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On the divisibility properties of sequences of integers (II)

by

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Let $a_1 < \ldots$ be a sequence of integers, we will denote it by A. Put $A(x) = \sum_{a_i<x} 1$ and denote

$$f(x) = \sum_{\substack{a_i \mid a_j \\ a_j < x}} 1.$$

In [1] we proved that if A has positive upper logarithmic density then for infinitely many x ($\log_k x$ denotes the k-fold iterated logarithm, $\exp z = e^z$)

$$f(x) > x\exp\big(c_1(\log_2 x)^{1/2}\log_3 x\big)$$

but there exists a sequence A of positive density for which for every x

$$f(x) < x\exp\big(c_2(\log_2 x)^{1/2}\log_3 x\big).$$

Throughout this paper $c_1, c_2, \ldots$ will denote positive constants, not necessarily the same at each occurrence. $\liminf_{x=\infty} [A(x)/x]$ will be called the lower density of the sequence A, η will denote numbers which can be chosen arbitrarily small not necessarily the same at each occurrence, $C_1, \ldots$ numbers which can be chosen arbitrarily large.

A natural question now was: What assumptions about A will insure that

$$\lim_{x=+\infty} \frac{f(x)}{x} = \infty \tag{1}$$

should hold?

It is easy to see that (1) does not have to hold if we assume that the logarithmic density of A is positive. It was stated in [1] that if A has positive density then (1) holds. It will turn out that the speed with which $f(x)/x$ tends to infinity depends in a curious way on the density of our sequence A. In fact we shall prove the following

THEOREM 1. *Let k be any integer and $1/(k+1) < a \leq 1/k$. Then there is a $c_1 = c_1(a)$ so that if the sequence A has lower density a then for all suffi-*

ciently large x

$$f(x) > x\exp\left(c_1(\log_{k+1}x)^{1/2}\log_{k+2}x\right). \tag{2}$$

It is rather surprising that this weird Theorem is nearly best possible.

THEOREM 2. *Let* $1/(k+1) < a < 1/k$. *Then there is a sequence of density* a *and a constant* $c_2 = c_2(a)$ *satisfying*

$$\liminf_{x=\infty} f(x)\left(x\exp\left(c_2(\log_{k+1}x)^{1/2}\log_{k+2}x\right)\right)^{-1} = 0. \tag{3}$$

$c_2 = c_2(a)$ *tends to* 0 *if* $a \to 1/(k+1)$ *and* k *is greater than* 1.

Let finally $a = 1/k$ *and* $g(x)$ *any function tending to infinity as slowly as we please. Then there exists a sequence of density* $1/k$ *for which*

$$\liminf_{x=\infty} f(x)\left(x\exp\left(g(x)(\log_{k+1}x)^{1/2}\log_{k+2}x\right)\right)^{-1} = 0. \tag{3'}$$

Denote by $L(a)$ the upper limit of the values of c_1 for which (2) holds. Clearly for every $c_2 > L(a)$ (3) holds. It would be of interest to determine $L(a)$ explicitly and to decide whether (2) or (3) holds for $c_2 = L(a)$. It seems possible that $L(a)$ tends to infinity as a tends to $1/k$. We already know from Theorem 2 that $L(a)$ tends to 0 if a tends to $1/(k+1)$ and $k > 1$. We can only prove that $L(a)$ tends to infinity if $a \to 1$.

$\Omega(n)$ will denote the number of prime factors of n multiple factors counted multiply and $\Omega_l(n)$ will denote the number of prime factors not exceeding l (multiple factors counted multiply). To prove Theorem 1 we need some lemmas.

LEMMA 1. *To every* $\eta > 0$ *there is a* $C_1 = C_1(\eta)$ *so that for every* $l \leq x$ *the number of integers* $n \leq x$ *for which*

$$|\Omega_l(n) - \log_2 l| < C_1(\log_2 l)^{1/2}$$

is less than ηx.

Lemma 1 can be proved easily by the method of Turán [5]. In fact it is easy to see that as $l \to \infty$ $\left(\Omega_l(n) - \log_2 l\right)/(\log_2 l)^{1/2}$ approaches the Gaussian distribution (see [2]).

LEMMA 2. *Let* $l < C(\log\log x)^{1/2}$. *Then we have uniformly in* x

$$\sum_{\substack{1 \leq t \leq x \\ \Omega(t) = l}} \frac{1}{t} = \left(1 + o(1)\right)(\log_2 x)^l/l!.$$

The proof of Lemma 2 is easy by complete induction with respect to l [4].

Using Lemmas 1 and 2 we now prove the crucial and difficult

LEMMA 3. *Let $d \leqslant 1$, $\delta < d$ be arbitrary numbers, $n^{1/2} < m < n < r$ depend on a parameter x and satisfy*

$$(4) \qquad \frac{r}{n} \to \infty, \quad \left(\log \frac{n}{m}\right)^2 > \frac{r}{n}.$$

Let further $m < b_1 < \ldots < b_s \leqslant n$ be an arbitrary sequence B of integers for which for every η there is a $C_\eta > 1/\eta$ so that for every $C_\eta m < t < n$ ($B(t) = \sum_{b_i \leqslant t} 1$, C_η depends only on η)

$$(5) \qquad B(t) > (d-\eta)t.$$

Denote by $g(u)$ the number of b's dividing u. Then there is an $\omega = \omega(\delta, d)$ so that the number of integers $n < u < r$ for which

$$(6) \qquad g(u) > \exp\left(\omega\left(\log_2 \frac{n}{m}\right)^{1/2} \log_3 \frac{n}{m}\right)$$

is greater than $(d-\delta)r$. Further for every fixed d, as δ tends to d, $\omega = \omega(\delta, d)$ can be made as large as we please. In other words for every C there is a $\delta_0 = \delta_0(C, d)$ so that for every $\delta_0 < \delta < d$ (6) holds with $\omega = C$.

To prove Lemma 3 denote by $b_1^*, \ldots, b_v^*$ the b's satisfying

$$(7) \qquad \left|\Omega_{n/m}(b_i^*) - \log_2 \frac{n}{m}\right| < C_1 \left(\log_2 \frac{n}{m}\right)^{1/2}.$$

By Lemma 1 and (5) we have

$$(8) \qquad B^*(t) > (d-2\eta)t$$

for every $C_2 m < t < n$.

Let now $\gamma = \gamma(\delta, d)$ be a number which will be determined later. Put

$$(9) \qquad k = \left[\gamma\left(\log_2 \frac{n}{m}\right)^{1/2}\right]$$

and consider the integers of the form $b_i^* q$, $i = 1, \ldots, v$, where q runs through all the integers satisfying

$$(10) \qquad \Omega(q) = k, \quad \frac{r}{n} < q < \frac{r}{C_2 m}.$$

Denote by $g^*(u)$ the number of solutions of $b_i^* q = u$ where q satisfies (10). Clearly

$$(11) \qquad g(u) \geqslant g^*(u),$$

so that to prove our lemma it will suffice to show that $g^*(u)$ satisfies (6) for at least $(d-\delta)r$ integers $n < u < r$.

Now we estimate from below $\sum_{n<u<r} g^*(u)$, or in other words we estimate from below the number of integers of the form $b_i^* q$ where q satisfies (10). Let q be a fixed integer satisfying (10). By (8) and (10) we have since $r/n \to \infty$

$$B^*\left(\frac{r}{q}\right) - B^*\left(\frac{n}{q}\right) > (d-2\eta)\frac{r}{q} - \frac{n}{q} > (d-3\eta)\frac{r}{q}. \tag{12}$$

From (12) we evidently have (in $\sum' q$ satisfies (10))

$$\sum_{n<u<r} g^*(j) = \sum_q{}' \left(B^*\left(\frac{r}{q}\right) - B^*\left(\frac{n}{q}\right)\right) > (d-3\eta) r \sum{}' \frac{1}{q}. \tag{13}$$

Now by Lemma 2 we have from (4) by a simple computation

$$\sum{}' \frac{1}{q} \geqslant \sum_{\substack{\Omega(q)=k \\ q \leqslant \frac{r}{C_2 m}}} \frac{1}{q} - \sum_{t \leqslant \frac{r}{n}} \frac{1}{t} > (1-\eta)\left(\log_2 \frac{r}{C_2 m}\right)^k / k! - 2\log \frac{r}{n} \tag{14}$$

$$> (1-2\eta)\left(\log_2 \frac{n}{m}\right)^k / k!.$$

From (13) and (14) we finally obtain

$$\sum_{n<u<r} g^*(u) > rd(1-5\eta)\left(\log_2 \frac{n}{m}\right)^k / k!. \tag{15}$$

Now we estimate $\max_{n<u<r} g^*(u)$ from above, in other words we estimate from above the number of solutions of

$$b_i^* q = u \tag{16}$$

where u satisfies (10). Clearly (16) has at most $\binom{T}{k}$ solutions where $T = \Omega_{n/m}(u)$ and k is defined by (9). Further by (16)

$$\Omega_{n/m}(u) = \Omega_{n/m}(b^*) + \Omega_{n/m}(q) < k + \log_2 \frac{n}{m} + C_1\left(\log_2 \frac{n}{m}\right)^{1/2} = L$$

or by a simple computation using (9)

$$g^*(u) < \binom{L}{k} < \frac{L^k}{k!} < \left(1 + \frac{C_1+\gamma}{\left(\log_2 \frac{n}{m}\right)^{1/2}}\right)^{\gamma\left(\log_2 \frac{n}{m}\right)^{1/2}} \frac{\left(\log_2 \frac{n}{m}\right)^k}{k!}. \tag{17}$$

Write

$$(18)\qquad \sum_{n<u<r} g^*(u) = \sum\nolimits_1 g^*(u) + \sum\nolimits_2 g^*(u)$$

where in $\sum_1$ the summation is extended over the $n<u<r$ satisfying

$$(19)\qquad g^*(u) \leqslant \eta d\left(\log_2 \frac{n}{m}\right)^k / k! = U$$

and in $\sum_2$ the opposite inequality holds. Clearly $\sum_1 g^*(u) \leqslant rU$. Thus from (15) and (18)

$$(20)\qquad \sum\nolimits_2 g^*(u) > dr(1-6\eta)\left(\log_2 \frac{n}{m}\right)^k / k!.$$

From (17) and (20) we obtain that the number of summands in $\sum_2 g^*(u)$ is greater than

$$(21)\qquad dr(1-6\eta)\left(1+\frac{C_1+\gamma}{\left(\log_2 \frac{n}{m}\right)^{1/2}}\right)^{-\gamma\left(\log_2 \frac{n}{m}\right)^{1/2}} = V.$$

Hence finally from (21), (19) and (11) there are at least $V = V(\gamma)$ integers $n<u<r$ satisfying

$$(22)\qquad g(u) > U.$$

Now since γ is at our disposal we can immediately obtain from (22) the statements of Lemma 3. If $\gamma = \gamma(d, \delta)$ is small enough then clearly $V > r(d-\delta)$ (if $\eta = \eta(d, \delta)$ is sufficiently small) and by a simple computation for a suitable $\omega = \omega(\gamma)$

$$U > \exp\left(\omega\left(\log_2 \frac{n}{m}\right)^{1/2} \log_3 \frac{n}{m}\right)$$

(the factor η of U causes no trouble since $\eta = \eta(\delta, d)$ is fixed and n/m tends to infinity). Thus (6) holds for $(d-\delta)r$ integers, as stated.

On the other hand for an arbitrarily large ω there is a $\gamma = \gamma(\omega)$ so that

$$U > \exp\left(\omega\left(\log_2 \frac{n}{m}\right)^{1/2} \log_3 \frac{n}{m}\right)$$

and for this value of γ a simple computation gives ($\varepsilon = \varepsilon(\gamma)$)

$$V > \varepsilon r > (d-\delta)r,$$

if $\delta > d-\varepsilon$. Thus the second statement of Lemma 3 is also proved.

Instead of assuming (4) we could prove our lemma by assuming $r/n > M(\delta, d)$ and $n/m > M_1^*(\delta, d)\exp(r/n)^{1/2}$ but our computations would be more complicated and Lemma 3 suffices for our purpose in its present form.

Perhaps a more interesting question is to what extent can the condition $\left(\log \frac{n}{m}\right)^2 > r/n$ be relaxed. It seems to us that the best possible condition for the truth of our lemma is that n/m and r/n tend to infinity so that for every fixed k eventually $n/m > (r/n)^k$ but we have not worked out the details of the proof.

Now we are ready to prove Theorem 1. Assume first $1/(k+1) < \alpha < 1/k$. We shall prove that (2) holds for a suitable $c_1 = c_1(\alpha)$ and all sufficiently large x. Put $\eta < [\alpha - 1/(k+1)]/3$ and let $c_1 = c_1(\eta)$ be a suitable constant.

Put

(23)
$$x_0 = x^{1/2}, \quad x_i = \frac{x}{\exp\left(c_1(\log_{i+1}x)^{1/2}\log_{i+2}x\right)}, \quad i = 1, \dots, k-1,$$
$$x_k = \frac{x}{\log_{k+1}x}, \quad x_{k+1} = x.$$

Now we define inductively the sequences A_i, $i = 1, \dots, k+1$ (the sequences A_i depend on x but since there is no danger of misunderstanding we do not indicate the dependence on x). A_1 is the sequence of a's belonging to the interval (x_0, x_1), in other words: $t \in A_1$ if and only if $x_0 < t < x_1$ and $t \in A$. Assume that for $i < j < k+1$ the sequence A_i has already been defined. The A_j is defined as follows: $A_j = A_j^{(1)} \cup A_j^{(2)}$ where $t \in A_j^{(1)}$ if and only if $x_{j-1} < t < x_j$ and $t \in A$, $t \in A_j^{(2)}$ if and only if $x_{j-1} < t < x_j$ and t has at least

(24)
$$\exp\left(2c_1(\log_j x)^{1/2}\log_{j+1}x\right) = T_j$$

divisors amongst the A_{j-1}. Now we prove

LEMMA 4. *Every integer $t \in A_j^{(2)}$, $2 \leqslant j \leqslant k+1$, has at least T_j divisors amongst the a's (i.e. amongst the members of the sequence A).*

The lemma is obvious for $j = 2$ and follows by a simple induction argument for $j > 2$. Assume that it holds for $j-1$ and we will prove it for j. Let $t \in A_j^{(2)}$. By (24) t has at least T_j divisors amongst $A_{j-1} = A_{j-1}^{(1)} \cup A_{j-1}^{(2)}$. Assume that t has D divisors in $A_{j-1}^{(1)}$, these divisors are a's in (x_{j-2}, x_{j-1}). If $D \geqslant T_j$ our lemma is proved. If $D < T_j$ our t is divisible by at least one $t' \in A_{j-1}^{(2)}$ which by our induction assumption is divisible by at least $T_{j-1} > T_j$ a's, hence Lemma 4 is proved.

Now we show that (2) holds for sufficiently small $c_1 = c_1(\alpha, \eta)$ if $x > x_0(c_1)$. Assume first that for some $2 \leqslant j \leqslant k+1$ ($|S|$ denotes the number of elements of the set S)

(25)
$$|A_j^{(1)} \cap A_j^{(2)}| \geqslant x_{j-1}.$$

(25) immediately implies (2). To see this observe that by $A_j^{(1)} \subset A$ (25) implies

(26) $$|A \cap A_j^{(2)}| \geqslant x_{j-1}.$$

(26), (24) and Lemma 4 clearly implies for $j \leqslant k$

$$f(x) \geqslant T_j|A \cap A_j^{(2)}| > x_{j-1}\exp\left(2c_1(\log_j x)^{1/2}\log_{j+1}x\right)$$
$$= x\exp\left(c(\log_j x)^{1/2}\log_{j+1}x\right) > x\exp\left(c(\log_{k+1}x)^{1/2}\log_{k+2}x\right)$$

which proves (2) for $j \leqslant k$. For $j = k+1$ (25) implies (2) by a simple computation which we leave to the reader.

Henceforth we can thus assume that for every $2 \leqslant j \leqslant k+1$ and all sufficiently large x

(27) $$|A_j^{(1)} \cap A_j^{(2)}| < x_{j-1}.$$

We shall show that (27) leads to a contradiction, and this will complete the proof of (2).

From (27) we deduce that for every $1 \leqslant j \leqslant k+1$ and every $\eta > 0$

(28) $$x_{j-1}\log_{k+1}x \leqslant z \leqslant x_j,$$

(29) $$A_j(z) > zj(a-3\eta)$$

if $c_1 = c_1(\eta)$ is sufficiently small and $x > x_0(c_1)$ is sufficiently large.

We prove (29) by induction with respect to j. First of all we remark that (23) implies $x_{j-1}\log_{k+1}x < x_j$, (29) clearly holds for $j = 1$ since A has lower density a. Assume that (29) holds for $j-1$, we will prove it for j. We apply Lemma 3 with $m = x_{j-2}\log_{k+1}x$, $n = x_{j-1}$, $r = z$, $B = A_{j-1}$. From (23) we deduce by a simple calculation that (4) is satisfied. By a further simple computation we obtain from (23) that for sufficiently large x

(30) $$\log_2\frac{n}{m} > \tfrac{1}{3}\log_j x.$$

Now we can use (6) of Lemma 3. By our induction hypothesis (29) holds for $j-1$ hence (5) holds with $d = (j-1)(a-3\eta)$ hence if we put $\delta = \eta$ we have by Lemma 3 if $c = c(\eta)$ is sufficiently small (use (6) and (24))

(31) $$A_j^{(2)}(z) > z\left((j-1)a-3(j-1)\eta-\eta\right) = z\left((j-1)a-(3j-2)\eta\right).$$

Further since the lower density of A is a

(32) $$A_j^{(1)}(z) \geqslant z\left(a+O(1)\right) > z(a-\eta).$$

From (31), (32) and (27) we finally obtain for sufficiently large x

$$A_j(z) = A_j^{(1)}(z)+A_j^{(2)}(z)-|A_j^{(1)} \cap A_j^{(2)}| > zj(a-3\eta)$$

which completes the proof of (29).

Now we show that (29) leads to a contradiction and this will complete the proof of (2). Apply (29) with $j = k+1$, $z = x = x_{k+1}$. We then obtain from $\eta < \big(a-1/(k+1)\big)/3$, $x \geq A_{k+1}(x) \geq (k+1)x(a-3\eta) > x$ an evident contradiction. Thus (2) is proved.

It follows immediately from Lemma 3 that to every C there is an ε so that if A has density $> 1-\varepsilon$ then

$$f(x) > x\exp\big(C(\log_2 x)^{1/2}\log_3 x\big).$$

We leave the simple proof to the reader. Thus as stated in the introduction, $L(a) \to \infty$ as $a \to 1$. Unfortunately we were unable to prove that $L(a) \to \infty$ as $a \to 1/k$ if $k > 1$ (from below).

Now we prove Theorem 2. Assume $1/(k+1) < a < 1/k$, $k \geq 2$. First of all we show that there exists a sequence of density a for which (3) holds for a suitable $c_2 = c_2(a)$. We will not give all the details but leave some of the simple arguments to the reader.

Let $\eta = \eta(a)$ be sufficiently small and let

$$p_i^{(j)}, \quad i = 1, 2, \ldots; \; j = 1, 2, \ldots, k-1, \tag{33}$$

be $k-1$ disjoint sequences of primes so that for every $j = 1, 2, \ldots, k-1$ the density of integers divisible by at least one of the $p_i^{(j)}$, $i = 1, 2, \ldots$ and none of the $p_i^{(s)}$, $i = 1, 2, \ldots$; $s < j$ is a.

It is easy to see that such a sequence of primes exists. It suffices to have for $j = 1, 2, \ldots, k-1$

$$\prod_i \left(1 - \frac{1}{p_i^{(j)}}\right) = \frac{1/a - j}{1/a - j + 1}. \tag{34}$$

A simple argument then shows that any set of $k-1$ disjoint sequences of primes satisfying (34) also satisfies (33).

Now we are ready to construct our sequence A of density a satisfying (3). Let $x_1 = 10$ and $\log_k x_r = x_{r-1}$. Assume that our sequence A has been defined up to x_{r-1}, we extend it up to x_r as follows:

Let $j = 1, 2, \ldots, k-1$. Put

$$x_r^{(0)} = x_{r-1}, \quad x_r^{(j)} = \frac{x_r}{\log_j x_r}. \tag{35}$$

Let now $x_r^{(j-1)} < t < x_r^{(j)}$. $t \in A$ if and only if $p_i^{(j)} | t$ for some i but $p_i^{(s)} \nmid t$ for every $s < j$ and every i. This defines the sequence A up to $x_r/\log_{k-1} x_r$. Clearly by (33) for every $z \leq x_r/\log_{k-1} x_r$, $A(z) = \big(a + o(1)\big)z$.

Now we have to define the sequence A in $(x_r/\log_{k-1}x_r, x_r)$. Denote by B the sequence of the integers in $(x_r/\log_{k-1}x_r, x_r)$ which are not divisible by any of the $p_i^{(j)}$, $i = 1, 2, \dots; 1 \leqslant j \leqslant k-1$. Clearly by (34) for every $x_r/\log_{k-1}x_r < z < x_r$

$$B(z) = \big(1+o(1)\big)\alpha\left(\frac{1}{\alpha} - k + 1\right)z. \tag{36}$$

Determine now c from the equation

$$\frac{1}{\sqrt{2\pi}}\int_{-c}^{c} e^{-x^2/2}dx = \left(\frac{1}{\alpha} - k + 1\right)^{-1}. \tag{37}$$

Let now $x_r/\log_{k-1}x_r < t < x_r$. $t \in A$ if and only if $t \in B$ and $(l = \log_{k-1}x_r)$

$$|\Omega_l(t) - \log_{k+1}x_r| < c(\log_{k+1}x_r)^{1/2}. \tag{38}$$

From (36), (37), (38) we easily obtain by the well known theorem of Erdös and Kac [2], that for every $z < x_r$ $A(z) = \alpha z + o(z)$. Thus the inductive definition of our sequence A is completed and our sequence A clearly has density α.

Now we show

$$f(x_r) < x_r \exp\big(c_2(\log_{k+1}x_r)^{1/2}\log_{k+2}x_r\big). \tag{39}$$

Observe that if $a_u | a_v$, $a_v \leqslant x_r$, then either $a_u \leqslant x_r^{(0)}$ or for some $0 < j \leqslant k-1$, $x_r^{(j-1)} < a_u \leqslant a_v \leqslant x_r^{(j)}$ or finally $x_r^{(k-1)} < a_u \leqslant a_v \leqslant x_r$. Thus

$$f(x) = \sum_{a_u|a_v}{}' 1 + \sum_{a_u|a_v}{}'' 1 + \sum_{a_u|a_v}{}''' 1 \tag{40}$$

where in $\sum'$ $a_u \leqslant x_r^{(0)} = x_{r-1}$ in $\sum''$ $x_r^{(j-1)} < a_u \leqslant a_v \leqslant x_r^{(j)}$ for some $j = 1, \dots, k-1$ and in $\sum'''$ $x_r^{(k-1)} < a_u \leqslant a_v \leqslant x_r$. We evidently have

$$\sum_{a_u|a_v}{}' 1 < x_r \sum_{t < x_{r-1}} 1/t < 2x_r \log x_{r-1} < 2x_r \log_{k+1} x_r. \tag{41}$$

Further (in $\sum^{(j)}$ $x_r^{(j-1)} < a_u \leqslant a_v \leqslant x_r^{(j)}$)

$$\sum_{a_u|a_v}{}^{(j)} 1 < \frac{x_r}{\log_j x_r} \sum_{t < \log_{j-1}x_r} 1/t < 2x_r. \tag{42}$$

Thus

$$\sum_{a_u|a_v}{}'' 1 < 2kx_r. \tag{43}$$

Now we estimate $\sum'''$. Let $x_r/\log_{k-1}x_r < a_v < x_r$. Then by (38)

$$\Omega_l(t) < \log_{k+1}x_r + c(\log_{k+1}x_r)^{1/2}.$$

Put

$$\log_{k+1} x_r + c(\log_{k+1} x_r)^{1/2} = T_1, \quad 2c\log_{k+1} x_r = T_2.$$

Clearly for fixed a_v the number of u's for which $a_u | a_v$, $x_r/\log_{k-1} x_r < a_u < x_r$ is at most

$$\sum_{i=0}^{T_2} \binom{T_1}{i} < \tfrac{1}{2}\exp\left(c_2(\log_{k+1} x_r)^{1/2}\log_{k+2} x_r\right) \tag{44}$$

for a suitable $c_2 = c_2(c) = c_2(a)$ (c by (37) is determined by a). Thus from (44)

$$\sum_{a_u|a_v}''' 1 < \frac{x_r}{2}\exp\left(c_2(\log_{k+1} x_r)^{1/2}\log_{k+2} x_r\right). \tag{45}$$

(40), (41), (43) and (45) clearly imply (39) and thus (3) is proved.

By the same method we can prove that (3′) holds, to see this it suffices to let in (38) c tend to infinity sufficiently slowly depending on $g(x)$. We leave the details to the reader.

To complete the proof of Theorem 2 we now have to show that if $k > 1$ and a tends to $1/(k+1)$ from above $c_2 = c_2(a)$ tends to 0. We will only outline the proof and leave some of the details to the reader.

More precisely we shall prove the following result: Let $k > 1$. To every ε there is a δ so that for every $1/(k+1) < a < 1/(k+1)+\delta$ there is a sequence A of density a for which (3) holds with a $c_2 = c_2(a) < \varepsilon$.

To prove this statement we define our sequence A in the interval $(x_{r-1}, x_r/\log_{k-1} x_r)$ just as previously. Thus the whole proof proceeds as previously until (36). Determine now η from the equation

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta} e^{-x^2/2} dx = \left(\frac{1}{a} - k + 1\right)^{-1}. \tag{46}$$

Clearly as a tends to $1/(k+1)$, η will tend to 0 from above.

Let now $x_r/\log_{k-1} x_r < t < x_r/\log_k x_r$. $t \in A$ if and only if $t \in B$ and ($l = \log_{k-1} x_r$)

$$\Omega_l(t) > \log_{k+1} x_r - \eta(\log_{k+1} x_r)^{1/2}. \tag{47}$$

Let finally $x_r/\log_k x_r < t < x_r$. Then $t \in A$ if and only if $t \in B$ and

$$\Omega_l(t) < \log_{k+1} x_r + \eta(\log_{k+1} x_r)^{1/2}. \tag{48}$$

From (46), (47) and (48) it follows by the theorem of Erdös and Kac [2] that A has density a (we use $\int_{-\infty}^{\eta} e^{-x^2/2} dx = \int_{-\eta}^{+\infty} e^{-x^2/2} dx$).

As in the previous proof we easily obtain for our sequence A (see (40), (41), (42) and (43))

$$f(x) < \sum_{a_u|a_v}^{*} 1 + O(x_r \log_{k+1} x_r) \tag{49}$$

where in $\sum^*$

$$\frac{x_r}{\log_{k-1} x_r} < a_u < \frac{x_r}{\log_k x_r} \quad \text{and} \quad \frac{x_r}{\log_k x_r} < a_v < x_r.$$

Put

$$\log_{k+1} x_r + \eta(\log_{k+1} x_r)^{1/2} = T_1', \quad 2\eta(\log_{k+1} x_r)^{1/2} = T_2'.$$

We have as in (44) and (45)

$$\sum_{a_u | a_v}^* 1 \leqslant \sum_{i=0}^{T_2'} \binom{T_1'}{i} < x_r \exp\left(\frac{\varepsilon}{2} (\log_{k+1} x_r)^{1/2} \log_{k+2} x_r\right) \tag{50}$$

if $\eta = \eta(\varepsilon)$ is sufficiently small. (49) and (50) clearly implies that (3) holds with a $c_2 < \varepsilon$ as stated. Thus the proof of Theorem 2 is complete.

Unfortunately we were unable to complete the proof in the case $k = 1$ (i.e. if $\alpha \to 1/2$ from above) and in fact are uncertain if the result continues to hold in this case.

For each x denote by $l(x)$ the smallest integer k for which $1 < \log_k x < e$. It seems to us that by the methods of this paper we can obtain the following results: Let $a_1 < a_2 < \ldots$ be a sequence of integers satisfying for a certain $\varepsilon > 0$ and all sufficiently large x

$$A(x) > (1+\varepsilon)\frac{x}{l(x)}.$$

Then $f(x)/x \to \infty$. On the other hand there exists a sequence $a_1 < \ldots$ satisfying for all large x

$$A(x) > (1-\varepsilon)\frac{x}{l(x)}$$

and nevertheless $\liminf\limits_{x=\infty} f(x)/x = 0$.

We have not in fact worked out the proof of these theorems and we can not be absolutely sure that they are correct.

The following result can be proved by the methods of [1]. Let c be a sufficiently large constant and assume that the sequence A satisfies

$$\limsup_{x=\infty} \frac{(\log\log x)^{1/2}}{\log x} \sum_{a_i < x} \frac{1}{a_i} > c.$$

Then $\limsup\limits_{x=\infty} f(x)/x = \infty$. Perhaps the following result holds: Assume that

$$\limsup_{x=\infty} \frac{(\log\log x)^{1/2}}{\log x} \sum_{a_i < x} \frac{1}{a_i} > 0. \tag{51}$$

Then $\limsup_{x=\infty} f(x)/x = \infty$. We proved that (51) implies that $a_i|a_j$ has infinitely many solutions [3].

Put $F(x) = \sum'_{\substack{a_i|a_j \\ a<x}} 1$ where the dash indicates that the summation is extended over those $a_i|a_j$ for which all prime factors of $a_j|a_i$ are greater than the greatest prime factor of a_i. It is easy to see that there is a sequence of positive density for which $\liminf_{x=\infty} F(x)/x = 0$ but for every such sequence

$$\limsup_{x=\infty} F(x)/x(\log\log x)^{1/2} > 0.$$

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