Reprinted from
ISRAEL JOURNAL OF MATHEMATICS
Volume 5, Number 1, January 1967

SOME REMARKS ON NUMBER THEORY. II.

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ABSTRACT

Like the previous paper of the same title [5] this note contains disconnected remarks on number theory.

1. Bellman and Shapiro in one of their papers [1] prove among others the following result: Denote by $Q(a,b)$ the number of squarefree integers n satisfying $a \leqq n < b$ and let $A(n)$ be a strictly monotone function tending to infinity together with n. Then if we neglect a sequence n_i of density 0 we have

$$Q(n, n + A(n)) = (1 + o(1))\frac{6}{\pi^2}A(n). \tag{1}$$

We will prove a more general theorem which will show that (1) remains true if the monotonicity of $A(n)$ is no longer required. In fact we will prove

THEOREM 1 *Let $f(k)$ be a real valued number-theoretic function satisfying*

$$\lim_{n=\infty} \frac{1}{n} \sum_{k=1}^{n} f(k) = \alpha \qquad (\alpha \neq \pm \infty). \tag{2}$$

Assume further that to every $\eta > 0$ there is a $g(\eta)$ so that for every $l > g(\eta)$ and every $n > 0$

$$\frac{1}{l}\sum_{k=0}^{l-1} f(n+k) < \alpha + \eta\ . \tag{3}$$

Then to every $\varepsilon > 0$ and $\delta > 0$ there is an $h(\varepsilon,\delta)$ so that for all but εx integers $n < x$ we have for every $l > h(\varepsilon,\delta)$

Received January 10, 1967.

$$\alpha - \delta < \frac{1}{l} \sum_{k=0}^{l-1} f(n+k) < \alpha + \delta. \tag{4}$$

Before we give the simple proof we make a few remarks. By the same method we could that if for every $l > g(\eta)$ and every $n > 0$

$$\frac{1}{l} \sum_{k=0}^{l-1} f(n+k) > \alpha - \eta \tag{3$'$}$$

then (4) holds.

It is easy to see that our Theorem implies that if for every $A(n) \to \infty$

$$\limsup_{n=\infty} \frac{1}{A(n)} \sum_{k=0}^{A(n)} f(n+k) \leqq \alpha$$

then for almost all $n \leqq x$ (i.e. all n neglecting a sequence of density 0)

$$\lim_{n=\infty} \frac{1}{A(n)} \sum_{k=0}^{A(n)} f(n+k) = \alpha. \tag{5}$$

Now we prove our Theorem. The upper bound in (4) is trivial since it follows from (3) so it is enough to prove the lower bound. Let us assume that (4) does not hold, then there is an $\varepsilon > 0$ and $\delta > 0$ so that for every t there are arbitrarily large values of x so that the number of integers $n_i \leqq x$ for which there is an $l_i > t$ satisfying

$$\frac{1}{l_i} \sum_{k=0}^{l_i-1} f(n_i+k) \leqq \alpha - \delta \tag{6}$$

is greater than εx. We shall now show that for $\eta < \frac{1}{2}\varepsilon\delta$, $t > g(\eta)$ (6) contradicts (3). To see this let m_i be the largest integer for which

$$\frac{1}{m_i - n_i} \sum_{t=n_i}^{m_i-n_i-1} f(t) \leqq \alpha - \delta. \tag{7}$$

By our assumption and by (2)

$$g(\eta) < m_i - n_i < \infty. \tag{8}$$

Consider now the sequence of intervals (n_i, m_i) (i.e. $n_i \leqq x < m_i$). There clearly exists a subsequence of disjoint intervals (n_{i_r}, m_{i_r}), $r = 1, 2, \cdots$ so that each n_i is covered by one of the intervals (n_{i_r}, m_{i_r}), $r = 1, 2, \cdots$. To see this put $n_{i_1} = n_1$, $m_{i_1} = m_1$ and assume that the intervals (n_{i_r}, m_{i_r}) $r < s$ have already been constructed. Let n_i be the least n_j greater than m_{i_r}(m_i can not be one of the n's since by (6) this would contradict the maximality property of m_{i_r}). Put $n_i = n_{i_s}$ $m_i = m_{i_s}$ and this sequence of intervals clearly has the required properties.

By our assumption $\sum_{n_i \leqq x} 1 > \varepsilon x$ holds for infinitely many x, hence if m_{i_s} is the; mallest $m_{i_r} \geqq x$ we evidently have

$$\sum_{r=1}^{s} (m_{i_r} - n_i) > \varepsilon m_{i_s}. \tag{9}$$

Put $m_{i_r} - n_{i_r} = \alpha_r$. From (9) we have either

$$\Sigma_1 \alpha_r > \frac{1}{2}\varepsilon m_{i_s} \text{ or } \Sigma_2 \alpha_r > \frac{1}{2}\varepsilon m_{i_s}$$

where in Σ_1, $r \equiv 1 \pmod 2$ and $r \leqq s$, in Σ_2, $r \equiv 0 \pmod 2$, and $r \leqq s$. Without loss of generality assume

$$\Sigma_1 \alpha_r > \frac{1}{2}\varepsilon m_{is}. \tag{10}$$

By (2) we have

$$\sum_{k=n_1}^{m_{i_s}} f(k) = (1 + o(1) \ \alpha m_{i_s} = \Sigma' + \Sigma'' \tag{11}$$

where in Σ'

$$n_{i_{2j+1}} \leqq k < m_{i_{2j+1}}, \quad 0 \leqq j \leqq \frac{s-1}{2}$$

and in Σ''

$$m_{i_{2j+1}} \leqq t < n_{i_{2j+3}}, \, 0 \leqq j \leqq \frac{s-3}{2}.$$

We have from (7)

$$\Sigma' \leqq (\alpha - \delta) \ \Sigma_1 \alpha_r \ . \tag{12}$$

We evidently have by (8)

$$\beta_j = n_{i_{2j+3}} - m_{i_{2j+1}} > m_{i_{2j}} - n_{i_{2j}} > g(\eta) \, .$$

Thus by (3)

$$\Sigma'' < (\alpha + \eta) \sum_{j=0}^{(s-3)/2} \beta_j = (\alpha + n) \left(m_{i_s} - \sum_{1} \alpha_r \right) + O(1) \tag{13}$$

since

$$\Sigma_1 \alpha_r + \sum_{j=0}^{(s-3/)2} \beta_j = m_{i_s} - n_1 = m_{i_s} + O(1).$$

Thus from (11), (12), (13) and (10) we have

$$\begin{aligned}(1 + o(1))\alpha m_{i_s} = \sum_{k=n_1}^{m_{i_s}} f(k) &\leqq (\alpha + \eta) m_{i_s} - (\eta + \delta) \ \Sigma_1 \alpha_r + O(1) \\ &\leqq (\alpha + \eta) m_{is} - \frac{1}{2}\varepsilon\delta m_{is} + O(1)\end{aligned}$$

an evident contradiction if $\eta < \frac{1}{2}\varepsilon\delta$. This completes the proof of our Theorem.

Corollary. Let $a_1 < a_2 < \cdots$ be any sequence of integers and let $b_1 < b_2 < \cdots$ the sequence of integers not divisible by any a_i. Assume that the b's have density α. Then if $U(n) \to \infty$ together with n we have for almost all n and every $l > U(n)$

$$\lim_{n=\infty} \frac{B(n+l)-B(n)}{l} = \alpha \left(B(m) = \sum_{b_i < m} 1 \right).$$

The corollary easily follows from our Theorem. Let $f(n) = 1$ if n is a b and $f(n) = 0$ otherwise. To prove our corollary we only have to show that our $f(n)$ satisfies (3). Denote by α_k the density of integers not divisible by any a_i, $1 \leqq i \leqq k$. Evidently α_k exists and $\alpha_1 \geqq \alpha_2 \geqq \cdots$. It is known [4] that if the b's have density α then

$$(14) \qquad \lim_{k=\infty} \alpha_k = \alpha$$

Let $f_k(n) = 1$ if $n \not\equiv 0 \pmod{a_i}$, $1 \leqq i \leqq k$ and $f_k(n) = 0$ otherwise. Clearly $f_k(n) \geqq f(n)$. $f_k(n)$ is periodic mod$[a_1, \cdots, a_k]$ thus $f_k(n)$ clearly satisfies (3) with α_k replacing α, hence finally by (14) $f(n)$ satisfies (3). If $\sum 1/a_i < \infty$ the proof of (14) is simple and direct and we do not need [4].

It is also easy to see that our Theorem applies for $f(n) = \sigma(n)/n$ or $f(n) = \phi(n)/n$. In fact it applies to every multiplicative function $f(n) \geqq 1$ which satisfies

$$\sum_p \frac{f(p)-1}{p} < \infty$$

we leave the details to the reader [6]. On the other hand our Theorem does not seem to imply Theorem 4 of [7].

2. In one of their papers Chowla and Vijayaraghavan [3] state that to every $\varepsilon > 0$ there is an A so that if $a_1 < \cdots < a_k \leqq x$ is a sequence of integers satisfying

$$\sum_{i=1}^{k} \frac{1}{a_i} \geqq A, \ (a_i, a_j) = 1$$

then the number of integers $n \leqq x$ not divisible by any a is $< \varepsilon x$.

This result indeed easily follows by Brun's method [8]. The number of integers $n \leqq x$, $n \not\equiv 0 \pmod{a_i}$, $1 \leqq i \leqq k$ is by Brun's method [8] less than $c_1 e^{-A} x$ (c_1 is an absolute constant independent of $a_1, \cdots, a_k$).

The following question seems to be of some interest:

Let $a_1 < \cdots$ be of any sequence of integers satisfying $\sum_i 1/a_i \leqq A$. Denote by $f(a_1, \cdots; x)$ the number of integers not exceeding x not divisible by any a_i. Put

$$F(A; x) = \min f(a_1, \cdots; x)$$

where the minimum is to be taken over all sequences satisfying $\Sigma_i 1/a_i \leqq A$. How large is $F(A;x)$ and which sequence $a_1 < \cdots$ gives the minimum? Let p_r be the largest prime $\leqq x$ and $p_r > p_{r-1} > \cdots$ the sequence of primes $\leqq x$. Define i by

$$\sum_{j=i}^{r} \frac{1}{p_j} \leqq A < \sum_{j=i-1}^{r} \frac{1}{p_j}.$$

It seems to me that perhaps $F(A,x) = f(p_i \cdots, p_r; x)$ or that at least

$$F(A;x) = (1 + o(1)) f(p_i, \cdots, p_r; x)\,. \tag{15}$$

It easily follows from the results of de Bruijn [2] that for $x > x_0(A)$ and $A > A_0$ $(\exp\ z = e^z)$

$$F(A;x) \leqq f(p_i, \cdots, p_r;\ x) < x \exp(-e^A)\ .$$

I do not see how to prove (15) and in fact I cannot even show that for some fixed $\varepsilon > 0$ (ε independent of A and x)

$$F(A,x) > \varepsilon f(p_i \cdots p_r;\ x),$$

in fact I have no satisfactory lower bound for $F(A;x)$.

3. We prove by Brun's method [8] the following

Theorem 2. *To every c_1 there is a $c_2 = c_2(c_1)$ so that if $a_1 < \cdots < a_k \leqq n$, $k > c_1 n$ is any sequence of integers then*

$$\Sigma_1 \frac{1}{d} > c_2 \log n$$

where in Σ_1 the summation is extended over all the integers d which are divisors of some a_i.

Let $\varepsilon = \varepsilon(c_1)$ be sufficiently small and write

$$f_\varepsilon(m) = \prod_{p^\alpha \| m} p^\alpha,\ p \leqq n^\varepsilon$$

where $p^\alpha \| m$ means $p^\alpha | m, p^{\alpha+1} \nmid m$, $d_1 < \cdots < d_r$ be the integers $f_\varepsilon(a_i)$, $i = 1, \cdots, k$. To prove our Theorem it will clearly suffice to show

$$\sum_{i=1}^{r} \frac{1}{d_i} > c_2 \log n\ . \tag{16}$$

We need two lemmas.

Lemma 1. *Let $\varepsilon < c_1/8$. Then for $n > n_0$ the number S of integers $m \leqq n$ for which $f_\varepsilon(m) > n^{1/2}$ is less than $c_1 n/2$.*

We evidently have by the well known result $\sum_{p<x} \frac{\log p}{p} = \log x + O(1)$

$$n^{S/2} \leqq \prod_{m=1}^{n} f_\varepsilon(m) < \prod_{p<n^\varepsilon} p^{n/p+n/p^2} + \cdots$$

$$= \prod_{p<n^\varepsilon} p^{n/_- -1} < \exp 2\varepsilon n \log n.$$

Thus $S < 4\varepsilon n < c_1 n/2$, which proves the lemma.

LEMMA 2. *Let $u \leqq n^{1/2}$. The number of integers $m \leqq n$ for which $f_\varepsilon(m) = u$ is less than $c_3 n/u\varepsilon \log n$.*

The integers $m \leqq n$ for which $f_\varepsilon(m) = u$ are of the form ut where

$$t \leqq \frac{n}{u}, \; t \not\equiv 0 (\text{mod } p), \; p \leqq n^\varepsilon \tag{17}$$

By Brun's method the number of integers t satisfying (17) is for $u \leqq n^{1/2}$ less than

$$c_4 \frac{n}{u} \prod_{p \leqq n^\varepsilon} \left(1 - \frac{1}{p}\right) < c_3/\varepsilon u \log n$$

which proves the Lemma.

By Lemma 1 the number of a's with $f_\varepsilon(a_i) \leqq n^{1/2}$ is greater than $c_1 \, n/2$. Thus we have for these a's by Lemma 2.

$$c_1 n/2 < \frac{c_3 n}{\varepsilon \log n} \sum_{i=1}^{r} \frac{1}{d_i}$$

hence

$$\sum_{i=1}^{r} \frac{1}{d_i} > \frac{\varepsilon c_1}{2c_3} \log n$$

which proves (16) and hence Theorem 2.

I have no reasonable estimate for c_2 as a function of c_1.

4. Straus asked me the following question:
What is the maximum number of integers $a_1 < \cdots < a_k \leqq x$ no two of which are relatively prime but every three of them are relatively prime? The question is perhaps a bit artificial but it seems to me of some interest that a simple and fairly precise answer can be given. Put max $k = f(x)$, then

$$f(x) = \left(\frac{1}{2} + o(1)\right) \frac{\log x}{\log \log x} . \tag{18}$$

To prove (18) observe that if $a_1 < \cdots < a_k \leqq x$ satisfies for every $1 \leqq i_1 < i_2 \leqq k$, $(a_i, a_j) \neq 1$ and for every $1 \leqq j_1 < j_2 < j_3 \leqq k$, $(a_{j_1}, a_{j_2}, a_{j_3}) = 1$ then to every $1 \leqq i < j \leqq k$ there corresponds a prime $p_{i,j}$ so that $p_{i,j} \mid a_i, p_{i,j} \mid a_j$ and for every other $r \leqq k$ $p_{i,j} \nmid a_r$, hence the $p_{i,j}$ are distinct for distinct $1 \leqq i < j \leqq k$ and we evidently have

$$x^k > \prod_{i=1}^{k} a_i \geqq \prod_{1 \leqq i < j \leqq k} p_{i,j}^2 \geqq \left(\prod_{r=1}^{\binom{k}{2}} q_r \right)^2 \tag{19}$$

where $2 = q_1 < \cdots$ are the sequence of consecutive primes. From the prime number theorem we have

$$\prod_{r=1}^{\binom{k}{2}} q_r = \exp((1 + o(1))\, k^2 \log k \tag{20}$$

Hence from (19) and (20) we have $k \log x \geqq (1 + o(1))\ 2k^2 \log k$ or

$$k \leqq (1 + o(1))\, \frac{\log x}{2 \log \log x}.$$

To complete the proof of (18) we now show that for every $\varepsilon > 0$ there is an x_0 so that if $x > x_0(\varepsilon)$ we can construct integers $a_1 < \cdots < a_k \leqq x$,

$$k > (1 - \varepsilon) \frac{\log x}{2 \log \log x}$$

so that no two a's should be relatively prime but every three of them are relatively prime. Put $k = [(1 - \varepsilon) \log x / 2 \log \log x]$ and let $q_1 < \cdots < q_{\binom{k}{2}}$ be the first $\binom{k}{2}$ consecutive primes. Form a symmetric matrix $|u_{i,j}|$ of size k from these primes the diagonal elements are all $1 - s$. a_i is the product of the primes in the i-th row each a_i is the product of $k - 1$ primes by the prime number theorem for every fixed ε and $x > x_0(\varepsilon)$ $a_i < q_{\binom{k}{2}} < x$. (a_i, a_j) is the prime $u_{i,j} = u_{j,i}$ and $(a_{\ }, a_j, a_r)$ is clearly always one.

Let r be fixed and x large. Denote by $f_r(x)$ the largest value of k for which there is a sequence $a_1 < \cdots < a_k \leqq x$ so that no r of them are relatively prime, but every $r + 1$ of them are relatively prime. In (18) we showed

$$f_2(x) = \left(\frac{1}{2} + o(1) \right) \log x / \log \log x.$$

By the same method we can prove

$$f_r(x) = (1 + o(1)) \left(\frac{(r-1)(r-1)}{r} \right)^{1/r-1} \left(\frac{\log x}{\log \log x} \right)^{1/r-1}.$$

REFERENCE

1. R. Bellman and H. N. Shapiro, *The distribution of squarefree integers in small intervals*, Duke Journal **21** (1954), 629–637.

2. N. G. de Bruijn, *On the number of positive integers* $\geqq x$ *and free of prime factors* $\geqq y$, Nederl. Akad. Wetensch. Proc. Ser. A. **A. 54** (1951), 50–60.

3. S. Chowla and T. Vijayaragharan, *On the largest prime divisors of numbers*, J. Indian Math. Soc. **11** (1947), 31–37.

4. H. Davenport and P. Erdös, *On sequences of positive integers*, J. Indian Math. Soc. **15** (1951), 19–24.

5. P. Erdös, *Some remarks on number theory*, Israel J. of Math. **3** (1965), 6–12.

6. P. Erdös, *Some remarks about additive and multiplicative functions*, Bull. Amer. Math. Soc., **52** (1946), 527–537, see in particular Theorem 8 p. 533.

7. P. Erdös, *Asymptotische Untersuchungen uber die Anzahl der peiler von n*, Math. Annalen, **169** (1967), 230–238.

8. See e.g. H. Halberstam and K. F. Roth, *Sequences*, Oxford and Clarendon Press, 1966.