesting conjecture is due to M. Kneser. Let S be a set, $|S| = 2n + k$, to each $S_1 \subset S$, $|S_1| = n$ make correspond a vertex. Two vertices are joined if the corresponding sets are disjoint. Is it true that this graph has chromatic number $k + 2$ (trivially it is less than or equal to $k + 2$)?

Another problem of Hajnal and myself states: Let G be a graph with $\chi(G) = \aleph_0$. Is it true that G has a subgraph G' with $\chi(G') = \aleph_0$ such that G' contains no C_3 (or more generally no C_k for $3 \le k \le n$)?

A finite version of this problem may be stated as follows. Is it true that to every k there is an $f(k)$ such that if $\chi(G) \geq f(k)$ then G contains a subgraph $G'\chi(G') \geq k$, and G' contains no C_3?

One final problem of Hajnal and myself: Is it true that for every $l \geq 3$ and $k \geq 2$ there is a graph G not containing K_{l+1} such that if we color its edges with k colors there is a K_l all of whose edges have the same color? Folkman [26] settled this conjecture for $k = 2$. Hajnal and I recently observed that the following question seems to be relevant here. Let G_n be a graph whose edges can be colored by two colors such that there is no C_3 all of whose edges have the same color. What can be said about $I(G_n)$? It is very easy to show that $I(G_n) > cn^{1/3}$; but perhaps $I(G_n) > cn^{(1/3)+\delta}$ also holds.

3. Miscellaneous Problems in Graph Theory

A well-known theorem of Turán [24] states that every $G(n; [n^2/4] + 1)$ contains a triangle. Turán raised the following questions, which seems very difficult.

Denote by $g(n; k, l)$ the smallest integer such that if in a set of n elements there are given $g(n; k, l)$ k-tuples, then there are always l elements all of whose

$$\binom{l}{k}$$

k-tuples are among the given ones. Turán determined $g(n; 2, l)$ for every l, but the problem is unsolved for every $3 \leq k < l$. It is easy to see that

$$\lim_{n=\infty} g(n; k, l)/n^k = c_{k,l}$$

exists for every k and l. Turán [24] proved $c_{2,l} = \frac{1}{2}(1 - 1/(l-1))$, but for $k > 2$ the value of $c_{k,l}$ is unknown.

Turán also conjectured that

$$g(2n; 3, 5) = n^2(n-1) + 1. \tag{18}$$

The proof of (18) is probably not easy; there are several ways of constructing $n^2(n-1) =$ triples of $2n$ elements so that there are no five elements all of whose triples are chosen.

I proved [8] that there is a constant c_3 such that every $G(n; c_3 n^{3/2})$ contains a subgraph of 7 vertices x_1; y_1, y_2, y_3; z_1, z_2, z_3 with the edges $(x_1, y_1), (x_1, y_2), (x_1, y_3), (z_1, y_1), (z_1, y_2), (z_2, y_1), (z_2, y_3), (z_3, y_2), (z_3, y_3)$.

More generally, I conjectured that for every k there is a c_k such that every $G(n; c_k n^{3/2})$ contains a graph having

$$1 + k + \binom{k}{2}$$

vertices

$$x_1; y_1, \ldots, y_k; z_1, \ldots, z_{\binom{k}{2}}$$

where x_1 is joined to all the y's, each z is joined to two y's, and no two z's are joined to the same two y's. I have not proved this even for $k = 4$.

Moon and Moser [22] posed the following question: Let G_n be a graph of n vertices. Denote by $g(n)$ the maximum number of different sizes of cliques that can occur in a G_n. They proved

$$n - [\log n/\log 2] - 2 \log\log n < g(n) \leq n - [\log n/\log 2].$$

I improved the lower bound to $n - (\log n/\log 2) - H(n) + O(1)$ where $H(n)$ is the smallest integer for which $\log_{H(n)} n < 1$ ($\log_r n$ denotes the r-fold interated logarithm). I expect that the lower bound is essentially the best possible, but I cannot even prove that $g(n) < n - (\log n/\log 2) - C$ for every C if $n > n_0(C)$ is sufficiently large [9].

Recently several papers have been published on extremal problems in graph theory (see [7, 11–13, 23]. Here I would like to mention one such question. Pósa proved that every $G(n; 2n - 3)$ contains a circuit with at least one diagonal and that $2n - 3$ is the best possible here. I had thought that every $G(n; kn - k^2 + 1)$ contains a circuit one vertex of which is the end point of at least $k - 1$ diagonals. Using Pósa's idea I proved this for $k = 3$ and $k = 4$, but Lewin proved (oral communication) that in general the conjecture is incorrect. I do not have any plausible conjecture to replace my original one.

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