SOME REMARKS ON SET THEORY, X.

by

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Let A be a set, G a class of subsets of A and $a=(a_n)_{n<\omega}\in A^\omega$ a sequence of elements of A. We say that G strongly cuts a if for every $n<\omega$ there exists an $X_n\in G$ such that $a_i\in X_n$ for $i<n$ and $a_i\notin X_n$ for $\omega>i\geqq n$. The *complement* $A(-)G$ of G is the system of all sets $A-X$ such that $X\in G$.

Now we are going to prove the following theorem.

THEOREM. *If A is an infinite set, G is a class of subsets of A such that $|G|>|A|$, then there exists an infinite sequence in A^ω which is strongly cut by G or by $A(-)G$.*

PROOF. Assume the conditions of the theorem. For a subset B of A and a class H of subsets of A we denote by $B(\cap)H$ the class of all sets $B\cap X$ with $X\in H$. We write $\overline{X}$ instead of $A-X$ (with the specified A). If $a_0,\dots,a_{n-1}\in A$ then $G_{a_0,\dots,a_{n-1}}$ will denote the class of sets in G containing all of $a_0,\dots,a_{n-1}$.

We distinguish two cases (i) and (ii).

(i) First we suppose

(1) For any $B\subset A$ and $H\subset G$ such that $|B(\cap)H|>m=|A|$ there is an X in H for which $|(B\cap\overline{X})(\cap)H|>m$.

In this case we prove that G strongly cuts a certain sequence in A^ω.

We define by induction a sequence $a_0,a_1,\dots$ of elements of A and a sequence $X_0,X_1,\dots$ of sets in G such that, for every $k<\omega$, the following two conditions hold:

$$a_0,\dots,a_{k-1}\in X_k;\ a_k,a_{k+1},\dots\notin X_k \tag{2}$$

and

$$|(\overline{X}_0\cap\overline{X}_1\cap\dots\cap\overline{X}_{k-1})(\cap)G_{a_0,\dots,a_{k-1}}|>m. \tag{3}$$

By (1) and the conditions of the theorem there is an X_0 in G with $|\overline{X}_0(\cap)G|>m$. Hence there is an element a_0 in $\overline{X}_0$ such that $|\overline{X}_0(\cap)G_{a_0}|>m$. (In the contrary case we would have that $\overline{X}_0(\cap)G\subseteqq\bigcup_{a\in\overline{X}_0}(\overline{X}_0(\cap)G_a)\cup\{0\}$ is of power at most $m\cdot m+1=m$).

Now assume in general that $n\geqq 1$ and we have defined $a_0,\dots,a_{n-1};X_0,\dots,X_{n-1}$ such that

$$a_0,\dots,a_{k-1}\in X_k;\ a_k,a_{k+1},\dots,a_{n-1}\notin X_k \tag{4}$$

for $k<n$ and (3) holds for $k=n$. Then applying (1) with $G_{a_0,\dots,a_{n-1}}$ in place of H

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we obtain a set X_n such that

(5) $$X_n \in G_{a_0,\dots,a_{n-1}}$$

and

$$|(\overline{X}_0 \cap \dots \cap \overline{X}_{n-1} \cap \overline{X}_n)(\cap)G_{a_0,\dots,a_{n-1}}| > m.$$

Thus we have an element a_n such that

(6) $$a_n \in \overline{X}_0 \cap \dots \cap \overline{X}_{n-1} \cap \overline{X}_n$$

and the class of the sets in $(\overline{X}_0 \cap \dots \cap \overline{X}_{n-1} \cap \overline{X}_n)(\cap)G_{a_0,\dots,a_{n-1}}$ containing a_n, is of power $> m$, i. e.

(7) $$|(\overline{X}_0 \cap \dots \cap \overline{X}_n)(\cap)G_{a_0,\dots,a_{n-1},a_n}| > m.$$

Considering (4), (5) and (6) we see that

(8) $$a_0, \dots, a_{k-1} \in X_k;\ a_k, a_{k+1}, \dots, a_{n-1}, a_n \notin X_k$$

for $k \leqq n$. (8) and (7) show that we have just the conditions for $n+1$ which were assumed for n. Thus by induction (and the axiom of choice) we have proved the existence of $(a_n)_{n<\omega}$ and $(X_n)_{n<\omega}$ with the required properties. In particular by (2) we see that G strongly cuts $(a_n)_{n<\omega}$.

(ii) Now we suppose that (1) does not hold, i. e. there is a subclass H of G and a subset B of A such that $|B(\cap)H| > m$ and for every $X \in H$ we have $|(B \cap \overline{X})(\cap)H| \leqq m$. In this case we prove that $A(-)G$ strongly cuts a certain sequence $(a_n)_{n<\omega} \in A^\omega$. First we show that we may assume $B = A$ and $H = G$, in other words that

(9) $$|\overline{X}(\cap)G| \leqq m \quad \text{for every} \quad X \in G.$$

Suppose that we have proved that the hypotheses of the theorem and (9) imply that $A(-)G$ strongly cuts a sequence in A^ω. Applying the suppositions of case (ii), we see that the conditions of the theorem and (9) hold for B and $B(\cap)H$ instead of A and G, respectively. Thus we have a $(b_n)_{n<\omega} \in B^\omega$ which is strongly cut by $B(-)(B(\cap)H)$, hence, a fortiori, $(b_n)_{n<\omega} \in A^\omega$ is a strongly cut by $A(-)G$.

Assuming (9), we shall show that (1) holds for $A(-)G$ instead of G, which implies by case (i) that $A(-)G$ strongly cuts a sequence in A^ω; this will complete the proof of our theorem. Indeed, suppose $B \subset A$, $H \subset G$ and $|B(\cap)(A(-)H)| > m$. This is equivalent to say that $|B(\cap)H| > m$. Then taking an *arbitrary* set X in H we have $|\overline{X}(\cap)H| \leqq m$ and *a fortiori* $|(B \cap \overline{X})(\cap)H| \leqq m$. But this implies $|(B \cap X)(\cap)H| > m$, because assuming $|(B \cap X)(\cap)H| \leqq m$ we would obtain $|B(\cap)H| \leqq m$. Indeed, every set in $B(\cap)H$ is the union of one in $(B \cap \overline{X})(\cap)H$ and one in $(B \cap X)(\cap)H$ and so we could have in $B(\cap)H$ at most $m \cdot m = m$ sets. Thus we really have $|(B \cap X)(\cap)H| > m$ which means that an arbitrary $\overline{X} \in A(-)H$ is suitable for the X of case (i), hence our proof is complete.

Now we state some unsolved problems.

A large "presque-disjoint" system G of subsets of a set A of power $\aleph_0$ [1] shows that the first alternative of the theorem is not always true.

[1] i. e. $|G| > \aleph_0$ and the intersection of any two sets in G is finite.

However the analogous question in the case of a set of power $\aleph_1$ remains open.

PROBLEM 1. Let $|A|=\aleph_1$, $|G|>\aleph_1$. (G is as above). Does G cut always strongly a sequence in A^ω?

If α is any ordinal, we may ask a similar question concerning the existence of $a\in A^\alpha$ strongly cut by a class H of subsets of A. We say that H strongly cuts $(a_\lambda)_{\lambda<\alpha}$ if for every $\nu<\alpha$ there is an $X_\nu\in H$ such that $a_\lambda\in X_\nu$ for $\lambda<\nu$ and $a_\lambda\notin X_\nu$ for $\alpha>\lambda\geqq\nu$. The same example as before shows that already for $\alpha=\omega+2$ the answer is negative if A is of power $\aleph_0$. We do not know what is the situation if the power of A is greater than $\aleph_0$, or if $\alpha=\omega+1$.

PROBLEM 2. Let $|A|=\aleph_1$, $|G|>\aleph_1$. Is there always a sequence $a\in A^{\omega+2}$ which is strongly cut by G or $A(-)G$? In this case perhaps the answer is positive even with ω_1 instead of $\omega+2$.

PROBLEM 3. Let $|A|=\aleph_0$, $|G|>\aleph_0$. Does there exist a sequence $a\in A^\omega$ such that one of the following holds: (i) a is strongly cut by G and there is an $X\in G$ which contains all the elements of a; or (ii) a is strongly cut by $A(-)G$.

Problem 3 arises essentially from the case $\alpha=\omega+1$ by weakening one of the alternatives.

A. MÁTÉ proved that a presque-disjoint system cannot be a counter-example for $\alpha=\omega+1$. To show this suppose that $|A|=\aleph_0$ and G is a large presque-disjoint system of infinite subsets of A, not equal to A. Then first there is an X in G such that every finite subset of X is contained in a set of G different from X. (In the contrary case we could associate a finite subset of X with every X in G in such a way that with different sets in G different finite sets are associated, this means that G is countable.) Starting from such an $X=X_\omega$ we choose an arbitrary X_0 in G. Then there is an a_0 in $X-X_0$ (since $X\cap X_0$ is finite) and an $X_1\in G$ such that $a_0\in X_1$. Then choosing an a_1 satisfying $a_1\in X-X_0-X_1$ we have an $X_2\in G$ with $a_0, a_1\in X_2$. Continuing in this manner we obtain a sequence $(a_n)_{n<\omega}$ and we can add an arbitrary element a_ω in $\overline{X}_\omega$. The resulting sequence $(a_\nu)_{\nu<\omega+1}$ is obviously strongly cut by G.

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