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ON SEQUENCES OF DISTANCES OF A SEQUENCE

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Let

$$A = \{a_1 < a_2 < a_3 \ldots\}$$

be a sequence of positive integers. We arrange all numbers of the form $|a_i - a_j|$ $(i \neq j)$ into a sequence

$$D(A) = \{d_1 < d_2 < d_3 < \ldots\}.$$

A subsequence

$$B = \{b_1 < b_2 < b_3 < \ldots\}$$

of A will be called *avoidable* if one can drop some terms in A so that 1° for the resulting sequence A' no term of B is contained in $D(A')$ and 2° the set A' is infinite. We ask about general conditions, sufficient or necessary for B to be avoidable. By "general" we mean conditions that do not depend on special choice of A or B. They should be expressed in terms of rarity of B in $D(A)$. This approach is by no means frustrated by the example $A = N = \{1, 2, 3, \ldots\}$ and $B = \{1, 3, 5, \ldots\}$, thus B being avoidable by removing all even (or all odd) numbers from A. The most natural assumption that B is of density 0 in $D(A)$ actually turns out to play an essential rôle, in view of the following

THEOREM 1. *For every A and every $\varepsilon > 0$ there is a sequence B of density $\leqslant \varepsilon$ in $D(A)$ which is not avoidable.*

Proof. Let ξ be a real number such that $\{d_n \xi\}$ is equidistributed mod 1 and that $a_i \xi \neq a_j \xi$ mod 1 for $i \neq j$. B may consist of all d_n's for which $\|d_n \xi\| < \varepsilon/2$, $\|\alpha\|$ denoting the distance of α to the nearest integer. Then B has the density ε in $D(A)$. To see that B is not avoidable assume the contrary and let $A' = \{a_1' < a_2' < \ldots\}$ be the (infinite) sequence which remains after removing suitable terms from A. Obviously, the set $\{a_n' \xi\}$ has a limit point mod 1. Hence there are pairs (a_i', a_j'), $i \neq j$, such that $\|(a_i' - a_j')\xi\| < \varepsilon/2$ (mod 1), and so $|a_i' - a_j'|$ is a b_n, this being a contradiction.

A kind of a converse is given by

THEOREM 2. *If A has positive lower density in N and B has lower density in N equal zero, then B is avoidable.*

Proof. If B were not avoidable, there would exist a finite segment $a_1, \ldots, a_l$ of A such that for $n > l$ we had $a_n - a_i \in B$ for some $i = 1, \ldots, l$. (The existence of such a "saturated" segment is not sufficient for B to be not avoidable as is shown by the example $A = \{1, 2, 4, 6, \ldots\}$ and $B = \{1, 3, 5, \ldots\}$, where B is clearly avoidable and the segment $\{1\}$ is saturated). Thus A would be contained, up to finitely many terms, in the union of finitely many translations of a set of lower density 0 and so would itself have lower density 0 contrary to the assumption.

The condition that A should have positive lower density is essential, in view of the following

THEOREM 3. *There exists a sequence A and a sequence $B \subset D(A)$ which has density 0 in $D(A)$ but is not avoidable.*

We proceed to the construction by putting

$$A = \bigcup_{k=1}^{\infty} [k^4, k^4 + k],$$

where $[m, n]$ denotes the set of integers $m, m+1, \ldots, n$. We have obviously $D(A) = N$. Now let

$$B = \bigcup_{i<k} [k^4 - i^4 - i, k^4 - i^4 + k].$$

One easily sees that B has density zero in N. However, B is not avoidable, because in every infinite subsequence A' of A there must be an $a'_r \in [k_1^4, k_1^4 + k_1]$ and an $a'_{r'} \in [k_2^4, k_2^4 + k_2]$, where $k_2 \neq k_1$. Then $|a'_r - a'_{r'}| \in B$.

A sufficient condition for avoidability is given by

THEOREM 4. *If $D(B) = \{c_1, c_2, \ldots\}$ has the property that the sequences $C_s^{\pm} = \{c_n \pm d_s\} \frown \{d_n\}$ are of lower density 0 in $D(A)$ for every s, then B is avoidable.*

Proof. If $a_1, \ldots, a_l$ is a segment of A such as in the proof of Theorem 2, then we have $a_{n_\nu} = a_{i_\nu} + b_{j_\nu}$ $(\nu = 1, 2)$ for n_1 and n_2 sufficiently large, for some $i_\nu = 1, \ldots, l$ and some j_ν. Hence $|a_{n_1} - a_{n_2}| = |(a_{i_1} - a_{i_2}) + (b_{j_1} - b_{j_2})|$ is of the form $c_n \pm d_s$, where s takes values from a finite set only. As $|a_{n_2} - a_{n_1}|$ is some d_n, we see that $D(A)$ is composed, up to a finite number of terms, of finitely many $C_s^{\pm}$'s, which contradicts the assumption of the Theorem.

Note that a sequence B satisfying this assumption has lower density zero in $D(A)$, since $B \setminus (b_1)$ is contained in $\{c_n + b_1\} \frown \{d_n\}$, the sequence $b_2 - b_1, b_3 - b_1, \ldots$ being a part of $D(B)$.

Remark. If B has positive upper (lower) density in $D(A)$, then this is not affected by adjoining the number 0 to A and thus making A to a subsequence of $D(A^*)$ $\big(A^* = A \cup (0)\big)$. In fact, it is easy to prove that those a_n's which do not appear in $D(A)$ constitute a subsequence of upper density $\leqslant \frac{1}{2}$ in $D(A^*)$.

We are not able to decide whether B is avoidable if $b_k = d_{n_k}$ with $n_{k+1} - n_k \to \infty$ (**P 594**) (1). This condition obviously implies that B has density 0 in $D(A)$, hence Theorem 2 shows its sufficiency if A has positive lower density in N. Without additional assumptions we do not even know whether $n_{k+1}/n_k \to \infty$ implies avoidability, we can but prove the following

THEOREM 5. *If the set $N \setminus D(A)$ is finite, $\lim\limits_{n} f(n) = \infty$ and*

$$n_{k+1} > n_k + f(n_k)(n_k \log n_k)^{1/2} \tag{*}$$

(e.g. if $n_k = k^s$, $s > 2$), then the sequence $B = \{d_{n_k}\}$ is avoidable.

Proof. We may suppose $D(A) = N$ and thus $d_n = n$. If $rx < n_j$ for some integer x and r, then, in view of $(*)$, the number of n_k's in the interval $(n_j, n_j + x)$ is

$$o\left(\frac{x^{1/2}}{r^{1/2}(\log x)^{1/2}}\right)$$

when $x \to \infty$. By the same argument, the same estimate is valid for the number of n_j's in $\big(rx, (r+1)x\big)$.

Therefore, there are not more numbers $n_k - n_j$ with $rx < n_j \leqslant (r+1)x$ and $n_j < n_k < n_k + x$ than $o(x/r \log x)$. Using $(*)$ once more we see that

$$o\left(\frac{x}{\log x}\right) \sum_{r<x} \frac{1}{r}$$

is an upper estimate of the number of all differences $n_k - n_j$ not exceeding x. Hence, there are only $o(x)$ such differences and the density of $D(B)$ in $D(A)$ turns out to be zero, the assumption of Theorem 4 being thus fulfilled.

(1) *Added in proof.* This problem has been recently solved in the affirmative by D. Rotenberg (to appear in Colloquium Mathematicum.)

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