SIMON STEVIN, *Wis- en Natuurkundig Tijdschrift*
40e Jaargang, Aflevering II (Oktober 1966)

ON THE NUMBER OF POSITIVE INTEGERS $\leqslant x$ AND FREE OF PRIME FACTORS $> y$

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1. Introduction

Let $\Psi(x,y)$ denote the number of integers specified in the title. A number of estimates and asymptotic formulae for this function have been given (cf. [1] and the literature mentioned there). Recently DE BRUIJN ([2]) proved an asymptotic formula for log $\Psi(x,y)$ which holds uniformly for $2 < y \leqslant x$. Part of the proof consisted of showing that

$$\Psi(x,y) \geqslant \binom{\pi(y)+u}{u} \text{ where } u = [(\log x)/(\log y)].$$

It is the purpose of this note to extend this inequality to an asymptotic formula (which is weaker than DE BRUIJN's result). In fact we shall prove:

Theorem 1: For $2 < y \leqslant x$ we have for $x \to \infty$, uniformly in y,

$$\log \Psi(x,y) \sim \log \binom{\pi(y)+u}{u} \tag{1}$$

where

$$u = [(\log x)/(\log y)].$$

We remark that this of course follows from DE BRUIJN's theorem. Our interest lies mainly in giving a short fairly straightforward proof. For some ranges of values of y (1) is nearly trivial and the most interesting part of the proof concerns the range $(\log x)^{\varepsilon} < y < (\log x)^{1+\varepsilon}$.

2. Proof of Theorem 1

We shall prove (1) by showing that for every $\varepsilon > 0$ we have

$$\binom{\pi(y)+u}{u} < \Psi(x,y) < \binom{\pi(y)+u}{u}^{1+\varepsilon} \quad \text{for } x > x_0(\varepsilon). \tag{2}$$

a. The first inequality immediately follows from the fact that $\binom{\pi(y)+u}{u}$ represents the number of solutions of

$$\sum_{p \leqslant y} \alpha_p \leqslant u = [(\log x)/(\log y)]$$

in nonnegative integers α_p and this number is less than the number of solutions of

$$\sum_{p \leqslant y} \alpha_p \log p \leqslant \log x$$

which is $\Psi(x,y)$ by definition.

In sections **b**, **c** and **d** we prove the second inequality of (2).

b. We now consider $y < (\log x)^{1+\varepsilon}$. We first remark that (2) is trivial for very small values of y, for example $y < (\log x)^{\varepsilon/2}$, because

$$\Psi(x,y) \leqslant \left(\frac{\log x}{\log 2} + 1\right)^{\pi(y)} < \binom{\pi(y)+u}{u}^{1+\varepsilon} \text{ if } y < (\log x)^{\varepsilon/2}.$$

Hence we can assume that $y > (\log x)^{\varepsilon/2}$.

Let N_1 denote the number of integers $\leqslant x$, free of prime factors $> y^{1-\varepsilon}$ and let N_2 denote the number of integers $\leqslant x$ all of whose prime factors satisfy $y^{1-\varepsilon} < p \leqslant y$. Then $\Psi(x,y) \leqslant N_1 N_2$.

Trivially $N_1 < \left(\frac{\log x}{\log 2} + 1\right)^{\pi(y^{1-\varepsilon})}$. Furthermore

$\binom{\pi(y)+u}{u} > \left(\frac{\pi(y)+u}{\pi(y)}\right)^{\pi(y)}$. From this it follows that

$$\frac{\log N_1}{\log \binom{\pi(y)+u}{u}} = O\left(\frac{\log\log x}{(\log x)^{\frac{1}{2}\varepsilon^2}}\right)$$

i.e. $N_1 < \binom{\pi(y)+u}{u}^{\varepsilon}$ for $x > x_1(\varepsilon)$.

Now N_2 is less than the number of solutions of

$$\sum_{y^{1-\varepsilon}<p\leqslant y} \alpha_p \leqslant \frac{\log x}{(1-\varepsilon)\log y}$$

in nonnegative integers α_p and this number does not exceed

$$\binom{\pi(y)+u'}{u'} \text{ where } u' = \left[\frac{\log x}{(1-\varepsilon)\log y}\right].$$

We now use the fact that if a, b and $b(1+\varepsilon)$ are positive integers then

$$\binom{a+b}{a}^{1+\varepsilon} = \prod_{i=0}^{a-1}\left(1+\frac{b}{a-i}\right)^{1+\varepsilon} > \prod_{i=0}^{a-1}\left(1+\frac{b(1+\varepsilon)}{a-i}\right) = \binom{a+b(1+\varepsilon)}{a}.$$

It follows that $N_2 < \binom{\pi(y)+u}{u}^{1+O(\varepsilon)}$.

Combining the estimates for N_1 and N_2 we find

$$\Psi(x, y) < \binom{\pi(y)+u}{u}^{1+O(\varepsilon)}$$

proving (2) for $y < (\log x)^{1+\varepsilon}$.

c. For $y > (\log x)^{n(\varepsilon)}$ where for instance $n(\varepsilon) = 2/\varepsilon$ the right-hand side of (2) is trivial because

$$\binom{\pi(y)+u}{u} > \left(\frac{\pi(y)}{u}\right)^u \text{ which implies} \binom{\pi(y)+u}{u}^{1+\varepsilon} > x.$$

d. The case $(\log x)^{1+\varepsilon} < y < (\log x)^{2/\varepsilon}$ will be treated by writing $y = (\log x)^{\alpha}$ and proving

$$\Psi(x, y) = x^{1-1/\alpha+o(1)} \tag{3}$$

which implies (2).

We first remark that $\binom{\pi(y)+u}{u} = x^{1-1/\alpha+o(1)}$ and the same holds for $\binom{\pi(y)}{u}$. So it remains to show that

$$\Psi(x, y) < x^{1-1/\alpha+o(1)}.$$

To do this we split the integers counted by $\Psi(x,y)$ into two classes. First those with at least u distinct prime factors. Their number is less than

$$\frac{x}{u!}\left(\sum_{p<x}\frac{1}{p}\right)^u < x\left(\frac{2e\log\log x}{u}\right)^u = x^{1-1/\alpha+o(1)}.$$

The number of integers $\leqslant x$ with less than u distinct prime factors $\leqslant y$ is less than

$$\binom{\pi(y)}{u}\binom{\frac{\log x}{\log 2}+u}{u}$$

because there are $\binom{\pi(y)}{u}$ different u-tuples of primes $\leqslant y$ and for each of these the sum of the exponents is less than $\dfrac{\log x}{\log 2}$. We have

$$\binom{\frac{\log x}{\log 2}+u}{u} = x^{o(1)}\left(\text{this follows from}\binom{n}{k} < \frac{n^k}{k!} < \left(\frac{ne}{k}\right)^k\right).$$

So the number of integers in the second class is also $x^{1-1/\alpha+o(1)}$.

This completes the proof of Theorem 1.

REFERENCES

[1] N.G. de Bruijn, On the number of positive integers $\leqslant x$ and free of prime factors $> y$, *Proc. Kon. Ned. Akad. v. Wetensch.*, *54*, 50-59 (1951).
[2] N.G. de Bruijn, On the number of positive integers $\leqslant x$ and free of prime factors $> y$, II, *Proc. Kon. Ned. Akad. v. Wetensch.*, *69*, 335-348 (1966).

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