## ON THE RECURRENCE OF A CERTAIN CHAIN[1]

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Let balls be placed successively and independently in urns $U_1$, $U_2, \cdots$, urn $U_i$ receiving each ball with probability $p_i$, $i=1, 2, \cdots$. After $n$ balls have been placed let $L_n$ be the number of urns containing an odd number of balls. The event $[L_n=0$ for infinitely many $n]$ has probability one or zero, termed respectively the "recurrent" and the "transient" cases. In [**1**, p. 94] it was stated that "it seems impossible to obtain a general criterion in terms of $\{p_k\}$ to ensure the recurrent case," and in [**2**] it was stated "it would appear that the necessary and sufficient conditions are rather delicate and not to be exhibited in neat form."

In this note we clarify matters, showing that the condition (1) given below, previously known to be sufficient for recurrence ([**1**] and [**2**]), is also necessary.

Without loss of generality we assume $p_i>0$, $i=1, 2, \cdots$, $p_1 \geqq p_2 \geqq p_3 \geqq \cdots$, and we set $f_n = p_n + p_{n+1} + \cdots$, so that $f_1=1$ and $f_n$ decreases monotonically to zero.

THEOREM. *A necessary and sufficient condition for recurrence is that*

$$\sum_1^\infty \frac{1}{2^n f_n} = \infty. \tag{1}$$

In the following $c_1, c_2, \cdots$ are suitable absolute positive constants. Let $X_1, X_2, \cdots$ be mutually independent Bernoulli random variables taking the values 0 or 1 with probabilities $\frac{1}{2}$ each, and set $S=\sum_{i=1}^\infty p_i X_i$. It was shown in [**2**] that a necessary and sufficient condition for recurrence is that $E(1/S)=\infty$. Let $N=\min\{n \mid X_n=1\}$ and let $E_n$ be the event $[N=n]$, so that $P(E_n)=2^{-n}$. Since $S \leqq f_N$ we have $E(1/S) \geqq E(1/f_N) = \sum_1^\infty 1/(2^n f_n)$, so that condition (1) is sufficient, and the necessity will follow if we show that $E(S^{-1} \mid E_n) \leqq c_1/f_n$.

Let $A_{nj} = [jS < f_n]$, $j=0, 1, \cdots; n=1, 2, \cdots$. We assert that it is sufficient to prove

$$\sum_{j=0}^\infty P(A_{nj} \mid E_n) \leqq c_2, \tag{2}$$

for if (2) is true

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$$c_2 \geqq \sum_{j=0}^{\infty} P(A_{nj} \mid E_n) = \sum_{j=0}^{\infty} P(f_n/S > j \mid E_n)$$
$$\geqq \int_0^{\infty} P(f_n/S > x \mid E_n) dx = E(f_n/S \mid E_n)$$
$$= f_n E(1/S \mid E_n).$$

Let now $T_{nk} = \sum_{i=n+1}^{n+k} X_i$, $n=0, 1, \cdots; k=1, 2, \cdots$, so that if $E_n$ occurs we obtain by partial summation,

$$S = \sum_{i=n}^{\infty} p_i X_i$$
$$= p_n + \sum_{k=1}^{\infty} T_{nk}(p_{n+k} - p_{n+k+1}),$$

and let $B_{nj} = \bigcup_{k \geqq j} [T_{nk} < k/4]$. We next assert that $A_{nj}E_n \subset B_{nj}E_n$ for $j \geqq 4$; $n=1, 2, \cdots$. For suppose $\overline{B}_{nj}E_n$ occurs. Then

$$S = p_n + \sum_{k=1}^{\infty} T_{nk}(p_{n+k} - p_{n+k+1})$$
$$\geqq p_n + (1/4) \sum_{k=j}^{\infty} k(p_{n+k} - p_{n+k+1})$$

and

$$f_n = p_n + \sum_{k=1}^{\infty} k(p_{n+k} - p_{n+k+1})$$
$$= p_n + \sum_{k=1}^{j-1} + \sum_{k=j}^{\infty} \leqq jp_n + \sum_{k=j}^{\infty} k(p_{n+k} - p_{n+k+1}).$$

Consequently

$$jS \geqq jp_n + (j/4) \sum_{k=j}^{\infty} k(p_{n+k} - p_{n+k+1}) \geqq f_n$$

for $j \geqq 4$, so that $S \geqq f_n/j$ and $\overline{A}_{nj}E_n$ occurs.

Hence $P(A_{nj} \mid E_n) \leqq P(B_{nj} \mid E_n)$, $j \geqq 4$, and to prove (2) it is sufficient to prove $\sum q_j < c_3$, where $q_j = P(B_{nj} \mid E_n)$, $q_j$ being independent of $n$.

Setting $V_j = T_{0j} = X_1 + X_2 + \cdots + X_j$ we have

$$q_j = P(V_k < k/4 \quad \text{for some } k \geqq j)$$
$$\leqq \sum_{k=j}^{\infty} P(V_k < k/4).$$

The terms in the last sum are well known to decrease exponentially (cf., e.g., Chernoff [**3**, Theorem 1]), so that $\sum q_j$ converges, (2) holds, and the theorem is proved.

## References

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3. H. Chernoff, *A measure of asymptotic efficiency of tests of hypotheses based on the sums of observations*, Ann. Math. Statist. **23** (1952), 493–507.

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