# PARTITION RELATIONS FOR CARDINAL NUMBERS

By

P. ERDŐS (Budapest), member of the Academy, A. HAJNAL (Budapest)

and R. RADO (Reading, England)

## 1. INTRODUCTION

In this paper our main object is the study of relations between cardinal numbers which are written in the form

$$a \to (b_0, b_1, \ldots)^r \quad \text{or} \quad a \to (b)_c^{<\aleph_0} \quad \text{or}$$

$$\binom{a}{b} \to \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix}. \tag{1}$$

Such relations were introduced in [3] and [1]. They are called I-relations, II-relations and III-relations respectively, and they will be defined in 3. 1, 3. 2 and 3. 3. In sections 18 and 19 we shall introduce certain generalizations of these relations. Our whole theory can be considered as having arisen out of the classical theorem of RAMSEY [18]. The most natural and direct generalization of the question settled by Ramsey's theorem is the problem of deciding whether for a given positive integer $r$ and given cardinals $a, b_0, b_1, \ldots$ the I-relation $a \to (b_0, b_1, \ldots)^r$ is true or false. Although the I-relation was only introduced in 1952 several papers had already appeared on such relations between 1933 and 1952. As far as we know the first of these papers is due to W. SIERPIŃSKI [19] who proved that $2^{\aleph_0} \nrightarrow (\aleph_1, \aleph_1)^2$. Two of the present authors have published several notes on such relations. All the results found by us before the present paper are contained in [1] where there will also be found references to previous papers by other writers except to those of G. KUREPA of which we had no knowledge at that time. Independently of P. ERDŐS and R. RADO and at about the same time KUREPA found several I-relations the most important of which, deduced under the assumption of the generalized continuum hypothesis, can be stated as follows:

$$\aleph_{\alpha+2} \to (\aleph_{\alpha+1}, \aleph_{\alpha+2})^2,$$

$$\aleph_{\alpha+r} \to (\aleph_{\alpha+1})^r_{\aleph_\alpha} \quad \text{for} \quad r \geqq 1,\ \alpha \geqq 0.$$

Furthermore, KUREPA proved independently but somewhat later than SIERPIŃSKI that

$$2^{\aleph_\alpha} \nrightarrow (\aleph_{\alpha+1}, \aleph_{\alpha+1})^2.$$

For these results see [20].

In this paper our first major aim is to discuss as completely as possible the relation I. Our most general results in this direction are stated in Theorems I and II, and the remaining open questions are stated in Problems 1 and 2. If we disregard cases when among the given cardinals there occur inaccessible numbers greater than $\aleph_o$, and if we assume the General Continuum Hypothesis, then our results are complete for $r=2$, and they are also complete for $r \geqq 3$ provided we restrict ourselves to finitely many numbers $b_\nu$. It seems that there are only two essentially different methods for obtaining positive partition formulae: those given in Lemma 1 (the ramification method) and those given in Lemma 3 (the canonization method). The idea underlying Lemma 1 has of course been known for a long time but the ramification method in its present form gives in some cases sharper and more general results than have been known before. We shall not always quote in the text the place where the previous weaker results appeared. Perhaps our most important new method is that given in Lemma 3 of which there are bound to be many further applications.

In Lemma 5 we state powerful new methods for constructing examples of particular partitions which yield negative I-relations. The simplest special case of these general constructions leads to a proof of

$$2^{2^{\aleph_0}} \nrightarrow (\aleph_1, \aleph_1)^3$$

which was the decisive step towards proving our general theorems. Throughout most of our work we assume the General Continuum Hypothesis. In some cases it will be obvious to the reader how the theorems should be formulated when this hypothesis is not made but in other cases unmanageable complications would arise if we were to abandon the continuum hypothesis.

In sections 17 we prove a negative result on II-relations which is sharper than that obtained in [4]. This result is perhaps not best possible but it is interesting on account of its connection with the abstract measure problem for inaccessible cardinals described in section 8.

In sections 18—20 we consider problems which are in various ways related to our original partition problem. Here we are very far from obtaining complete results, and in some cases we are not even able to give a complete discussion of the many open questions. The proofs and constructions of counter examples are similar to those used in sections 1—16 but the results show many new and interesting features such as those present in Theorems 22 and 23.

Our second major aim is an investigation of the polarized partition relation III. Such relations were formally introduced in [3] but an earlier result of SIERPIŃSKI [22] implicitly contains the formula

$$\binom{\aleph_0}{\aleph_1} \nrightarrow \begin{pmatrix} \aleph_0 & \aleph_0 \\ \aleph_1 & \aleph_1 \end{pmatrix}.$$

In 3. 3 we define a very general type of polarized relation but we in fact restrict ourselves to relations of type (1). This severe restriction is probably not essential and it is justified only because even in this special case our discussion is not complete. Several important problems remain unsolved some of which, such as Problems 10 and 12, seem difficult and may require new methods. It may well be that new phenomena arise with more general polarized partitions but we have not investigated these.

# PART I

## 2. NOTATION AND DEFINITIONS

Unless the contrary is stated, Roman capitals denote sets, small Greek letters as well as $k, l, m, n$ denote ordinal numbers (ordinals), and small Roman letters other than $k, l, m, n, x, y, z$ denote cardinal numbers (cardinals). The sequence of infinite cardinals is $\aleph_0, \aleph_1, \aleph_2, \ldots$, and the corresponding initial ordinals are $\omega_0, \omega_1, \omega_2, \ldots$. We always put $\omega = \omega_0$. No distinction is made between finite ordinals and the corresponding finite cardinals. We shall always assume that

$$r, s < \omega.$$

For every $\alpha$, the symbol $\alpha^*$ denotes the order type obtained from the type $\alpha$ by reversing all order relations.

Bold type letters $\mathbf{A}, \mathbf{F}, \ldots$ are used to denote sets of sets or families of sets. As usual, these last two notions differ in that in a family the multiplicity of occurrence of any particular set is essential whereas this is not so in a set of sets. If $A$ is a set of ordinals and $A \neq \emptyset$ then $\min A$ denotes the least element of $A$.

Set inclusion in the wide sense, union and intersection are denoted by

$$A \subset B, \quad A + B, \quad AB$$

respectively, and the symbols

$$\Sigma(v \in N) A_v, \quad \Pi(v \in N) A_v$$

denote union and intersection of any family of sets. Without fear of creating confusion we use the same notation for sums and products of cardinals as for union and intersection of sets. The symbol

$$\Sigma'(v \in N) A_v$$

denotes the set $\Sigma(v \in N) A_v$ and, at the same time, expresses the fact that

$$A_\mu A_v = \emptyset \quad \text{for} \quad \mu, v \in N;\ \mu \neq v.$$

We shall make use of the *obliteration operator* ^ whose effect on a well-ordered sequence of elements is to remove that element above which it is placed. Thus

$$x_0, x_1, \ldots, \hat{x}_n \tag{1}$$

denotes the sequence of type $n$ whose $v$th term is $x_v$. We use this notation even in cases where an element $x_n$ has not been defined at all. The symbol

$$a_0 + a_1 + \ldots + \hat{a}_n \tag{2}$$

denotes the sum of all cardinals $a_v$ for wich $v < n$, and other similar uses of the operator ^ will be easily interpreted. We shall nearly always omit the customary three dots ... to indicate a continuation of the symbols as indicated in front of them, so that (1) and (2) are more simply written as $x_0,, \hat{x}_n$ and $a_0 + + \hat{a}_n$ respectively.

If $\Phi(x)$ is a proposition involving the general element $x$ of $A$ then for $B \subset A$ the symbol

$$B\{x: \Phi(x)\}$$

denotes the set of all $x \in B$ such that $\Phi(x)$ is true. In the formula for $\Phi(x)$ we may use the logical signs $\wedge$ for "and", $\vee$ for "or", $\supset$ for "implies", $\exists x$ for "there is $x$", and $\forall x$ for "for all $x$".

The cartesian product of sets $A_0,, \hat{A}_n$ is

$$A_0 \times \times \hat{A}_n = \{(x,, \hat{x}_n): v < n \supset x_v \in A_v\} = \{f: v < n \supset f(v) \in A_v\}.$$

If $A_0,, \hat{A_n}$ are pairwise disjoint sets each of which is ordered then the symbol

$$A_0 + + \hat{A}_n(tp)$$

denotes the set $A_0 + + \hat{A}_n = A$ and, at the same time, indicates that $A$ is ordered by the rule that for $x, y \in A$ we have $x < y$ if and only if

either (i) there is $\nu < n$ with $x, y \in A_\nu$, and $x < y$ in $A_\nu$,
or (ii) there are $\mu < \nu < n$ with $x \in A_\mu$ and $y \in A_\nu$.

If $x_0,, \hat{x}_n \in A$, and if a binary relation $x \prec y$ is defined on $A$ then each of the symbols

$$\{x_0,, \hat{x}_n\}_\prec, \quad \{x_\nu : \nu < n\}_\prec$$

denotes the set $\{x_0,, \hat{x}_n\}$ and, in addition, expresses the fact that $x_\alpha \prec x_\beta$ for $\alpha < \beta < n$.

The cardinal of $A$ is $|A|$, and if $A$ is ordered then the order type of $A$ is denoted by tp $A$. If the same set $A$ is ordered by several order relations $R_\nu$ then tp $(A, R_\nu)$ denotes the order type of $A$ under $R_\nu$. If tp $A = \alpha$ then we put $|\alpha| = |A|$. For any $A$ and $B$ we put

$$A - B = A\{x : x \notin B\}.$$

If $\alpha \leqq \beta$ then we put

$$[\alpha, \beta) = \{\nu : \alpha \leqq \nu < \beta\}.$$

Every $\alpha$ has a unique representation $\alpha = \omega\beta + r$. We put

$$\alpha \dot{-} s = \begin{cases} \omega\beta + (r-s) & \text{if } r \geqq s, \\ \omega\beta & \text{if } r < s. \end{cases}$$

Ordinals of the form $\gamma + 1$ are said to be of the *first kind*, all others of the *second kind*. Ordinals of the form $\omega\beta$, where $\beta \geqq 1$, are *limit numbers*, all others *isolated*. If $a = \aleph_\alpha$ then we put

$$a^+ = \aleph_{\alpha+1}; \; a^- = \aleph_{\alpha \dot{-} 1}.$$

If $a < \omega$ then $a^+ = a + 1$ and $a^- = a \dot{-} 1$.

If $a \geqq \aleph_0$ then $a'$ denotes throughout this paper the least cardinal $b$ such that

$$a = a_0 + + \hat{a}_n$$

for some suitable $n$, $a_\nu$ such that $|n| = b$ and $a_0,, \hat{a}_n < a$. If $a = \aleph_\alpha$ then $a' = \aleph'_\alpha = \aleph_{\mathrm{cf}(\alpha)}$ where $\mathrm{cf}(\alpha)$ is the *cofinality function* introduced by TARSKI. Whenever the notation $a'$ is used it is tacitly assumed that $a \geqq \aleph_0$.

The cardinal $a$ is called *weakly inaccessible* if

$$a = a' = a^-,$$

and it is *strongly inaccessible* if

$$a = a', \text{ and } b < a \text{ implies } 2^b < a.$$

These two notions will mainly be used when the General Continuum Hypothesis

(3) $$2^{\aleph_\nu} = \aleph_{\nu+1} \text{ for all } \nu$$

is assumed in which case strongly and weakly inaccessible cardinals form the same class and will simply be referred to as inaccessible. The number $\aleph_0$ is strongly inaccessible, and no other inaccessible number is known.

We put

$$[S]^a = \{X: X \subset S \wedge |X| = a\},$$
$$[S]^{<a} = \Sigma(b < a)[S]^b,$$
$$[S]^{\geqq a} = \Sigma(a \leqq b \leqq |S|)[S]^b,$$
$$\mathbf{P}(S) = \{X: X \subset S\}.$$

The following notation will be very useful. If $I \subset \mathbf{P}(S)$ then

$$[I]_r = \{|X|: X \subset S \wedge [X]^r \subset I\}.$$

Clearly, $[I]_r$ depends on $I$ and $r$ only and not on $S$. We shall use the notation

$$[S_0,, \hat{S}_n]^{a_0,, \hat{a}_n} = \mathbf{P}(S_0 + + \hat{S}_n)\{X: (\forall v)(v < n \supset |XS_v| = a_v)\}.$$

For any $a$ we denote by $\omega(a)$ the *initial ordinal* whose cardinal is $a$, i. e. we put

$$\omega(a) = \min\left(|n| = a\right)n.$$

The statement of propositions in whose proof the General Continuum Hypothesis (3) is assumed, will be prefixed by the symbol $(\ast)$. The same applies to problems which are of interest only when (3) is assumed.

## 3. THE PARTITION RELATIONS I, II, III

**3. 1. The ordinary partition relation (relation I).** A *partition* of $A$ is any sequence $(A_0,, \hat{A}_n)$ such that $A = A_0 + + \hat{A}_n$. We shall also use the version $A = \Sigma(v \in N)A_v$. The $A_v$ are the classes of the partition. The partition is called *disjoint* if $A = \Sigma' A_v$. An *r-partition* of $A$ is, by definition, a partition of $[A]^r$.

The *power* of a partition $\Delta = (A_0,, \hat{A}_n)$ of $A$ is the cardinal

$$|\Delta| = |\{v: v < n \wedge A_v \neq \emptyset\}|.$$

Thus a 1-partition of $A$ is a sequence $(I_0,, \hat{I}_n)$ such that $[A]^1 = \Sigma I_v$. Then $A = \Sigma J_v$ where $J_v = A\{x: \{x\} \in I_v\}$ for $v < n$. When there is no danger of a misunderstanding we shall identify the partition $(I_0,, \hat{I}_n)$ of $[A]^1$ with the partition $(J_0,, \hat{J}_n)$ of $A$.

If $\Delta$ is a partition of $A$, and if $B \subset A$, then the statement

$$|\Delta| \leqq 1 \text{ in } B$$

means, by definition, that $B$ lies in some class of $\Delta$, and "$|\Delta| > 1$ in $B$" is the negation of this statement.

Trivial results in [1] show that there is no loss of generality in assuming that in every partition the classes are indexed by a set of ordinals of the form $[0, n)$. Occasionally we shall employ a notation in which the classes of a partition are indexed in some other way.

We define the *ordinary partition relation (I-relation)*

$$a \to (b_0,, \hat{b}_n)^r \quad (\text{or: } a \to (b_\nu)^r_{\nu<n}) \tag{1}$$

as follows. The relation (1) expresses the fact that whenever

$$|S| = a; \quad [S]^r = I_0 + + \hat{I}_n \tag{2}$$

then there are a number $\nu < n$ and a set $X \subset S$ such that

$$|X| = b_\nu; \quad [X]^r \subset I_\nu.$$

More simply, this means that (2) implies that

$$b_\nu \in [I_\nu]_r \text{ for some } \nu < n.$$

The logical negation of (1), and similarly for all partition relations to be introduced later, is written as

$$a \nrightarrow (b_0,, \hat{b}_n)^r \quad (\text{or: } a \nrightarrow (b_\nu)^r_{\nu<n}). \tag{3}$$

The relations (1) and (3) are only of interest if

$$r \geqq 1; \quad n \geqq 2; \quad r < b_0,, \hat{b}_n \leqq a.$$

Here is a systematic discussion of the degenerate cases of (1) and (3). Define four disjoint subsets of $[0, n)$ whose union is $[0, n)$:

$$N_{++} = \{\nu : r \leqq b_\nu \leqq a\},$$
$$N_{+-} = \{\nu : r \leqq b_\nu \nleqq a\},$$
$$N_{-+} = \{\nu : r \nleqq b_\nu \leqq a\},$$
$$N_{--} = \{\nu : r \nleqq b_\nu \nleqq a\}.$$

*Case* 1. $N_{-+} \neq \emptyset$. Then (1) holds. For if (2) holds then we choose $\nu \in N_{-+}$. Then $r \nleqq b_\nu \leqq a$, and we can choose $X \in [S]^{b_\nu}$. Then $[X]^r = \emptyset \subset I_\nu$ and (1) follows.

*Case* 2. $N_{-+} = \emptyset$.

*Case* 2*a*. $N_{+-} + N_{--} \neq \emptyset$. Then (3) holds. For if $|S| = a$ then there is a partition (2) such that $I_\nu = \emptyset$ for $\nu \notin N_{+-} + N_{--}$. Now suppose there are a number $\nu < n$ and a set $X \in [S]^{b_\nu}$ such that $[X]^r \subset I_\nu$. Then $b_\nu \leqq a$; $\nu \in N_{++} + N_{-+} = N_{++}$; $\emptyset \neq [X]^r \subset I_\nu$; $\nu \in N_{+-} + N_{--}$ which is a contradiction. Hence (1) is false.

*Case* 2*b*. $N_{+-} + N_{--} = \emptyset$. Then $r_\nu \leqq b_\nu \leqq a$ for $\nu < n$, and it follows that this is the only case worth studying. Hence when discussing (1) or (3) we may if we wish assume $r \leqq b_0,, \hat{b}_n \leqq a$. We mention that even this last case can be further reduced by omitting those $b_\nu$ for which $b_\nu = r$. For we have the following simple proposition: If $m \leqq n$ and $b_0 = = \hat{b}_m = r$, then the relations $a \to (b_0,, \hat{b}_n)^r$ and $a \to (b_m,, \hat{b}_n)^r$ are equivalent. If $b_\nu = b$ for $\nu < n$ we write (1) also in the form

$$a \to (b)^r_c \text{ or } a \to (b)^r_n \tag{4}$$

where $c=|n|$. This notation is justified since if all $b_\nu$ are equal to each other then the truth of (1) does not depend on the ordinal number $n$ but only on the value of $|n|$. More generally, the relation

$$a \to ((b_0)_{c_0}, (b_1)_{c_1})^r \tag{5}$$

has the following meaning. Let $|n_0|=c_0$; $|n_1|=c_1$;

$$d_\nu = b_0 \text{ for } \nu < n_0$$
$$b_1 \text{ for } n_0 \leqq \nu < n_0+n_1.$$

Then

$$a \to (d_\nu)^r_{\nu<n_0+n_1}.$$

We may, of course, have more than two groups of equal entries in (5). Similar remarks apply to (3).

REMARK. Concerning a further extension of the relation (1) to cover partitions of $[S]^e$ for $e \geqq \aleph_0$ it is proved in [3], p. 434, that every such analogue of (1) is false.

**3. 2. Partition relations with multiple exponents (II-relations).** Let $r_0,, \hat{r}_l, a, b, c$ be given cardinals. Then we say that the relation

$$a \to (b)_c^{r_0,, \hat{r}_l} \tag{6}$$

holds if, whenever $|S|=a$ and

$$[S]^r = \Sigma(\nu<\omega(c))I(r,\nu) \quad (\text{partition } \Delta_r)$$

for all $r$ then there always exists a set $X \in [S]^b$ such that

$$|\Delta_{r_\lambda}| \leqq 1 \text{ in } [X]^{r_\lambda} \text{ for every } \lambda<l.$$

We call (6) a *II-relation* or, more explicitly, a *II* $(r_0,, \hat{r}_l)$*-relation*. The II $(r_0)$-relations coincide with the ordinary I-relations. When studying (6) we may always suppose if we wish that $l \leqq \omega$ and $r_0 < \cdots < \hat{r}_l$. If $\{s_0,, \hat{s}_m\} \subset \{r_0,, \hat{r}_l\}$ then (6) implies $a \to (b)_c^{s_0,, \hat{s}_m}$. In 5. 3 we shall see that if $b \geqq \aleph_0$ and $\sup(\mu<m)s_\mu=\omega$ then the converse implication holds so that for $b \geqq \aleph_0$ all relations (6) with fixed $a, b, c$ and arbitrary $r_0,, \hat{r}_l$ with $\sup(\lambda<l)r_\lambda=\omega$ are equivalent. The relation (6) is only of interest if $c \geqq 2$ and $a \geqq b \geqq r_0,, \hat{r}_l$. In particular, (6) is false if $a<b$; $c>0$; $l>0$ and (6) is true if $a \geqq b$ and $b<r_\lambda$ for some $\lambda<l$. The logical negation of (6) is written as

$$a \nrightarrow (b)_c^{r_0,, \hat{r}_l}.$$

Another type of II-relation, which we shall discuss in some more detail is written as

$$a \to (b)_c^{<\aleph_0}$$

and expresses the following condition. Let $|S|=a$;

$$[S]^r = \Sigma(\nu<\omega(c))I(r,\nu) \quad (\text{partition } \Delta_r)$$

for every $r$. Then there always are a set $X \in [S]^b$ and a number $r_0$ such that

$$|\Delta_r| \leqq 1 \text{ in } [X]^r \text{ for } r \geqq r_0.$$

**3. 3. Polarized partition relations (III-relations).** Let $s \geqq 2$; $r_0,, r_{s-1} \geqq 1$, and let $a_\sigma$ and $b_{\sigma\nu}$ be cardinals, for $\sigma < s$; $\nu < n$. Then the relation

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{s-1} \end{pmatrix} \to \begin{pmatrix} b_{0\nu} \\ b_{1\nu} \\ \vdots \\ b_{s-1,\nu} \end{pmatrix}^{r_0, r_1,, r_{s-1}}_{\nu < n} \tag{7}$$

is said to hold if the following condition is satisfied. Let $|S_\sigma| = a_\sigma$ for $\sigma < s$, and $S_\sigma S_\tau = \varnothing$ for $\sigma < \tau < s$. Let

$$[S_0,, S_{s-1}]^{r_0,, r_{s-1}} = I_0 + + \hat{I}_n.$$

Then there are always sets $X_\sigma \subset S_\sigma$ for $\sigma < s$ and a number $\nu < n$ such that

$$|X_\sigma| = b_{\sigma\nu} \text{ for } \sigma < s, \text{ and } [X_0,, X_{s-1}]^{r_0,, r_{s-1}} \subset I_\nu.$$

The relation (7) is only of interest if

$$r_\sigma \leqq b_{\sigma\nu} \leqq a_\nu \text{ for } \sigma < s; \nu < n,$$

and we shall frequently when considering (7), assume these conditions to hold. In the present paper we shall mainly consider the very special case

$$s = 2; \quad r_0 = r_1 = 1; \quad n = 2$$

which corresponds to the partitioning of even graphs into two subgraphs. We shall however also investigate a generalization of this special case in a different direction Let $a, b, c_\nu, d_\nu, e_\nu, f_\nu$ be cardinals, for $\nu < n$. Then the "*relation with alternatives*"

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} c_\nu \vee d_\nu \\ e_\nu \vee f_\nu \end{pmatrix}^{1,1}_{\nu < n} \tag{8}$$

is said to hold whenever the following statement is true. Let $AB = \varnothing$; $|A| = a$; $|B| = b$; $[A, B]^{1,1} = I_0 + + \hat{I}_n$. Then there are always sets $X \subset A$; $Y \subset B$ and a number $\nu < n$ such that $[X, Y]^{1,1} \subset I_\nu$ and either (i) $|X| = c_\nu$; $|Y| = e_\nu$ or (ii) $|X| = d_\nu$; $|Y| = f_\nu$. It is worth noting here that the alternatives for the sets $X$ and $Y$ are not independent so that for instance the possibility $|X| = c_\nu$; $|Y| = f_\nu$ is not permitted. The relation (8) is only of interest if

$$1 \leqq c_\nu, d_\nu \leqq a; \ 1 \leqq e_\nu, f_\nu \leqq b \text{ for } \nu < n,$$

and we shall frequently assume these inequalities.

REMARKS. 1. Theorems proved in sections 24 and 26 will show that alternatives are essential for the investigation of even graphs. Clearly there is no need for introducing more than two alternatives for each class.

2. It would of course be possible to define more general relations with alternatives but then it would not be easy to determine the minimum number of alternatives required for a complete discussion.

3. If $c_\nu = d_\nu$ and $e_\nu = f_\nu$ for $\nu < n$ then (8) is equivalent to

$$\binom{a}{b} \to \binom{c_\nu}{e_\nu}^{1,1}_{\nu<n}.$$

4. If in (8) we have, for $\nu < n$, $c_\nu = c$; $d_\nu = d$; $e_\nu = e$; $f_\nu = f$, then we also write (8) in the form

$$\binom{a}{b} \to \binom{c \vee d}{e \vee f}^{1,1}_{|n|}.$$

5. On reflection it will become obvious that in studying any of the relations I, II, III there will be no loss of generality in assuming that all partitions involved in the arguments are disjoint.

6. The left hand sides of all our partition relations will nearly always be assumed to contain infinite cardinals only.

## 4. PRELIMINARIES ON PARTITION RELATIONS I, II, III

**4. 1. Invariance under permutations of the arguments.** Let $|m| = |n|$, and let $\nu \to \pi(\nu)$ be a one-one map of $[0, m)$ onto $[0, n)$. Then the relations $a \to (b_\nu)^r_{\nu<n}$ and $a \to (b_{\pi(\nu)})^r_{\nu<m}$ are equivalent, and similarly the relations

$$\binom{a}{b} \to \binom{c_\nu \vee d_\nu}{e_\nu \vee f_\nu}^{1,1}_{\nu<n} \quad \text{and} \quad \binom{a}{b} \to \binom{c_{\pi(\nu)} \vee d_{\pi(\nu)}}{e_{\pi(\nu)} \vee f_{\pi(\nu)}}^{1,1}_{\nu<m}$$

are equivalent. For the proof see [1], Theorem 17.

**4. 2. Monotonicity properties.** We say that one of our relations is *increasing (decreasing)* in one of its arguments if whenever the relation holds it continues to hold when this particular argument is increased (decreased) while the remaining arguments remain constant. It is easy to see that every one of our partition relations I, II, III is increasing in every cardinal variable on the left hand side and decreasing in every cardinal variable on the right hand side, with the exception of the "exponent" $r$ and a cardinal indicating the number of classes. For the proof see [1], Theorem 12.

**4. 3. Substitution rules.** (i) Let $a \to (b_0,, \hat{b}_n)^r$; $n \geqq 1$; $b_0 \to (c_0,, \hat{c}_m)^r$. Then $a \to (c_0,, \hat{c}_m, b_1,, \hat{b}_n)^r$. For the proof see [1], Theorem 16.

(ii) Substitution in a relation with two alternatives may lead to a relation with more than two alternatives. Thus it is easy to see that the three relations (in which on the right only the first row is written out)

$$\binom{a}{b} \to \binom{a_0 \vee a_1,\ a_2 \vee a_3}{b_0 \vee b_1,\ \ldots\ldots}; \quad \binom{a_0}{b_0} \to \binom{a_4 \vee a_5, a_6 \vee a_7}{\ldots\ldots\ldots\ldots}; \quad \binom{a_1}{b_1} \to \binom{a_8 \vee a_9,\ a_{10} \vee a_{11}}{\ldots\ldots\ldots\ldots}$$

imply the new relation

$$\binom{a}{b} \to \binom{a_4 \vee a_5 \vee a_8 \vee a_9,\ a_6 \vee a_7 \vee a_{10} \vee a_{11},\ a_2 \vee a_3}{\ldots\ldots\ldots\ldots\quad \ldots\ldots\ldots\ldots\quad \ldots\ldots}$$

and that in general nothing stronger than this can be asserted.

## 5. FURTHER REMARKS ON PARTITIONS

**5. 1. Product of partitions.** Let $k \geqq 1$, and let, for each $\varkappa < k$, $\Delta_\varkappa$ be a partition

$$S = \Sigma(\nu < n_\varkappa) A_\varkappa(\nu)$$

of $S$. Then the *product*

$$\Delta_0 \ldots \hat{\Delta}_k = \Pi(\varkappa < k)\Delta_\varkappa$$

denotes the partition

$$S = \Sigma(\varkappa < k \supset \nu_\varkappa < n_\varkappa) B(\nu_0,, \nu_k)$$

where

$$B(\nu_0,, \hat{\nu}_k) = \Pi(\varkappa < k) A_\varkappa(\nu_\varkappa).$$

It follows that

$$|\Pi\Delta_\varkappa| \leqq \Pi|\Delta_\varkappa|.$$

Also, if every $\Delta_\varkappa$ is disjoint then $\Pi\Delta_\varkappa$ is disjoint. By convention, an empty product of partitions of $S$ is the partition which has only one class.

**5. 2. Induced partitions.** Let $\Delta$ be a disjoint partition of $S$. Then the relation

$$x_0 \equiv x_1(\cdot\Delta)$$

denotes, by definition, the fact that $x_0, x_1 \in S$ and that $x_0$ and $x_1$ lie in the same class of $\Delta$. Let

$$f: y \to f(y)$$

be a map from a set $T$ into $S$. Then a partition $\Delta'$ of $T$ is defined by the rule

$$\Delta'(y) = \Delta(f(y)) \text{ for } y \in T$$

which, by definition, means that

$$y_0 \equiv y_1(\cdot\Delta')$$

if and only if

$$y_0, y_1 \in T \text{ and } f(y_0) \equiv f(y_1)(\cdot\Delta).$$

We call $\Delta'$ the partition of $T$ *induced* by $\Delta$ and $f$. Clearly, $|\Delta'| \leqq |\Delta|$. Frequently the multiplication of a number of induced partitions is the effective tool for proving results on partitions.

We now prove the assertion made in 3. 2 about the equivalence of various II $(r_0,, \hat{r}_l)$-relations.

**5. 3.** *Let*

$$a \to (b)_c^{r_0,, \hat{r}_l};\ b \geqq \aleph_0;\ \sup(\lambda < l) r_\lambda = \omega.$$

*Then*

$$a \to (b)_c^{0,, \hat{\omega}}.$$

Proof. Let $\omega(a) = n$; $S = [0, n)$; $|N| = c$, and let

$$[S]^r = \Sigma(\nu \in N) I(r, \nu) \text{ (partition } \Delta_r)$$

for all $r$.

By the remark made in 3. 2 we may assume that $r_0 < \, < \hat{r}_l$. Define the $r$-partition $\Delta'_r$ of $S$ by

$$\Delta'_r(\{x_0,, \hat{x}_{r_<}\}) = \Delta_\lambda(\{x_0,, \hat{x}_\lambda\}) \quad \text{for} \quad r_\lambda \leqq r < r_{\lambda+1}, \quad \lambda < \omega.$$

Then obviously $|\Delta'_r|=c$. Hence, by hypothesis, there is $X\subseteq S$ such that $\mathrm{tp}X=\omega(b)$ and

$$|\Delta'_{r_\lambda}|\leqq 1 \text{ in } [X]^{r_\lambda} \text{ for } \lambda<l.$$

Now let $s<\omega$; $\{x_0,, \hat{x}_\lambda\}_<, \{y_0,, \hat{y}_\lambda\}_< \subset X$. Then $r_\lambda\geqq\lambda$ and we can choose $x_\lambda,, \hat{x}_{r_\lambda}$, $y_\lambda,, \hat{y}_{r_\lambda}\in X$ such that $x_0<\dots<\hat{x}_{r_\lambda}$ and $y_0<\dots<\hat{y}_{r_\lambda}$. Then

$$\{x_0,, x_{r_\lambda}\} \equiv \{y_0,, y_{r_\lambda}\}(\cdot\Delta'_{r_\lambda})$$

and hence, by definition of $\Delta'_r$,

$$\{x_0,, \hat{x}_\lambda) \equiv \{y_0,, \hat{y}_\lambda\}(\cdot\Delta_\lambda).$$

Hence $|\Delta_\lambda|\leqq 1$ in $[X]^\lambda$, and the assertion follows.

## 6. THE RAMIFICATION LEMMA

We shall now describe a mode of reasoning which constitutes the core of many arguments about partitions. It has been used, in one form or another, in a number of papers already. In the rather more general form stated below it is a very powerful tool for obtaining partition relations.

LEMMA 1. *Let $\varrho=\varrho\dotplus 1>0$. Let $S(v_0,, v_\sigma)$ and $F(v_0,, \hat{v}_\sigma)$ be sets and $n(v_0,, \hat{v}_\sigma)$ be ordinals defined for $\sigma\leqq\varrho$. Put*

$$N=\{(v_0,, \hat{v}_\sigma)\colon \sigma\leqq\varrho\wedge(\tau<\sigma\supset v_\tau<n(v_0,, \hat{v}_\tau))\};\ S'(v_0,, \hat{v}_\sigma)=S\Pi(\tau<\sigma)S(v_0,, v_\tau)$$

*for $\sigma\leqq\varrho$.*

*Suppose that*

$$S'(v_0,, \hat{v}_\sigma)=F(v_0,, \hat{v}_\sigma)+'\Sigma(v_\sigma<n(v_0,, \hat{v}_\sigma))S(v_0,, v_\sigma)$$

*for $\sigma<\varrho$; $(v_0,, \hat{v}_\sigma)\in N$. Then we have:*

(i) $$F(v_0,, \hat{v}_\tau)F(v_0,, \hat{v}_\sigma)=\emptyset \text{ for } \tau<\sigma<\varrho \text{ and } (v_0,, \hat{v}_\sigma)\in N.$$

(ii) $$S=\Sigma(\sigma<\varrho\wedge(v_0,, \hat{v}_\sigma)\in N)F(v_0,, \hat{v}_\sigma)+\Sigma((v_0,, \hat{v}_\varrho)\in N)S'(v_0,, \hat{v}_\varrho).$$

(iii) *Let $|S|\geqq a\geqq\aleph_0$ and $|\varrho|<a'$; $|F(v_0,, \hat{v}_\sigma)|<a$ for $(v_0,, \hat{v}_\sigma)\in N$.*

*Suppose that there are cardinals $c_\sigma$ such that $c_\sigma^{|\sigma|}<a'$ $\sigma<\varrho$, and $|n(v_0,, \hat{v}_\tau)|\leqq c_\sigma$ whenever $\tau<\sigma<\varrho$ and $(v_0,, \hat{v}_\sigma)\in N$. Then*

(1) *there is $(v_0,, \hat{v}_\varrho)\in N$ with $S'(v_0,, \hat{v}_\varrho)\neq\emptyset$.*

(iv) *Let $|S|\geqq b^+>\aleph_0$; $\varrho\leqq\omega(b')$; $|n(v_0,, \hat{v}_\sigma)|\leqq b$ and*

$$|F(v_0,, \hat{v}_\sigma)|\leqq b \text{ for } \sigma<\varrho \text{ and } (v_0,, \hat{v}_\sigma)\in N. \text{ Let } 2^c\leqq b \text{ for } c<b.$$

*Then* (1) *holds.*

(v) *Let $|S|\geqq a$; $|\varrho|<a$; $|n(v_0,, \hat{v}_\sigma)|<a$ and $|F(v_0,, \hat{v}_\sigma)|<a$ for $\sigma<\varrho$ and $(v_0,, v_\sigma)\in N$. Suppose that $a$ is strongly inaccessible. Then* (1) *holds.*

PROOF OF (i). $F(v_0,, \hat{v}_\tau)F(v_0,, \hat{v}_\sigma)\subset F(v_0,, \hat{v}_\tau)S'(v_0,, \hat{v}_\sigma)\subset F(v_0,, \hat{v}_\tau)S(v_0,, v_\tau)=\emptyset$.

PROOF OF (ii). Put $\Sigma(\sigma<\varrho\wedge(v_0,, \hat{v}_\sigma)\in N)F(v_0,, \hat{v}_\sigma) = G$; $S-G = T$. For $\sigma<\varrho$ and $(v_0,, \hat{v}_\sigma)\in N$ put $S(v_0,, v_\sigma)-G = T(v_0,, v_\sigma)$. Then $T\Pi(\tau<\sigma)T(v_0,, v_\tau) = \Sigma((v_0,, v_\sigma)\in N)T(v_0,, v_\sigma)$ for $\sigma<\varrho$; $(v_0,, \hat{v}_\sigma)\in N$. Hence we have, for $\pi<\sigma<\varrho$ and $(v_0,, v_\sigma)\in N$,

$$T(v_0,, v_\sigma)\subset\Sigma((v_0,, \hat{v}_\sigma, \lambda)\in N)T(v_0,, \hat{v}_\sigma, \lambda) = \Pi(\tau<\sigma)T(v_0,, v_\tau)\subset T(v_0,, v_\pi).$$

Now suppose that there is an element

$$x\in T-\Sigma((v_0,, \hat{v}_\varrho)\in N)\Pi(\sigma<\varrho)T(v_0,, v_\sigma).$$

Put $N^* = N\{(v_0,, v_\sigma)\colon \sigma<\varrho\wedge x\in T(v_0,, v_\sigma)\}$. Then $N^*\neq\emptyset$. Define a partial order on $N^*$ by putting $(v_0,, v_\tau)<(v_0,, v_\sigma)$ whenever $\tau<\sigma$. We want to apply Zorn's lemma to $N^*$. Let $l = l\dot{-}1>0$ and $(v_0(\lambda),, v_{\sigma_\lambda}(\lambda))<(v_0(\mu),, v_{\sigma_\mu}(\mu))$ for $\lambda<\mu<l$. Then $\sigma_0<<\hat{\sigma}_l<\varrho$; $\omega\leqq\sup(\lambda<l)\sigma_\lambda=\pi\leqq\varrho$, and there are numbers $v_0,, \hat{v}_\pi$ such that $v_\sigma(\lambda)=v_\sigma$ for $\lambda<l$ and $\sigma\leqq\sigma_\lambda$. Then $(v_0,, \hat{v}_\pi)\in N$. Also, $x\in\Pi(\varkappa<\pi)T(v_0,, v_\varkappa)$ and hence, by definition of $x$, $\pi<\varrho$. Then

$$x\in\Pi(\varkappa<\pi)T(v_0,, v_\varkappa) = \Sigma((v_0,, v_\pi)\in N)T(v_0,, v_\pi),$$

and there is $v_\pi$ such that $(v_0,, v_\pi)\in N$ and $x\in T(v_0,, v_\pi)$. Then $(v_0(\lambda),, v_{\sigma_\lambda}(\lambda))<(v_0,, v_\pi)$ for $\lambda<l$, and hence the partial order on $N^*$ is inductive. Thus Zorn's lemma applies and gives a maximal element $(v_0,, v_\alpha)$ in $N^*$. Then $\alpha<\varrho$; $\alpha+1<\varrho$; $x\in T(v_0,, v_\alpha) = \Pi(\beta<\alpha+1)T(v_0,, v_\beta) = \Sigma((v_0,, v_{\alpha+1})\in N)T(v_0,, v_{\alpha+1})$, and there is $v_{\alpha+1}$ such that $(v_0,, v_{\alpha+1})\in N$ and $x\in T(v_0,, v_{\alpha+1})$. But then $(v_0,, v_{\alpha+1})>(v_0,, v_\alpha)$ which contradicts the maximality of $(v_0,, v_\alpha)$. Hence there is no such element $x$, and therefore

$$S-G = \Sigma((v_0,, \hat{v}_\varrho)\in N)\Pi(\sigma<\varrho)(S(v_0,, v_\sigma)-G).$$

This implies (ii).

PROOF OF (iii). Let (1) be false. Then, by (ii),

$$|S|\leqq\Sigma(\sigma<\varrho\wedge(v_0,, \hat{v}_\sigma)\in N)|F(v_0,, \hat{v}_\sigma)|. \tag{2}$$

The number of terms of the sum in (2) is at most

$$\Sigma(\sigma<\varrho)c_\sigma^{|\sigma|}<a',$$

by the regularity of $a'$. Hence, using the definition of $a'$, we deduce from (2) that $|S|<a$ which is the required contradiction.

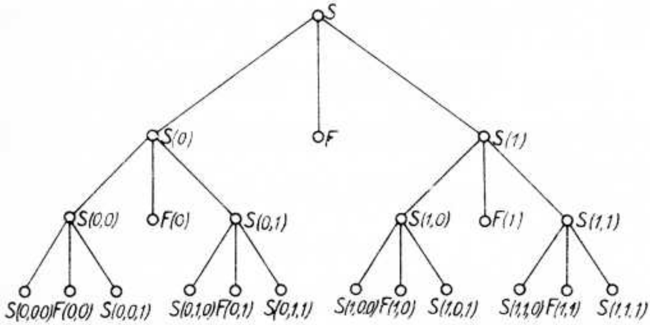
PROOF OF (iv). Put, in (iii), $a=b^+$ and $c_\sigma=b$. Then the hypothesis of (iii) holds since, if $\sigma<\varrho$, we have $|\sigma|<b'$ and hence

$$c_\sigma^{|\sigma|}=b^{|\sigma|}\leqq b<a = a'.$$

PROOF OF (v). Let (1) be false. Then, again, (2) holds. Put $N_\sigma = \{(v_0,, \check{v}_\sigma)\colon (v_0,, \hat{v}_\sigma)\in N\}$ for $\sigma<\varrho$. We prove by induction that $|N_\sigma|<a$. Let $\tau<\varrho$, and suppose that $|N_\sigma|<a$ for $\sigma<\tau$. Then we have: If $\tau=0$, then $|N_\tau|=1<a$. If $\tau = \pi+1$, then $|N_\tau| = \Sigma((v_0,, \hat{v}_\pi)\in N_\pi)|n(v_0,, \hat{v}_\pi)|<a$. Now let $\tau = \tau\dot{-}1>0$ and $(v_0,, \hat{v}_\tau)\in N_\tau$. Then $(v_0,, \hat{v}_\sigma)\in N_\sigma$ for $\sigma<\tau$ and hence $|N_\tau|\leqq\Pi(\sigma<\tau)|N_\sigma|<a$. Thus $|N_\sigma|<a$ for

$\sigma < \varrho$, and again (2) leads to the contradiction $|S| < a$. This proves (v) and completes the proof of Lemma 1.

We call the system $\mathbf{R}$ of sets $N$, $F(v_0,, \hat{v}_\sigma)$, $S(v_0, v_\sigma)$ a *ramification system* on $S$ of *length* $\varrho$. If all $n(v_0,, \hat{v}_\sigma)$ have the same value $n$ we call $n$ the *order* of the ramification system. With every ramification system there is associated a tree whose lowest branches, in the case of order 2, are shown in the following diagram:




In applications we shall always use the notation $S'(v_0,, \hat{v}_\sigma)$ as defined in the lemma, and also $N_\sigma$ as defined in the proof of (v). We shall construct $\mathbf{R}$ inductively. Assuming that $S'(v_0,, \hat{v}_\sigma)$ has already been defined for some fixed $\sigma$, $v_0,, \hat{v}_\sigma$ we shall define $n(v_0,, \hat{v}_\sigma)$, $F(v_0,, \hat{v}_\sigma)$, $S(v_0,, v_\sigma)$.

## 7. POSITIVE THEOREMS FOR I-RELATIONS IN THE CASE $a = \aleph_{\alpha+1}$; $r = 2$

THEOREM 1. *Let* $a \to (a_0,, \hat{a}_m)^2$; $b \to (b_0,, \hat{b}_k)^1$; $b \geqq \aleph_0$; $a, b^+ \leqq c = c'$, *and suppose that* $(a^- |k|)^d < c$ *for all* $d < b$. *Then*

$$c \to (a_0,, \hat{a}_m, b_0,, \hat{b}_k)^2.$$

($*$) COROLLARY 1. $\aleph_{\alpha+1} \to (\aleph_{\alpha+1}, (\aleph'_\alpha)_k)^2$ *for* $\alpha \geqq 0$; $|k| < \aleph'_\alpha$.

*Deduction of the Corollary from the Theorem.* Put $m = 1$; $a = a_0 = \aleph_{\alpha+1}$; $b = b_0 = = \hat{b}_k = \aleph'_\alpha$; $c = \aleph_{\alpha+1}$. Then the hypothesis of Theorem 1 holds, and the Corollary follows.

PROOF OF THEOREM 1. Let $|S| = c$,

$$[S]^2 = \Sigma(\mu < m) K_\mu + \Sigma(\varkappa < k) L_\varkappa.$$

We may assume that

(1) $\qquad$ if $S' \in [S]^a$, then $[S']^2 \not\subset \Sigma(\mu < m) K_\mu$.

We have to find a number $\varkappa < k$ and a set $S'' \subset S$ such that $[S'']^2 \subset L_\varkappa$ and $|S''| \geqq b_\varkappa$.

We define inductively a ramification system $\mathbf{R}$ on $S$ of length $\varrho = \omega(b)$ and order $n = \omega(a^- |k|)$. By (1), $a \geqq 2$. Let $\sigma < \varrho$, and let $S'(v_0,, \hat{v}_\sigma)$ be defined for some

$v_0,, \hat{v}_\sigma$. We write $v$ in place of $v_0,, \hat{v}_\sigma$. Choose as $F(v)$ a maximal subset of $S'(v)$ such that

$$[F(v)]^2 \subset K_0 + + \hat{K}_m.$$

Then, by (1), $|F(v)| < a$, and by the maximality of $F(v)$ we have

$$S'(v) - F(v) = \Sigma(v_\sigma < n)\, S(v, v_\sigma)$$

where, for each $v_\sigma < n$, either $S(v, v_\sigma) = \emptyset$ or, for some $y(v, v_\sigma) \in F(v)$ and $\varkappa(v, v_\sigma) < k$, we have

$$(2) \qquad S(v, v_\sigma) = \big(S'(v) - F(v)\big)\{x : \{y(v, v_\sigma), x\} \in L_{\varkappa(v, v_\sigma)}\}.$$

Here we have used the fact that the number of pairs $(y, \varkappa)$ with $y \in F(v)$ and $\varkappa < k$ does not exceed $a^-|k| = |n|$.

This defines **R**. Now Lemma 1 (iii) applies. For: $|S| = c$ is regular; $|\varrho| = b < c$. If $\sigma < \varrho$, then

$$|n|^{|\sigma|} = \big(a^-|k|\big)^{|\sigma|} < c$$

since $|\sigma| < b$. Hence by Lemma 1 (iii) there are numbers $v_0,, \hat{v}_\varrho < n$ such that $S'(v_0,, \hat{v}_\varrho) \neq \emptyset$. Then (2) holds for $\sigma < \varrho$. Put $x_\sigma = y(v_0,, v_\sigma)$ for $\sigma < \varrho$. If $\sigma < \tau < \varrho$ then

$$x_\tau = y(v_0,, v_\tau) \in F(v_0,, \hat{v}_\tau) \subset S'(v_0,, \hat{v}_\tau) \subset S(v_0,, v_\sigma);$$

$$\{x_\sigma, x_\tau\} \in L_{\varkappa(v_0,, v_\sigma)}.$$

Put $M_\varkappa = [0, \varrho)\{\sigma : \varkappa(v_0,, v_\sigma) = \varkappa\}$ for $\varkappa < k$. Then $[0, \varrho) = M_0 + + \hat{M}_k$, and by $b \to (b_\varkappa)^1_{\varkappa < k}$ there is $\varkappa < k$ such that $|M_\varkappa| \geqq b_\varkappa$. Put $S'' = \{x_\sigma : \sigma \in M_\varkappa\}$. Then $|S''| \geqq b_\varkappa$; $[S'']^2 \subset L_\varkappa$, and Theorem 1 follows.

REMARK. The method used in the proof of Theorem 1 actually yields a result of a more general type: see Theorem 39 in section 22. We have described here the proof for this special case in order to prepare the complete discussion of relation I without applying results on polarized partitions.

($*$) THEOREM 2. *If $c < a'$, then $a^+ \to (a)^2_c$.*

PROOF. Let $|S| = a^+$; $[S]^2 = I_0 + + \hat{I}_n$; $n = \omega(c)$. We define a ramification system **R** on $S$ of length $\varrho = \omega(a)$ and order $n$. Let $\sigma < \varrho$, and let $S'(v_0,, \hat{v}_\sigma)$ be defined. Again, instead of $v_0,, \hat{v}_\sigma$ we write $v$. If $S'(v) = \emptyset$ then put $F(v) = S(v, v_\sigma) = \emptyset$ for $v_\sigma < n$. Now let $S'(v) \neq \emptyset$. Choose $x(v) \in S'(v)$ and put $F(v) = \{x(v)\}$,

$$S(v, v_\sigma) = \big(S'(v) - F(v)\big)\{y : \{x(v), y\} \in I_{v_\sigma}\} \qquad \text{for} \quad v_\sigma < n.$$

This defines **R**. Lemma 1 (iii) applies. For $|S| = a^+$ is regular; $|\varrho| = a < a^+$; $|F(v)| \leqq 1 < a^+$, and if $\sigma < \varrho$ then

$$|n|^{|\sigma|} = c^{|\sigma|} < a^+.$$

Hence there are numbers $v_0,, \hat{v}_\varrho < n$ with $S'(v_0,, \hat{v}_\varrho) \neq \emptyset$. Then $F(v_0,, \hat{v}_\sigma) = \{x_\sigma\}$; $x_\sigma = x(v_0,, \hat{v}_\sigma)$ for $\sigma < \varrho$. If $\sigma < \tau < \varrho$ then, by definition of $S(v_0,, v_\sigma)$,

$$\{x_\tau\} = F(v_0,, \hat{v}_\tau) \subset S'(v_0,, \hat{v}_\tau) \subset S(v_0,, v_\sigma);\ \{x_\sigma, x_\tau\} \in I_{v_\sigma}.$$

If $M_\varkappa=[0,\varrho)\{\sigma: v_\sigma=\varkappa\}$ for $\varkappa<n$, then $[0,\varrho)=M_0++\hat{M}_n$, and since $|n|=c<a'=|\varrho|'$, there is $\varkappa<n$ such that $|M_\varkappa|=|\varrho|=a$. Put $S''=\{x_\sigma:\sigma\in M_\varkappa\}$. Then $|S''|=a$; $[S'']^2\subset I_\varkappa$, and Theorem 2 follows.

REMARK. If $a$ is regular then the conclusion of Theorem 2 follows from Corollary 1.

## 8. FURTHER POSITIVE THEOREMS FOR I-RELATIONS. REMARKS AND PROBLEMS FOR THE CASE OF INACCESSIBLE CARDINALS

**8. 1.** ($*$) LEMMA 2 *(the stepping-up lemma). Let* $r\geqq 1$; $a\geqq\aleph_0$; $a\to(b_\nu)^r_{\nu<m}$. *Then*

(1) $$a^+\to(b_\nu+1)^{r+1}_{\nu<m}.$$

PROOF. We may assume that $r<b_0,,\hat{b}_m\leqq a$.

*Case* 1. $m\geqq\omega(a)$. Then there is a partition

$$[S]^r=L_0++\hat{L}_m,\quad\text{where } |S|=a,$$

such that $|L_\nu|\leqq 1$ for $\nu<n$. Then, by hypothesis, there are $\nu<m$ and $Y\in[S]^{b_\nu}$ such that $[Y]^r\subset L_\nu$. Then $|[Y]^r|\leqq|L_\nu|\leqq 1$; $b_\nu\leqq r$, which is a contradiction.

*Case* 2. $m<\omega(a)$. Then (1) follows by [1], Theorem 39 (ii). For the convenience of the reader we briefly describe the proof.

Let $|S|=a^+$, and

$$[S]^{r+1}=I_0+'+'\hat{I}_m\quad\text{(partition } \varDelta).$$

We define inductively a ramification system $\mathbf{R}$ on $S$ of length $\varrho=\omega(a)$. Let $\sigma<\varrho$, and let $S'(\nu_0,,\hat{\nu}_\sigma)$ be defined. Write $\nu$ in place of $\nu_0,,\hat{\nu}_\sigma$, for some fixed numbers $\nu_0,,\hat{\nu}_\sigma$. If $S'(\nu)=\emptyset$, put $F(\nu)=\emptyset$ and $n(\nu)=0$. Now let $S'(\nu)\neq\emptyset$. Choose $F(\nu)=\{x(\nu)\}\subset S'(\nu)$ and put

$$\varDelta_\nu(y)=\Pi\big(X\in[\Sigma(\tau\leqq\sigma)F(\nu_0,,\hat{\nu}_\tau)]^r\big)\,\varDelta(X+\{y\})$$

for $y\in S'(\nu)-F(\nu)$. Put $n(\nu)=\omega(|\varDelta_\nu|)$. Then

$$S'(\nu)-F(\nu)=\Sigma\big(\nu_\sigma<n(\nu)\big)\,S(\nu,\nu_\sigma),$$

where $|\varDelta_\nu|\leqq 1$ on each $S(\nu,\nu_\sigma)$. This defines $\mathbf{R}$. Now Lemma 1 (iii) applies. For we have, for $\tau<\sigma$,

$$|n(\nu_0,,\hat{\nu}_\sigma)|\leqq|\varDelta|^{|\sigma+1|^r}=c_\sigma<a;\quad |n(\nu_0,,\hat{\nu}_\tau)|\leqq c_\sigma.$$

Also, $c_\sigma^{|\sigma|}\leqq a<a^+$; $|S|=a^+=a^{+\prime}$; $|\varrho|<a^+$; $|F(\nu)|\leqq 1<a^+$. Hence, by Lemma 1 (iii), there is $(\nu_0,,\hat{\nu}_\varrho)\in N$ such that $S'(\nu_0,,\hat{\nu}_\varrho)\neq\emptyset$. Choose $x_\varrho\in S'(\nu_0,,\hat{\nu}_\varrho)$. Then we can write $F(\nu_0,,\hat{\nu}_\sigma)=\{x_\sigma\}$ for $\sigma<\varrho$, and we have $\{x_0,,x_\varrho\}_{\neq}\subset S$. Now

$$\varDelta_\nu(y)=\Pi\big(X\in[\{x_0,,x_\sigma\}]^r\big)\varDelta\big(X+\{y\}\big)\ \text{ for }\ y\in S'(\nu)-\{x_\sigma\},$$

for $\sigma<\varrho$. By definition of $S(\nu,\nu_\sigma)$,

$$\{x_{\sigma_0},,x_{\sigma_r}\}\equiv\{x_{\sigma_0},,\hat{x}_{\sigma_r},x_\varrho\}\ (\cdot\varDelta)$$

for $\sigma_0 < < \sigma_r < \varrho$. We have $[[0, \varrho)]^r = K_0 + + \hat{K}_m$, where $K_\mu = [[0, \varrho)]^r\{X: X + \{x_\varrho\} \in I_\mu\}$ for $\mu < m$. Since $|\varrho| = a \to (b_\mu)^r_{\mu < m}$, there are $\mu$ and $M$ such that $\mu < m$; $M \subset [0, \varrho)$; $|M| = b_\mu$; $[M]^r \subset K_\mu$. Put $Q = \{x_\sigma : \sigma \in M\} + \{x_\varrho\}$. Then

$$Q \subset S;\ \ |Q| = b_\mu + 1;\ \ [Q]^{r+1} \subset I_\mu,$$

and (1) follows. This proves Lemma 2.

($*$) THEOREM 3. *If* $\alpha \geqq 0$; $c < \aleph'_\alpha$; $r \geqq 2$, *then*

$$\aleph_{\alpha+(r-1)} \to (\aleph_{\alpha+1}, (\aleph'_\alpha)_c)^r.$$

($*$) THEOREM 4. *If* $\alpha \geqq 0$; $c < \aleph'_\alpha$; $r \geqq 1$, *then* $\aleph_{\alpha+(r-1)} \to (\aleph_\alpha)^r_c$.

PROOFS. Theorem 4 is trivial for $r = 1$. Both, Theorem 3 and Theorem 4 are true for $r = 2$, by Corollary 1 and Theorem 2 respectively. For general $r \geqq 2$ both theorems follow from Lemma 2 by induction over $r$.

($*$) COROLLARY 2. *If* $\aleph_\alpha$ *is regular;* $c < \aleph_\alpha$; $r \geqq 2$, *then*

$$\aleph_{\alpha+(r-1)} \to (\aleph_{\alpha+1}, (\aleph_\alpha)_c)^r.$$

REMARK. For regular $\aleph_\alpha$ Theorem 4 follows from Theorem 3.
We shall need the well known

THEOREM OF RAMSEY [18]. (i) *For* $c < \aleph_0$ *and* $r \geqq 1$, $\aleph_0 \to (\aleph_0)^r_c$.
(ii) *For* $b, c, r < \aleph_0$ *there is* $R(b, c, r) < \aleph_0$ *such that*

$$R(b, c, r) \to (b)^r_c.$$

($*$) THEOREM 5. *Let* $a$ *be inaccessible;* $m < \omega(a)$; $b_0,, \hat{b}_m < a$. *Then*

$$a \to (a, b_0,, \hat{b}_m)^2.$$

The case $m < \omega$ of this theorem is [1], Theorem 8.

PROOF. Let

$$|S| = a;\ \ [S]^2 = L + L_0 + + \hat{L}_m;\ \ a \notin [L]_2.$$

We define a ramification system $\mathbf{R}$ on $S$ of length $\varrho = \omega((b_0 + + \hat{b}_m)^+)$. We note that

$$|\varrho| \to (b_0,, \hat{b}_m)^1. \tag{2}$$

Let $\sigma < \varrho$, and let $S'(v_0,, \hat{v}_\sigma) = S'(v)$ be defined. We take as $F(v)$ a maximal subset of $S'(v)$ such that $[F(v)]^2 \subset L$. Since $a \notin [L]_2$ we have $|F(v)| < a$. Then there are a number $n(v)$, elements $x(v, v_\sigma)$ of $F(v)$ and numbers $\mu(v, v_\sigma) < m$ such that

$$S'(v) - F(v) = \Sigma(v_\sigma < n(v))\, S(v, v_\sigma),$$

where $\{x, x(v, v_\sigma)\} \in L_{\mu(v, v_\sigma)}$ for $v_\sigma < n(v)$; $x \in S(v, v_\sigma)$. This follows from the maximality of $F(v)$. We can make

$$|n(v)| \leqq |F(v)| \cdot |m| < a.$$

This defines $\mathbf{R}$. Lemma 1 (iv) applies and yields $(v_0,, \hat{v}_\varrho) \in N$ such that $S'(v_0,, \hat{v}_\varrho) \neq \varnothing$.

Then $n(v_{0},, \hat{v}_\sigma) \geqq 1$ for $\sigma < \varrho$, and there exists $x_\sigma = x(v_{0},, v_\sigma)$ for $\sigma < \varrho$. Then

$$\{x_\tau, x_\sigma\} \in L_{\mu(v,, v_\tau)} \text{ for } \tau < \sigma < \varrho,$$

$$[0, \varrho) = K_0 + + \hat{K}_m,$$

where $K_\mu = \{\tau : \tau < \varrho \wedge \mu(v_{0},, v_\tau) = \mu\}$ for $\mu < m$. By (2) there are $M \subset [0, \varrho)$ and $\mu < m$ such that $|M| = b_\mu$ and $M \subset K_\mu$. Put $X = \{x_\sigma : \sigma \in M\}$. Then $|X| = b_\mu$ and $[X]^2 \subset L_\mu$ which proves Theorem 5.

**8. 2.** The following remarks are relevant in connection with the discussion of partition relations involving inaccessible cardinals.

We want to consider the following propositions involving a cardinal $a \geqq \aleph_0$:

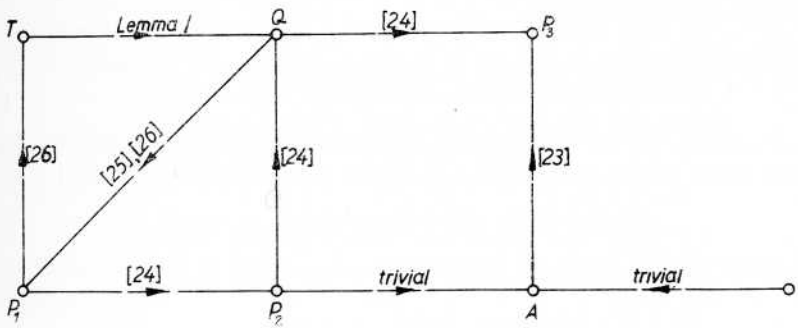
A: If $a$ is strongly inaccessible, then $a \nrightarrow (a)_2^{<\aleph_0}$.

B: If $a$ is strongly inaccessible, then $a \nrightarrow (\aleph_0)_2^{<\aleph_0}$.

$P_1, P_2, P_3$, Q: These are defined in [24].

T: $a \nrightarrow (a, 4)^3$.

The following diagram shows implication relations known to hold between these propositions together with the corresponding references:




Furthermore, (i) if $P_3 \Rightarrow Q$, then Q holds for all $a$ [28]; (ii) if Gödel's constructability axiom is assumed then $P_3$ holds for all $a$ [27]. (iii) Q holds for a very wide class of strongly inaccessible cardinals [5].

By (iii) $a \nrightarrow (a, a)^2$, $a \nrightarrow (a, 4)^3$ and $a \nrightarrow (a)_2^{<\aleph_0}$ hold for many inaccessible cardinals.

The following problems remain open.

Problem (A) Is it true that $a \nrightarrow (a, a)^2$, $a \nrightarrow (a, 4)^3$ or $a \nrightarrow (a)_2^{<\aleph_0}$ hold for every strongly inaccessible cardinal?

(B) Is there any strongly inaccessible cardinal for which $a \nrightarrow (\aleph_0)_2^{<\aleph_0}$ holds?

*Added in proof (23. III. 1965.).* For a more detailed discussion of the recent results concerning Q, $P_1$, $P_2$, $P_3$ see [30].

It has been recently proved by F. Rowbottom that Gödel's constructibility axiom implies that $a \nrightarrow (\aleph_1)_2^{<\aleph_0}$ holds for every cardinal $a$.

## 9. CANONICAL PARTITIONS. CANONIZATION LEMMA

**9. 1.** DEFINITION. Let $\Delta$ be a disjoint partition of $[S]^r$, and let $\Sigma'(\nu<n)\, S_\nu \subset S$. Then $\Delta$ is called *canonical* in $(S_0,, \hat{S}_n)$ if for $X, Y \in [S_0 + + \hat{S}_n]^r$ the relations

$$|XS_\nu| = |YS_\nu| \text{ for } \nu < n$$

imply

$$X \equiv Y(\cdot \Delta).$$

We remark here that this notion of canonicity differs from that used in [1]. There we were considering ordinal canonicity whereas here we have cardinal canonicity.

**9. 2.** ($*$) LEMMA 3 *(Canonization lemma). Let* $|S| = a > a'$; $\varrho = \omega(a')$; $r \geqq 1$; $a_0 < < \hat{a}_\varrho < a = \sup(\sigma < \varrho) a_\sigma$, *and let* $\Delta$ *be a disjoint partition of* $[S]^r$ *such that* $|\Delta| < a$. *Then there are sets* $S_\sigma$ *such that* $|S_\sigma| = a_\sigma$ *for* $\sigma < \varrho$ *and* $\Sigma'(\sigma < \varrho)\, S_\sigma \subset S$, *and* $\Delta$ *is canonical in* $(S_0,, \hat{S}_\varrho)$. *If we are, in addition, given any representation*

$$S = \Sigma'(\sigma < \varrho)\, S'_\sigma$$

*such that* $|S'_\sigma| < a$ *for* $\sigma < \varrho$, *then we can stipulate that, in addition,* $S_\lambda \subset S'_{\sigma(\lambda)}$ *for* $\lambda < \varrho$, *where* $\sigma(0) < < \hat{\sigma}(\varrho) < \varrho$.

PROOF. Put

$$R = \{(r_0,, \hat{r}_s, t) : r_0,, \hat{r}_s \geqq 1 \wedge r_0 + + \hat{r}_s = r \wedge t < s\}.$$

Let $(r_0,, \hat{r}_s, t) \in R$. We say that $\Delta$ is $(r_0,, \hat{r}_s, t)$-canonical in $(A_0,, \hat{A}_\varrho)$ if the following conditions hold: $\Sigma'(\sigma < \varrho) A_\sigma \subset S$; $|A_\sigma| = a_\sigma$ for $\sigma < \varrho$. Whenever $X, Y \in [A_0 + + \hat{A}_\varrho]^r$ and

$$XA_\sigma = X_\sigma;\ YA_\sigma = Y_\sigma;\quad |X_\sigma| = |Y_\sigma| \text{ for } \sigma < \varrho,$$

$$\{\sigma : X_\sigma \neq \emptyset\} = \{\sigma_0,, \hat{\sigma}_s\}_<;\quad |X_{\sigma_\lambda}| = r_\lambda\ (\lambda < s);$$

$$X_\sigma = Y_\sigma \text{ for } \sigma \neq \sigma_t,$$

then $X \equiv Y(\cdot \Delta)$. Our aim is to find sets $S_0,, \hat{S}_\varrho$ such that $\Delta$ is $(r_0,, \hat{r}_s, t)$ — canonical in $(S_0,, \hat{S}_\varrho)$ for every $(r_0,, r_s, t)$ of $R$ simultaneously.

Let $(r_0,, \hat{r}_s, t) \in R$, and let $\Delta$ be $(r_0,, \hat{r}_s, u)$-canonical in a fixed system $(A_0,, \hat{A}_\varrho)$ for every $u$ such that $t < u < s$. This is for instance true if $t = s-1$ and $\Sigma'(\sigma < \varrho) A_\sigma \subset S$; $|A_\sigma| = a_\sigma$ for $\sigma < \varrho$. It suffices to deduce that there are numbers $\lambda_\sigma$ and sets $B_\sigma$ such that $\lambda_0 < < \hat{\lambda}_\varrho < \varrho$ and $B_\sigma \subset A_{\lambda_\sigma}$ for $\sigma < \varrho$, and $\Delta$ is $(r_0,, \hat{r}_s, t)$-canonical in $(B_0,, \hat{B}_\varrho)$. For, the passage from $(A_0,, \hat{A}_\varrho)$ to $(B_0,, \hat{B}_\varrho)$ does not destroy any canonicity $\Delta$ may have possessed in $(A_0,, \hat{A}_\varrho)$, and a finite number* of steps as described above starting with the sets $S'_0,, \hat{S}'_\varrho$, leads to the required system $(S_0,, \hat{S}_\varrho)$.

We well-order the set $[S]^{<\aleph_0}$ and denote, for $s < \omega$; $X \subset S$; $|X| \geqq s$, by $\pi(X, s)$ the first element of $[X]^s$. We now define $\lambda_\sigma$ and $B_\sigma$ by simultaneous induction. Let $\tau < \varrho$, and let $\lambda_\sigma$, $B_\sigma$ be defined for $\sigma < \tau$, and suppose that $\lambda_\sigma < \varrho$ and $B_\sigma \in [A_{\lambda_\sigma}]^{a_\sigma}$

* In fact, $|R| = (r+1)\, 2^{r-2}$.

for $\sigma < \tau$. We define $\lambda_\tau$ and $B_\tau$ as follows. There is $\lambda_\tau < \varrho$ such that $\lambda_\sigma < \lambda_\tau$ for $\sigma < \tau$. This follows from $|\varrho|' = |\varrho|$. Put

$$\Delta_{\lambda_\tau}(X) = \Pi\big(P \in [A_{\lambda_0} + + \hat{A}_{\lambda_\tau}]^{r_0 + + \hat{r}_t} \wedge \lambda_\tau < \hat{\varrho}_t < < \hat{\varrho}_s < \varrho\big)$$
$$\Delta\big(P + X + \hat{\pi}(A_{\varrho_t}, r_t) + + \hat{\pi}(A_{\varrho_s}, r_s)\big) \quad \text{for} \quad X \in [A_{\lambda_\tau}]^{r_t}.$$

Then

$$|\Delta_{\lambda_\tau}| \leqq |\Delta|^{(a_{\lambda_0} + + \hat{a}_{\lambda_\tau})^{r_0 + + \hat{r}_t}(a')^{s-t-1}} = b_\tau < a,$$

where $b_\tau$ is independent of the choice of $\lambda_\tau$. We have $a_\tau + b_\tau < \aleph_\alpha < a$ for some $\alpha = \beta + 1$. By Theorem 4,

$$\aleph_{\alpha + (r_t - 1)} \to (\aleph_a)^{r_t}_{b_t},$$

and $\aleph_{\alpha + (r_t - 1)} < a$. We can choose $\lambda_\tau$ so that, in addition, $a_{\lambda_\tau} \geqq \aleph_{\alpha + (r_t - 1)}$. Then we shall have

$$a_{\lambda_\tau} \to (a_\tau)^{r_t}_{|\Delta_{\lambda_\tau}|},$$

and therefore there is a set $B_\tau \in [A_{\lambda_\tau}]^{a_\tau}$ such that

$$|\Delta_{\lambda_\tau}| \leqq 1 \quad \text{in} \quad [B_\tau]^{r_t}.$$

This completes the definition of $\lambda_\sigma$ and $B_\sigma$ for $\sigma < \varrho$. We have $\lambda_0 < < \hat{\lambda}_\varrho < \varrho$ and $B_\sigma \in [A_{\lambda_\sigma}]^{a_\sigma}$ for $\sigma < \varrho$. We now show that $\Delta$ is $(r_0,, \hat{r}_s, t)$-canonical in $(B_0,, \hat{B}_\varrho)$.

Let

$$X, Y \in [B_0 + + \hat{B}_\varrho]^r; \quad XB_\sigma = X_\sigma; \quad YB_\sigma = Y_\sigma \text{ for } \sigma < \varrho;$$
$$\{\sigma: X_\sigma \neq \varnothing\} = \{\sigma_0,, \hat{\sigma}_s\}_<;$$
$$|X_{\sigma_\lambda}| = r_\lambda \text{ for } \lambda < s, \text{ and } X_\sigma = Y_\sigma \text{ for } \sigma \neq \sigma_t.$$

Then we have, in view of the $(r_0,, \hat{r}_s, u)$-canonicity for every $u$ in the range $t < u < s$, the relation

$$X = X_{\sigma_0} + + \hat{X}_{\sigma_s} \equiv X'(\cdot\Delta),$$

where

$$X' = X_{\sigma_0} + + X_{\sigma_t} + \hat{\pi}(A_{\lambda_{\sigma_t}}, r_t) + + \hat{\pi}(A_{\lambda_{\sigma_s}}, r_s).$$

By definition of $\Delta_{\lambda_{\sigma_t}}$ and $B_{\sigma_t}$ we have $X' \equiv X''(\cdot\Delta)$, where

$$X'' = X_{\sigma_0} + + \hat{X}_{\sigma_t} + Y_{\sigma_t} + \hat{\pi}(A_{\lambda_{\sigma_t}}, r_t) + + \hat{\pi}(A_{\lambda_{\sigma_s}}, r_s).$$

Finally, again by the $(r_0,, \hat{r}_s, u)$-canonicity for $t < u < s$,

$$X'' \equiv X_{\sigma_0} + + \hat{X}_{\sigma_t} + Y_{\sigma_t} + \hat{Y}_{\sigma_t} + + \hat{Y}_{\sigma_s} = Y(\cdot\Delta).$$

This completes the proof of Lemma 3. We note that the final clause of the lemma is also proved by our construction.

**9.3.** (*) Lemma 3A *(polarized canonization lemma). Let* $a > a'$; $AB = \varnothing$; $|A| = |B| = a$; $\varrho = \omega(a')$; $a_0 < < \hat{a}_\varrho < a = \sup(\sigma < \varrho)a_\sigma$;

$$[A, B]^{1,1} = I_0 +' I_1.$$

*Then there are sets $A_\sigma$, $B_\sigma$ and numbers $h(\sigma, \tau) < 2$ such that*

$$\Sigma'(\sigma < \varrho) A_\sigma \subset A;\ \Sigma'(\sigma < \varrho) B_\sigma \subset B;$$

$$|A_\sigma| = |B_\sigma| = a_\sigma \text{ for } \sigma < \varrho;$$

$$[A_\sigma, B_\tau]^{1,1} \subset I_{h(\sigma,\tau)} \text{ for } \sigma, \tau < \varrho.$$

PROOF. Let $\omega(a) = n$; $A = \{x_0,, \hat{x}_n\}_{\neq}$; $B = \{y_0,, \hat{y}_n\}_{\neq}$; $S = [0, n)$. Then $[S]^2 = \Sigma'(\varkappa, \lambda < 2) I(\varkappa, \lambda)$, where

$$I(\varkappa, \lambda) = \{\{\mu, \nu\}_< : \{x_\mu, y_\nu\} \in I_\varkappa \wedge \{x_\nu, y_\mu\} \in I_\lambda\} \text{ for } \varkappa, \lambda < 2.$$

By Lemma 3 there are sets $S_\sigma$ and numbers $h_0(\sigma, \tau), h_1(\sigma, \tau) < 2$ such that $S_0 + + \hat{S}_\varrho(tp) \subset S$; $|S_\sigma| = a_\sigma$ for $\sigma < \varrho$;

$$[S_\sigma, S_\tau]^{1,1} \subset I(h_0(\sigma, \tau), h_1(\sigma, \tau)) \text{ for } \sigma < \tau < \varrho.$$

Put $A_\sigma = \{x_\mu : \mu \in S_{2\sigma}\}$; $B_\sigma = \{y_\nu : \nu \in S_{2\sigma+1}\}$ for $\sigma < \varrho$. Then $|A_\sigma| = a_{2\sigma} \geqq a_\sigma$; $|B_\sigma| = a_{2\sigma+1} > a_\sigma$ for $\sigma < \varrho$;

$$A_\sigma A_\tau = B_\sigma B_\tau = \varnothing \text{ for } \sigma < \tau < \varrho.$$

Now let $\sigma, \tau < \varrho$; $x_\mu \in A_\sigma$; $y_\nu \in B_\tau$. Then $\mu \in S_{2\sigma}$; $\nu \in S_{2\tau+1}$. If $\sigma \leqq \tau$, then $2\sigma < 2\tau + 1$; $\{\mu, \nu\}_< \in [S_{2\sigma}, S_{2\tau+1}]^{1,1} \subset I(h_0(2\sigma, 2\tau+1), h_1(2\sigma, 2\tau+1))$;

$$\{x_\mu, y_\nu\} \in I_{h_0(2\sigma, 2\tau+1)}.$$

If $\sigma > \tau$, then $2\tau + 1 < 2\sigma$; $\{\nu, \mu\}_< \in [S_{2\tau+1}, S_{2\sigma}]^{1,1} \subset I(h_0(2\tau+1, 2\sigma), h_1(2\tau+1, 2\sigma))$;

$$\{x_\mu, y_\nu\} \in I_{h_1(2\tau+1, 2\sigma)}.$$

Hence the assertion holds if we put

$$h(\sigma, \tau) = \begin{cases} h_0(2\sigma, 2\tau+1) & \text{if } \sigma \leqq \tau \\ h_1(2\tau+1, 2\sigma) & \text{if } \sigma > \tau. \end{cases}$$

REMARK. There are, of course, more general versions of polarized canonization procedures.

**9. 2. Super-canonicity.** DEFINITION. Let $\Delta$ be a disjoint $r$-partition of $S$, and let $\Sigma'(\nu \in n) S_\nu \subset S$. Then $\Delta$ is called *super-canonical* in $(S_0,, \hat{S}_n)$ if the following condition holds. Whenever

$$X, Y \in [S_0 + + \hat{S}_n]^r;\quad \{\mu : XS_\mu \neq \varnothing\} = \{\mu_0,, \hat{\mu}_s\}_<;$$

$$\{\nu : YS_\nu \neq \varnothing\} = \{\nu_0,, \hat{\nu}_s\}_<;\quad |XS_{\mu_\sigma}| = |YS_{\nu_\sigma}| \text{ for } \sigma < s,$$

then $X \equiv Y(\cdot\Delta)$. It follows that every super canonical partition is canonical. It is easy to prove, by induction over $r$, that the number of systems $(r_0,, \hat{r}_s)$ such that $r_0,, \hat{r}_s \geqq 1$; $r_0 + + \hat{r}_s = r$ is $2^{r-1}$. It follows that if $r \geqq 1$ and $\Delta$ is super-canonical then $|\Delta| \leqq 2^{r-1}$.

LEMMA 3B *(Super-canonization lemma). Let* $|S| = a > a'$; $m < \omega(a') = n$. *Suppose that either* (i) $a' = \aleph_0$, *or* (ii) $a' > \aleph_0$, *and* $a'$ *is measurable*. Let* $r \geqq 1$ *and*

$$[S]^r = I_0 +' +' \hat{I}_m \quad (\textit{partition } \Delta).$$

*Then there are sets* $S_\nu$ *and cardinals* $a_\nu$ *such that*

$$a_0 << \hat{a}_n < a = \sup(\nu < n) a_\nu;$$

$$\Sigma'(\nu < n) S_\nu = S; \quad |S_\nu| = a_\nu \ \textit{for} \ \nu < n;$$

$$|\Sigma(\nu < n) S_\nu| = a,$$

*and* $\Delta$ *is super-canonical in* $(S_0,, \hat{S}_n)$.

PROOF. There is only one step where the proof in case (ii) differs from that in case (i).

By Lemma 3, there is a set

$$B = \Sigma'(\nu < n) B_\nu \subset S$$

such that $\Delta$ is canonical in $(B_0,, \hat{B}_n)$; $|B_\nu| = b_\nu$ for $\nu < n$; $b_0 << \hat{b}_n < a = \sup(\nu < n) b_\nu$. Now there is a partition

(1) $$[[0, n)]^r = \Sigma'(\lambda < l) J_\lambda,$$

where $l < \omega$, such that two elements $\{\mu_0,, \hat{\mu}_r\}_<$, $\{\nu_0,, \hat{\nu}_r\}_<$ of $[[0, n)]^r$ belong to the same class $J_\lambda$ if and only if, whenever

$$X \in [\Sigma(\varrho < r) B_{\mu_\varrho}]^r; \quad Y \in [\Sigma(\varrho < r) B_{\nu_\varrho}]^r,$$

and $|XB_{\mu_\varrho}| = |YB_{\nu_\varrho}|$ for $\varrho < r$, then $X$ and $Y$ lie in the same class $I_\nu$. We now apply to the partition (1) the relation

$$a' \to (a')^r_l.$$

In case (i) this relation is Ramsey's theorem, and in case (ii) it follows from [4], Theorem 9. We obtain a set $\{\sigma_0,, \hat{\sigma}_n\}_< \subset [0, n)$ and a number $\lambda < l$ such that $[\{\sigma_0,, \hat{\sigma}_n\}]^r \subset J_\lambda$. Then $\Delta$ is super-canonical in $(B_{\sigma_0},, \hat{B}_{\sigma_n})$, and the assertion holds if we put $S_\nu = B_{\sigma_\nu}$ for $\nu < n$.

## 10. POSITIVE I-RELATIONS IN THE CASE $r = 2$; $a > a'$

**10. 1.** ($*$) LEMMA 4. *Let* $a \geqq \aleph_0$, *and let* $b_0,, \hat{b}_m$ *be any cardinals. Then the relations*

$$a \to (a, b_0,, \hat{b}_m)^2; \quad a' \to (a', b_0,, \hat{b}_m)^2$$

*are equivalent.*

* i. e. that $a'$ does not possess the property $P_3$ of [24]. In fact, the weaker condition $a' \to (a', a')^r$ already suffices. After recent results discussed in 8. 2 the existence of such cardinals, other than $\aleph_0$, has been rendered unlikely.

PROOF. The case $a=a'$ is trivial. Now let $a>a'$. The case when $b_0<2$ is trivial. Now let $b_0,, \hat{b}_m \geqq 2$, and put $n=\omega(a')$.

1. Let $a' \to (a', b_0,, \hat{b}_m)^2$. Then $m<a'$, and $b_0,, \hat{b}_m \leqq a'$. Let $|S|=a$ and $[S]^2 = I+'\Sigma'(\mu<m)I_\mu$ (partition $\Delta$). Assume that $b_\mu \notin [I_\mu]_2$ for $\mu<m$. By Lemma 3 there are sets $S_\nu$ such that $a' \leqq |S_\nu| < |S_\lambda| < a$ for $\nu<\lambda<n$;

$$\Sigma'(\nu<n)\, S_\nu \subset S; \quad |S_0 ++ \hat{S}_n| = a,$$

and $\Delta$ is canonical in $(S_0,, \hat{S}_n)$. If $\mu<m$; $\nu<n$; $[S_\nu]^2 \subset I_\mu$, then $|S_\nu|<b_\mu$ whereas in fact $|S_\nu| \geqq a' \geqq b_\mu$. This contradiction shows that $[S_\nu]^2 \subset I$ for $\nu<n$. Choose $S' \subset S$ such that $|S'S_\nu|=1$ for $\nu<n$. Then $|S'| = a' \to (a', b_0,, \hat{b}_m)^2$, and we have at least one of the following cases.

*Case* 1. There are a number $\mu<m$ and a set $S'' \in [S']^{b_\mu}$ such that

$$[S'']^2 \subset I_\mu.$$

This contradicts $b_\mu \notin [I_\mu]_2$.

*Case* 2. There is $S'' \in [S']^{a'}$ such that $[S'']^2 \subset I$. Put

$$S''' = \Sigma(S''S_\nu \neq \varnothing)\, S_\nu.$$

Then $|S'''|=a$; $[S''']^2 \subset I$. This proves $a \to (a, b_0,, \hat{b}_m)^2$.

2. Let $a \to (a, b_0,, \hat{b}_m)^2$;

$$[[0, n)]^2 = J+'\Sigma'(\mu<m)J_\mu.$$

Let $S=\Sigma'(\nu<n)\, S_\nu$ and $|S_\nu|<|S|=a$ for $\nu<n$. Then

$$[S]^2 = I+'\Sigma'(\mu<m)I_\mu,$$

where

$$I_\mu = \Sigma(\{\alpha, \beta\}_< \in J_\mu)[S_\alpha, S_\beta]^{1,1} \text{ for } \mu<m.$$

Then $|S| \to (a, b_0,, \hat{b}_m)^2$, and we have at least one of the cases:

*Case* 1. There is $S' \in [S]^a$ with $[S']^2 \subset I$. Put $N = \{\nu\colon S'S_\nu \neq \varnothing\}$. Then $|N|=a'$; $[N]^2 \subset J$.

*Case* 2. There are a number $\mu<m$ and a set $S' \in [S]^{b_\mu}$ with $[S']^2 \subset I_\mu$. Define $N$ as in case 1. Then $|S'S_\nu| \leqq 1$ for $\nu<n$; $|N|=|S'|=b_\mu$; $[N]^2 \subset J_\mu$. This proves $a' \to (a', b_0,, \hat{b}_m)^2$ and completes the proof of Lemma 4.

**10. 2.** (∗) THEOREM 6. *Let* $a>a' = b^+$; $c<a'$. *Then*

$$a \to (a, (b')_c)^2.$$

PROOF. By Corollary 1, $a' \to (a', (b')_c)^2$, and the assertion follows from Lemma 4.

(∗) COROLLARY 3. *Let* $\alpha > \mathrm{cf}(\alpha) = \beta+1$ *and* $c<\aleph'_\alpha$. *Then*

$$\aleph_{\alpha+(r-2)} \to (\aleph_\alpha, (\aleph'_\alpha)_c)^r \text{ for } r \geqq 2.$$

This follows from Theorem 6 and Lemma 2.

## 11. THE SETS OF VECTORS USED AS COUNTER EXAMPLES

**11. 1. The vector discrepancy.** For any two distinct "vectors" $x=(x_0,, \hat{x}_n)$; $y=(y_0,, \hat{y}_n)$ of the same "length" $n$ we define the *discrepancy* $\delta(x, y)$ by putting

$$\delta(x, y) = \min (x_\nu \neq y_\nu) \nu.$$

When no confusion can arise we shall instead of $\delta(x, y)$ write more simply $xy$. If $r \geqq 3$, and $x_0,, x_{r-1}$ are vectors of the same length, $x_\varrho \neq x_{\varrho+1}$ for $\varrho+1<r$, then we put

$$\delta(x_0,, x_{r-1}) = (x_0x_1, x_1x_2,, x_{r-2}x_{r-1}); \quad \delta'(x_0,, x_{r-1}) = \{x_\varrho x_{\varrho+1} : \varrho < r-1\}.$$

Thus $\delta(x_0,, x_{r-1})$ is a vector of length $r-1$, and $\delta'(x_0,, x_{r-1})$ is a set of ordinals.

Every set $X$ of vectors of the same length whose components $x_\nu, y_\nu$ are ordinals will be ordered lexicographically by putting

$$x < y \text{ whenever } x_{xy} < y_{xy}.$$

By $\operatorname{tp}(X)$ we always mean $\operatorname{tp}(X, <)$.

**11. 2. The set** $V(\alpha)$. We put

$$V(\alpha) = \{(x_0,, \hat{x}_{\omega_\alpha}) : x_0,, \hat{x}_{\omega_\alpha} < 2\}.$$

If $(*)$ is assumed then $|V(\alpha)| = 2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

**11. 3. The set** $V'(\alpha)$. Let $\alpha > \operatorname{cf}(\alpha)$ and $\omega_{\operatorname{cf}(\alpha)} = n$. For every such $\alpha$ for which we shall want to consider the set $V'(\alpha)$ we shall choose, without mentioning this explicitly, a fixed sequence $\alpha_0,, \hat{\alpha}_n$ such that

$$\operatorname{cf}(\alpha) < \alpha_0 < < \hat{\alpha}_n < \alpha = \sup (\nu < n) \alpha_\nu.$$

We put $V'(\alpha) = \{(x_0,, \hat{x}_n) : (\forall \nu)(\nu < n \supset x_\nu < \omega_{\alpha_\nu})\}$. Then $V(\operatorname{cf}(\alpha)) \subset V'(\alpha)$. The set $V'(\alpha)$ is only defined if $\alpha > \operatorname{cf}(\alpha)$. If $(*)$ is assumed then

$$\aleph_\alpha = \Sigma \aleph_{\alpha_\nu} < \Pi \aleph_{\alpha_\nu} \leqq \aleph_\alpha^{|n|} = \aleph_{\alpha+1}$$

so that $|V'(\alpha)| = \Pi \aleph_{\alpha_\nu} = \aleph_{\alpha+1}$.

**11. 4.** Sierpiński **partitions.** Let $(V(\alpha), \prec)$ and $(V'(\alpha), \prec')$ be well-orders of the sets $V(\alpha)$ and $V'(\alpha)$ respectively, of types $\omega(|V(\alpha)|)$ and $\omega(|V'(\alpha)|)$ respectively. We define the Sierpiński *partitions*

$$\Delta_S : [V(\alpha)]^2 = T_0 +' T_1,$$

$$\Delta_S' [V'(\alpha)]^2 = T_0' +' T_1'$$

by putting

$$T_0 = [V(\alpha)]^2 \{\{x, y\}_< : x \prec y\},$$

$$T_0' = [V'(\alpha)]^2 \{\{x, y\}_< : x \prec' y\}.$$

**11. 5.** We shall need the following well known facts.

$(*)$ (i) If $X \subset V(\alpha)$, and either $\operatorname{tp} X = \sigma$ or $\operatorname{tp} X = \sigma^*$, then $|X| \leqq \aleph_\alpha$.

(∗) (ii) If $X \subset V'(\alpha)$ and tp $X = \sigma$, then $|X| \leqq \aleph_\alpha$; if $X \subset V'(\alpha)$ and tp $X = \sigma^*$, then $|X| \leqq \aleph'_\alpha$. The results (i) and (ii) can be briefly expressed by

$$\omega_{\alpha+1}, \omega^*_{\alpha+1} \nleqq \text{tp}\,(V(\alpha));\ \omega_{\alpha+1}, \omega^*_{\text{cf}(\alpha)+1} \nleqq \text{tp}\,(V'(\alpha)).$$

We shall make frequent use, without reference, of the following simple propositions.

**11. 6.** (i) Let either $\{x, y, z\}_{\neq} \subset V(\alpha)$ or $\{x, y, z\}_{\neq} \subset V'(\alpha)$. Then $xz \geqq \min(xy, yz)$, and here is equality if $xy \neq yz$.

(ii) If $\{x, y, z\}_< \subset V(\alpha)$, then $xy \neq yz$ and $xz = \min(xy, yz)$.

PROOF. Put $x = (x_0,, \hat{x}_n)$, and similarly for $y, z$. Here $n = \omega_\alpha$ or $n = \omega_{\text{cf}(\alpha)}$.

PROOF OF (i). If $v < xy, yz$, then $x_v = y_v = z_v$. Hence $xz \geqq \min(xy, yz)$. If $xy = v_0 < yz$, then $x_{v_0} \neq y_{v_0} = z_{v_0}$; $xz \leqq v_0 = \min(xy, yz)$. If $xy > v_1 = yz$, then $x_v = y_{v_1} \neq z_{v_1}$; $xz \leqq v_1 = \min(xy, yz)$.

PROOF OF (ii). If $xy = yz = v_0$, then $0 \leqq x_{v_0} < y_{v_0} < z_{v_0} < 2$ which is impossible. Hence $xy \neq yz$ and, by (i), $xz = \min(xy, yz)$.

## 12. COUNTER EXAMPLES FOR $r = 2$

(∗) THEOREM 7. *$a^+ \nrightarrow (a^+, a'^+)^2$ for $a \geqq \aleph_0$.*

This follows from [1] Theorem 7 (ii) which states: If $a \geqq \aleph_0$ and $b = \min(a^c > a)c$, then $a^b \nrightarrow (a^+, b^+)^2$. For if (∗) is assumed then $b = a'$ and $a^b = a^+$. For convenience here is the proof. Let $a = \aleph_\alpha$.

*Case* 1. $a = a'$. Consider the partition $\Delta_S$ of 11. 4. It follows from 11. 5 (i) that $\aleph_{\alpha+1} \notin [T_0]_2 + [T_1]_2$. This proves the assertion.

*Case* 2. $a > a'$. Consider the partition $\Delta'_S$ of 11. 4. It follows from 11. 5 (ii) that $\aleph_{\alpha+1} \notin [T'_0]_2$ and $\aleph'^+_\alpha \notin [T'_1]_2$. This proves the assertion, and Theorem 7 follows.

THEOREM 8. *If $n \geqq 1$ and $a_v < b_v$ for $v < n$, then $a_0 \ldots \hat{a}_n \nrightarrow (b_0,, \hat{b}_n)^2$.*

REMARKS. The case $a_0 = = \hat{a}_n = 2$ is due to Gödel. The case: $n = 2$ and arbitrary $a_v, b_v$ is [1] Theorem 36 (iii).

PROOF. Let $|A_v| = a_v$ for $v < n$, and let $S$ be the cartesian product of the sets $A_v$. Then $|S| = a_0 \ldots \hat{a}_n$, and

$$[S]^2 = \Sigma(v < n) I_v,$$

where

$$I_v = [S]^2 \{\{f, g\} : f(v) \neq g(v)\} \quad \text{for} \quad v < n.$$

If $S' \subset S$ and $[S']^2 \subset I_v$, then

$$|S'| = |\{f(v) : f \in S'\}| \leqq |A_v| < b_v$$

which completes the proof of Theorem 8.

(∗) THEOREM 9. *If $3 \leqq b_0,, \hat{b}_n \leqq a$ and $a' < a < b_0 \ldots \hat{b}_n$, then*

(1) $$a^+ \nrightarrow (b_0,, \hat{b}_n)^2.$$

PROOF. *Case* 1. $|n| \geqq a$. Then by Theorem 8, $2^a \nrightarrow (3)^2_a$ and hence, a fortiori, (1).

*Case* 2. $|n| < a$. Put $m = \omega(a')$ and choose $a_0,, \hat{a}_m$ such that

$$a_0 < < \hat{a}_m < a = \sup(\mu < m) a_\mu.$$

We can find inductively $\{v_0,, \hat{v}_m\}_{\neq} \subset [0, n)$ such that $b_{v_\mu} > a_\mu$ for $\mu < m$. For let $\lambda < m$, and suppose that $v_\mu$ has been defined for $\mu < \lambda$, such that $v_\mu < n$ for $\mu < \lambda$. Then there is $v_\lambda \in [0, n) - \{v_0,, \hat{v}_\lambda\}$ such that $b_{v_\lambda} > a_\lambda$, since otherwise we would obtain the contradiction

$$a^+ \leqq \Pi(v < n) b_v \leqq \left(\Pi(\mu < \lambda) b_{v_\mu}\right) a_\lambda^{|n|} \leqq a^{|\lambda|} a_\lambda^{|n|} \leqq a.$$

This completes the inductive definition of $v_0,, \hat{v}_m$. By Theorem 8,

$$a_0 \dots \hat{a}_m \nrightarrow (b_{v_\mu})^2_{\mu < m}.$$

Since $a_0 \dots \hat{a}_m = a^+$, the relation (1) follows, and Theorem 9 is proved.

($*$) COROLLARY 4. $a^+ \nrightarrow (a)^2_{a'}$ *if* $a > a'$.

($*$) THEOREM 10. *Let* $a > a' > \aleph_0$. *Then* $a^+ \nrightarrow (a^+, (3)_{a'})^2$.

PROOF. We shall apply the following theorem of ERDŐS and FODOR [6]: Let $\aleph_0 \leqq b + c < a$; $|S| = a$,

$$x \notin f(x) \subset S \text{ and } |f(x)| < b \text{ for } x \in S;$$

$$S_\mu \in [S]^a \text{ for } \mu < \omega(c).$$

Then there is a set $S' \subset S$ such that $S' f(S') = \emptyset$ and $|S' S_\mu| = a$ for $\mu < \omega(c)$. In [6] this is proved in the special case when $S_\mu S_v = \emptyset$ for $\mu < v < \omega(c)$ but the general case then follows since quite generally, whenever $|T_\mu| = a \geqq \aleph_0$ for $\mu < \omega(a)$, there are sets $T'_\mu \in [T_\mu]^a$ for $\mu < \omega(a)$ such that $T'_\mu T'_v = \emptyset$ for $\mu < v < \omega(a)$.

Now let $l = \omega(a')$; $m = \omega(a)$; $n = \omega(a^+)$; $S = [0, n)$. Then $l < m < n$, and we can write $[S]^a = \{A_0,, \hat{A}_n\}$. Put $\mathbf{K}_\varrho = \{A_v : v < \varrho \wedge A_v \subset [0, \varrho)\}$ for $\varrho < n$. Then

$$[S]^a = \Sigma(\varrho < n) \mathbf{K}_\varrho; \quad |\mathbf{K}_\varrho| \leqq a \text{ for } \varrho < n. \tag{2}$$

Let $a_0 < < \hat{a}_l < a = \sup(\lambda < l) a_\lambda$. Then there are sets $\mathbf{K}_{\varrho\lambda}$ such that

$$\mathbf{K}_\varrho = \Sigma(\lambda < l) \mathbf{K}_{\varrho\lambda}; \quad |\mathbf{K}_{\varrho\lambda}| \leqq a_\lambda \text{ for } \varrho < n; \lambda < l. \tag{3}$$

Let $\lambda$ be fixed, $\lambda < l$. We define $f_\lambda(\varrho)$ by induction over $\varrho$. Put $f_\lambda(\varrho) = \emptyset$ for $\varrho < m$. Now let $m \leqq \sigma < n$, and suppose that $f_\lambda(\varrho)$ has been defined for $\varrho < \sigma$ in such a way that

$$\varrho \notin f_\lambda(\varrho) \subset S; \quad |f_\lambda(\varrho)| < a_{\lambda+1} \text{ for } \varrho < \sigma.$$

Then $|\sigma| = a$, and by the theorem of ERDŐS and FODOR, applied to the set $[0, \sigma)$, there is $F_{\sigma\lambda}$ such that

$$F_{\sigma\lambda} \subset [0, \sigma); \quad |F_{\sigma\lambda}| \leqq a_\lambda; \quad F_{\sigma\lambda} f_\lambda(F_{\sigma\lambda}) = \emptyset, \tag{4}$$

$$F_{\sigma\lambda} A_v \neq \emptyset \text{ if } A_v \in \mathbf{K}_{\sigma\lambda}. \tag{5}$$

We put $f_\lambda(\sigma)=F_{\sigma\lambda}$. Then $\sigma\notin f_\lambda(\sigma)\subset S$; $|f_\lambda(\sigma)|\leqq a_\lambda<a_{\lambda+1}$. This completes the definition of $f_\lambda(\varrho)$ for $\lambda<l$; $\varrho<n$. Now we have

$$[S]^2 = I+'\Sigma(\lambda<l)I_\lambda,$$

where

$$I_\lambda = \{\{\varrho,\sigma\}:\varrho<\sigma<n\wedge\varrho\in f_\lambda(\sigma)\}\qquad \text{for } \lambda<l.$$

1. Let $S'\subset S$; $|S'|=a^+$. Then there is $\varrho<n$ such that $A_\varrho\subset S'$. Now there is $\sigma\in S'$ such that $\sigma>\varrho$ and $A_\varrho\subset[0,\sigma)$. Then $\sigma\geqq m$. By (3) there is $\lambda<l$ such that $A_\varrho\in\mathbf{K}_{\sigma\lambda}$. By (5), $f_\lambda(\sigma)A_\varrho = F_{\sigma\lambda}A_\varrho\neq\emptyset$, and there is $\tau\in f_\lambda(\sigma)A_\varrho$. Then $\tau\in A_\varrho\subset S'$; $\{\tau,\sigma\}_<\in I_\lambda$; $[S']^2\not\subset I$; $a^+\notin[I]_2$.

2. Let $\lambda<l$ and $[\{\varrho'',\varrho',\varrho\}_<]^2\subset I_\lambda$. Then $\{\varrho'',\varrho'\}\subset f_\lambda(\varrho)=F_{\varrho\lambda}$; $\varrho''\in f_\lambda(\varrho')$ and therefore $\varrho''\in F_{\varrho\lambda}f_\lambda(F_{\varrho\lambda})$ which contradicts (4). Hence $3\notin[I_\lambda]_2$, and Theorem 10 follows.

## 13. COUNTER EXAMPLES FOR $r\geqq 3$. PRELIMINARIES

**13. 1.** The positive and negative results proved so far enable us to give an almost complete discussion for the case $r=2$. This will be done in section 15. Lemma 2, the stepping-up lemma, gives us a method to obtain positive relations for $r\geqq 3$. This seems to be the only method for proving positive relations in these cases. Thus our aim would be to prove a converse of Lemma 2 i. e. to show that for $r\geqq 2$ and $a\geqq\aleph_0$ the relation $a\nrightarrow(b_\nu)^r_{\nu<n}$ implies $2^a\nrightarrow(b_\nu+1)^{r+1}_{\nu<n}$. However, we can prove this only under some restrictions and using different methods to cover the various cases. We are now going to prove several lemmas which assure this implication under various conditions.

First we need some more definitions and some preliminary results concerning the set $V(\alpha)$.

**13. 2.** Let $\alpha$ be fixed. For $\{x,y\}_\prec\subset V(\alpha)$ we put $\eta(x,y)=0$ if $x<y$, and $\eta(x,y)=1$ if $x>y$. Thus, in the notation of 11. 4,

$$\{x,y\}\in T_{\eta(x,y)} \text{ if } \{x,y\}_\prec\subset V(\alpha).$$

If $r\geqq 3$ and $\{x_0,,x_{r-1}\}_\prec\subset V(\alpha)$, we put

$$\eta(x_0,,x_{r-1}) = \big(\eta(x_0,x_1),,\eta(x_{r-2},x_{r-1})\big).$$

Let $r\geqq 3$; $1\leqq s\leqq r-1$; $k_0,,k_{s-1}<2$. Denote by

$$K_{k_0,,k_{s-1}}(\alpha,r)$$

the set of all sets $\{x_0,,x_{r-1}\}_\prec\subset V(\alpha)$ such that

$$\eta(x_0,,x_{r-1})=(k_0,,k_{r-2}) \text{ for some } \hat{k}_{s-1},,\hat{k}_{r-1}.$$

Thus $K_{010}(\alpha,r)$ is defined for $r\geqq 4$ and is the set of all sets $\{x_0,,x_{r-1}\}_\prec\subset V(\alpha)$ such that $x_0<x_1>x_2<x_3$. The symbol $K_{k,,k}$ stands for $K_{k_0,,k_{r-2}}$ where $k_0==$ $==k_{r-2}=k$ We put

$$K(\alpha,r) = K_{0,,0}(\alpha,r)+K_{1,,1}(\alpha,r).$$

We have the following simple result. If $X \subset V(\alpha)$, and if tp $(X, \prec)$ is a limit number then, for every choice of $s, k_0,, k_{s-1}$ with $1 \leqq s \leqq r-1$, the relation

$$[X]^r \subset K_{k_0,,k_{s-1}}(\alpha, r)$$

holds if and only if, $k_0 = = k_{s-1}$ and $[X]^r \subset K_{k_0}$. This follows at once if one consideres any set $\{x_0,, x_r\}_\prec \subset X$ and the meaning of the satement

$$\{x_0,, x_{r-1}\}, \{x_1,, x_r\} \in K_{k_0,,k_{s-1}}(\alpha, r).$$

See also 13. 5.

**13. 3.** Let $\{x_0,, x_{r-1}\}_\prec \in K(\alpha, r)$. Then $x_\varrho x_{\varrho+1} \neq x_{\varrho+1} x_{\varrho+2}$ for $\varrho < r-2$. For we have either $x_0 << x_{r-1}$ or $x_0 >> x_{r-1}$, and in either case the assertion follows from 11. 6 (ii).

**13. 4.** For $\delta_0 < \delta_1 < \omega_\alpha$ we put $\xi(\delta_0, \delta_1) = 0$ and $\xi(\delta_1, \delta_0) = 1$. If $r \geqq 3$ and $\delta_0,, \delta_{r-1} < \omega_\alpha$; $\delta_\varrho \neq \delta_{\varrho+1}$ for $\varrho < r-1$, we put

$$\xi(\delta_0,, \delta_{r-1}) = (\xi(\delta_0, \delta_1),, \xi(\delta_{r-2}, \delta_{r-1})).$$

Now let $1 \leqq s \leqq r-2$ and $k_0,, k_{s-1} < 2$. Denote by

$$P_{k_0,,k_{s-1}}(\alpha, r)$$

the set of all sets $\{x_0,, x_{r-1}\}_\prec \subset K(\alpha, r)$ for which

$$\xi(\delta(x_0,, x_{r-1})) = (k_0,, k_{r-3}) \text{ for some } \hat{k}_{s-1},, \hat{k}_{r-2}.$$

We note that $\xi(\delta(x_0,, x_{r-1}))$ exists by 13. 3.

Thus $P_{010}(\alpha, r)$ is defined for $r \geqq 5$ and denotes the set of all sets $\{x_0,, x_{r-1}\}_\prec \subset \subset V(\alpha)$ such that

(i) either $x_0 << x_{r-1}$ or $x_0 >> x_{r-1}$

and

(ii) $x_0 x_1 < x_1 x_2 > x_2 x_3 < x_3 x_4$.

Throughout the rest of this paper whenever the arguments of any of the sets $V, V', K, P, K_{k_0,,k_{s-1}}, P_{k_0,,k_{s-1}}$ are $\alpha, r$ they will not be shown.

The symbol $P_{k,,k}$ stands for $P_{k_0,,k_{r-3}}$ where $k_0 = = k_{r-3} = k$. Put $P = P_{0,,0} + + P_{1,,1}$. We have $P_{k_0,,k_{s-1}} \subset K$ for $1 \leqq s \leqq r-2$ and $k_0,, k_{s-1} < 2$.

We shall now deduce some properties of the sets $K_{k_0,,k_{s-1}}$ and $P_{k_0,,k_{s-1}}$.

**13. 5.** Let $r \geqq 3$ and $X \subset V$. Then $|X| < r+1$ provided at least one of the following conditions (a), (b), (c) holds:

(a) $[X]^r \subset K_{01}$;

(b) $[X]^r \subset K_{10}$;

(c) $[X]^r \subset K_{k_0,,k_{r-2}}$ for some $(k_0,, k_{r-2}) \neq (k_0,, k_0)$.

PROOF. Let $\{x_0,, x_r\}_\prec \subset X$; $\eta_\varrho = \eta(x_\varrho, x_{\varrho+1})$ for $\varrho < r$. In case (a) we have $\{x_0,, x_{r-1}\} \in K_{01}$ and hence $\eta_0 = 0$; $\eta_1 = 1$. But we also have $\{x_1,, x_r\} \in K_{01}$ and hence $\eta_1 = 0$; $\eta_2 = 1$. This is a contradiction. In case (b) we find, in the same way, $\eta_0 = 1$; $\eta_1 = 0$ and also $\eta_1 = 1$; $\eta_2 = 0$, i. e. a contradiction. In case (c) we have at the same time $\eta_\varrho = k_\varrho$ and $\eta_{\varrho+1} = k_\varrho$ for all $\varrho \leqq r-2$. These equations imply $k_0 = = = k_{r-2}$ which is the desired contradiction.

**13. 6.** (a). Let either (i) $r \geqq 3$ and $[X]^r \subset K$ or (ii) $r \geqq 4$ and $|X| \geqq r+1$; $[X]^r \subset \subset K_{010}+K$. Then $[X]^r \subset K_{s,,s}$ for some $s<2$.

(b) Let $r \geqq 4$ and $[X]^r \subset P$. Then $[X]^r \subset P_{t,,t}$ for some $t<2$.

PROOF. We may assume $|X| \geqq r+1$.

*Proof of* (a). Let $[X]^r \not\subset K_{s,,s}$ for $s<2$. Then there are sets $A_0, A_1 \in [X]^r$ such that $A_s \notin K_{s,,s}$ for $s<2$. We can choose $x \in X-A_0$. Put $\{x\}+A_0+A_1 = \{x_0,, x_{n-1}\}_<$. Then $n \geqq r+1$. Put $\eta(x_\nu, x_{\nu+1}) = \eta_\nu$ for $\nu \leqq n-2$.

*Case* 1. $\{x_\nu,, x_{\nu+r-1}\} \in K$ for $\nu+r-1 \leqq n-1$. Then $\eta_\nu = = \eta_{\nu+r-2}$ and, since $|[\nu, \nu+r-1)[\nu+1, \nu+r)| = r-2 \geqq 1$, we have $\eta_0 = = \eta_{n-2}$. We deduce that $A_0, A_1 \in K_{\eta_0,,\eta_0}$ which is a contradiction.

*Case* 2. There is $\nu_0 \leqq n-r$ with $\{x_{\nu_0},, x_{\nu_0+r-1}\} \notin K$. Then (ii) holds, and $\{x_{\nu_0},, x_{\nu_0+r-1}\} \in K_{010}$. If $\nu_0=0$ then $\{x_{\nu_0+1},, x_{\nu_0+r}\} \notin K_{010}+K$, and if $\nu_0 \geqq 1$ then $\{x_{\nu_0-1},, x_{\nu_0+r-2}\} \notin K_{010}+K$. In either of these cases we obtain a contradiction against the hypothesis.

*Proof of* (b). We have $[X]^r \subset P \subset K$. Hence, by (a), there is $s<2$ with $[X]^r \subset \subset K_{s,,s}$. Let us assume that $[X]^r \not\subset P_{t,,t}$ for $t<2$. Then there are sets $A_0, A_1 \in [X]^r$ such that $A_t \notin P_{t,,t}$ for $t<2$. Let $A_0+A_1 = \{x_0,, x_{n-1}\}_<$. Then $n \geqq r$, and we have either $x_0 < < x_{n-1}$ or $x_0 > > x_{n-1}$. Also, $\{x_\nu,, x_{\nu+r-1}\} \in P_{t_\nu,,t_\nu}$ for $\nu+r-1 \leqq n-1$. Put $\xi(x_\nu x_{\nu+1}, x_{\nu+1}x_{\nu+2}) = \xi_\nu$ for $\nu+2 \leqq n-1$. Then $\xi_\nu = = \xi_{\nu+r-3} = t_\nu$, say, for $\nu \leqq n-r$. Since $r \geqq 4$ these equations imply $t_0 = = t_{n-r}$, and we obtain the contradiction $A_0, A_1 \in P_{t_0,,t_0}$. This completes the proof.

**13. 7.** Let $X \subset V$ and $|X| = b \geqq \aleph_0$. Then there are a set $X' \in [X]^b$ and a number $t<2$ with $[X']^r \subset K_{t,,t}$, provided at least one of the following conditions (a), (b), (c) holds:

(a) $r \geqq 3$ and $[X]^r K_{01} = \varnothing$;

(b) $r \geqq 3$ and $[X]^r K_{10} = \varnothing$;

(c) $r \geqq 4$ and $[X]^r K_{010} = \varnothing$.

PROOF. Suppose there are no $X'$ and $t$ with the required properties. We may assume that $X = \{x_0,, \hat{x}_n\}_<$ where $n = \omega(b)$. Put $[\mu, \nu] = \eta(x_\mu, x_\nu)$ for $\mu < \nu < n$. We show that

(1) there is $\{\nu_0, \nu_1, \nu_2, \nu_3\}_< \subset [0, n)$ with $[\nu_0, \nu_1] = 0$; $[\nu_1, \nu_2] = 1$; $[\nu_2, \nu_3] = 0$.

Let us assume that (1) is false. Then we have the following two cases (all ordinals are in $[0, n)$):

*Case* 1. There are $\mu < \nu < \lambda$ with $[\mu, \nu] = 0$; $[\nu, \lambda] = 1$. Then there are $\varrho, \sigma$ with $\mu < \nu < \lambda < \varrho < \sigma$ and $[\varrho, \sigma] = 0$. But then $[\lambda, \varrho] = 1$, and we have $[\mu, \nu] = 0$; $[\nu, \varrho] = 1$; $[\varrho, \sigma] = 0$ which is impossible.

*Case* 2. Whenever $\mu < \nu < \lambda$ and $[\mu, \nu] = 0$, then $[\nu, \lambda] = 0$. Then there are $\mu < \nu$ with $[\mu, \nu] = 0$. Now there are $\lambda, \varrho$ with $\mu < \nu < \lambda < \varrho$ and $[\lambda, \varrho] = 1$. But then $[\nu, \lambda] = 0$, and hence $[\lambda, \varrho] = 0$ which is a contradiction. This proves (1) so that there are $\nu_0, \nu_1, \nu_2, \nu_3$ such that the conditions in (1) hold. If $r \geqq 4$ then we can choose $\nu_4,, \nu_r$ such that $\nu_0 < < \nu_r < n$. Then all three conditions (a), (b), (c) are violated since

$$\{x_{\nu_0},, x_{\nu_{r-1}}\} \in K_{01}; \ \{x_{\nu_1},, x_{\nu_r}\} \in K_{10}$$

and, if $r \geqq 4$, then $\{x_{\nu_0},, x_{\nu_{r-1}}\} \in K_{010}$. This shows that our first assumption was false so that there are $X'$ and $t$ such that the required conditions hold.

**13. 8.** Let $r \geqq 3$; $X \subset V$; $|X| \geqq b = b'$; $[X]^r \subset K_{t,,t}$; $\alpha \geqq 0$. Then there is $X' \in [X]^b$ with $[X']^r \subset P_{0,,0}$.

PROOF. We may assume $X = \{x_0,, \hat{x}_n\}_<$, where $n = \omega(b)$. Then

(2) $$\text{either } x_0 << \hat{x}_n \text{ or } x_0 >> x_n.$$

We define inductively numbers $\mu_0,, \hat{\mu}_n, \delta_0,, \hat{\delta_n}$. Let $\sigma < n$, and suppose that $\mu_0,, \hat{\mu}_\sigma, \delta_0,, \hat{\delta}_\sigma$ have been defined such that $\mu_0 << \hat{\mu}_\sigma < n$ and $\delta_0 << \hat{\delta}_\sigma < \omega_\alpha$. Put

(3) $$A_\varrho = \{x_\nu: \nu > \mu_\varrho \wedge x_{\mu_\varrho} x_\nu \neq \delta_\varrho\} \quad \text{for} \quad \varrho < \sigma.$$

Let us suppose that

(4) $$|A_\varrho| < b \text{ for } \varrho < \sigma.$$

Put

(5) $$B_\sigma = A_0 + + \hat{A}_\sigma + \{x_\nu: (\exists \varrho)(\varrho < \sigma \wedge \nu \leqq \mu_\varrho)\}.$$

By (4) and $b = b'$ we have

(6) $$|B_\sigma| < b.$$

There is a least number $\bar{\mu}_\sigma < n$ such that

(7) $$B_\sigma \subset \{x_\nu: \nu < \bar{\mu}_\sigma\}.$$

Put

(8) $$D_\sigma = \{x_\mu x_\nu: \bar{\mu}_\sigma \leqq \mu < \nu < n\},$$

(9) $$\delta_\sigma = \min D_\sigma.$$

There is a least number $\mu_\sigma$ such that $\bar{\mu}_\sigma \leqq \mu_\sigma < n$ and $x_{\mu_\sigma} x_\nu = \delta_\sigma$ for at least one $\nu > \mu_\sigma$. There is a least number $\bar{\bar{\mu}}_\sigma > \mu_\sigma$ such that $x_{\mu_\sigma} x_{\bar{\bar{\mu}}_\sigma} = \delta_\sigma$. Put

$$A_\sigma = \{x_\nu: \mu_\sigma < \nu \wedge x_{\mu_\sigma} x_\nu \neq \delta_\sigma\}.$$

If $\varrho < \sigma$ then $x_{\mu_\varrho} \in B_\sigma$, and $\mu_\varrho < \bar{\mu}_\sigma \leqq \mu_\sigma$ by (7). We have

(10) $$x_{\mu_\sigma} x_\nu = \delta_\sigma \text{ for } \nu \geqq \bar{\bar{\mu}}_\sigma.$$

For let $\nu \geqq \bar{\bar{\mu}}_\sigma$, and put $x_{\mu_\sigma} = x$; $x_{\bar{\bar{\mu}}_\sigma} = y$; $x_\nu = z$. Then $\bar{\mu}_\sigma \leqq \mu_\sigma < \bar{\bar{\mu}}_\sigma \leqq \nu$; $xy = \delta_\sigma$; $xz \in D_\sigma$. Hence, by (9), $xz \geqq \delta_\sigma$. Thus if (10) were false then $xz > \delta_\sigma$. Then $\bar{\bar{\mu}}_\sigma < \nu$ and $xy < xz$. On the other hand we have, by (2), either $x < y < z$ or $x > y > z$. Hence, by 11. 6 (ii), $xz = \min(xy, yz) \leqq xy$ which is a contradiction. This proves (10). It now follows from (10) that $|A_\sigma| < b$. Finally we note that if $\varrho < \sigma$ then, by (5), $x_{\mu_\varrho} \in B_\sigma$ and hence, by (7), $\mu_\varrho < \bar{\mu}_\sigma \leqq \mu_\sigma$. This completes the inductive definition of $\mu_0,, \hat{\mu}_n, \delta_0,, \hat{\delta}_n$ such that $\mu_0 << \hat{\mu}_n$. Put $x_{\mu_\nu} = y_\nu$ for $\nu < n$. Then

(11) $$y_\varrho y_\sigma = \delta_\varrho \text{ for } \varrho < \sigma.$$

For let $\varrho < \sigma$. If $y_\sigma \in B_\sigma$ then, by (7), $\mu_\sigma < \bar{\mu}_\sigma$ which is false. Hence $y_\sigma \notin B_\sigma$, and it follows from (5) that $y_\sigma \notin A_\varrho$. Since $\mu_\sigma > \mu_\varrho$ we conclude from (3) that (11) holds. We now assert that

(12) $$\delta_\varrho < \delta_\sigma \text{ for } \varrho < \sigma.$$

To see this, let $\varrho < \sigma$ and $\delta_\varrho \geqq \delta_\sigma$. Then, by (11), $y_\varrho y_\sigma = \delta_\varrho$ and $y_\sigma y_{\sigma+1} = \delta_\sigma$. By (2), either $y_\varrho < y_\sigma < y_{\sigma+1}$ or $y_\varrho > y_\sigma > y_{\sigma+1}$. In either case it follows from (11) and 11. 6 (ii) that

$$\delta_\varrho = y_\varrho y_{\sigma+1} = \min(y_\varrho y_\sigma, y_\sigma y_{\sigma+1}) = \min(\delta_\varrho, \delta_\sigma) = \delta_\sigma$$

and $y_\varrho y_\sigma = y_\sigma y_{\sigma+1}$ which contradicts 11. 6 (ii). This proves (12). Put $X' = \{y_0,, \hat{y}_n\}$. Then

(13) $$|X'| = b;\ [X']^r \subset [X]^r \subset K_{t,,t} \subset K.$$

Let $\sigma_0 < < \sigma_{r-1}$ and $A = \{y_{\sigma_0},, y_{\sigma_{r-1}}\}$. Then, by (11) and (12), $y_{\sigma_\varrho} y_{\sigma_{\varrho+1}} = \delta_{\sigma_\varrho} < < \delta_{\sigma_{\varrho+1}} = y_{\sigma_{\varrho+1}} y_{\sigma_{\varrho+2}}$ for $\varrho + 2 \leqq r-1$ so that $\xi(y_{\sigma_\varrho} y_{\sigma_{\varrho+1}}, y_{\sigma_{\varrho+1}} y_{\sigma_{\varrho+2}}) = 0$ for $\varrho \leqq r-3$. This, together with (13), shows that $[X']^r \subset P_{0,,0}$ and proves that $X'$ has the required properties.

**13. 9.** Let $r \geqq 3$; $X \subset V$; $|X| \geqq r+1$; $[X]^r \subset P_{k_0,, k_{r-3}}$. Then $k_0 = = k_{r-3}$.

Proof. There is a set $\{x_0,, x_r\}_< \subset X$. Put $\delta_\varrho = x_\varrho x_{\varrho+1}$ for $\varrho \leqq r-1$, and $\xi_\varrho = \xi(\delta_\varrho, \delta_{\varrho+1})$ for $\varrho \leqq r-2$. Then we have, by definition of $P_{k_0,, k_{r-3}}$, $(\xi_0,, \xi_{r-3}) = = (\xi_1,, \xi_{r-2}) = (k_0,, k_{r-3})$ and hence $k_0 = = k_{r-3}$.

**13. 10.** Let $r \geqq 4$; $X \subset V$; $[X]^r \subset P$. Then there is $t < 2$ with $[X]^r \subset P_{t,,t}$.

Proof. Let $A, B \in [X]^r$. It suffices to show that there is $t < 2$ such that $A, B \in P_{t,,t}$. Let $A + B = \{x_0,, x_{p-1}\}_<$. Then $p \geqq r$ and $\{x_\pi,, x_{\pi+r-1}\} \in P_{s_\pi,, s_\pi}$ for $\pi \leqq p-r$. Since $\{x_{\pi+1},, x_{\pi+3}\} \subset \{x_\pi,, x_{\pi+r-1}\}\{x_{\pi+1},, x_{\pi+r}\}$ for $\pi \leqq p-r-1$ it follows from the definition of $P_{s_\pi,, s_\pi}$ that $s_0 = = s_{p-r} = s$, say. Put $\delta_\nu = x_\nu x_{\nu+1}$ for $\nu \leqq p-2$.

*Case* 1. $s = 0$. Then $\delta_0 < < \delta_{p-2}$, and repeated application of 11. 6 (ii) yields $x_\mu x_\nu = \delta_\mu$ for $\mu < \nu < p$. Hence $A, B \in P_{0,,0}$.

*Case* 2. $s = 1$. Then $\delta_0 > > \delta_{p-2}$. Then, similarly, $A, B \in P_{1,,1}$.

**13. 11.** Let $r \geqq 4$; $X \subset V$; $|X| = b \geqq \aleph_0$; $[X]^r \subset K_{t,,t}$. Then there is $X' \in [X]^b$ with $[X']^r \subset P_{0,,0}$ provided at least one of the following conditions (a), (b), (c), (d) holds:

(a) $[X]^r P_{01} = \emptyset$;

(b) $[X]^r P_{10} = \emptyset$;

(c) $r \geqq 5$ and $[X]^r P_{010} = \emptyset$;

(d) $r \geqq 5$ and $[X]^r P_{101} = \emptyset$.

Proof. We may assume $X = \{x_0,, \hat{x}_n\}_<$, where $n = \omega(b)$. In what follows we always suppose the letters $u, v, w, y, z$ to denote elements of $X$. We shall use freely, without reference, 11. 6 (ii).

*Case* 1. $t = 0$. Then $x_0 < < \hat{x}_n$. Let $s \geqq 3$, and suppose that $y_0 < < y_{s-1}$, and $\xi(y_\nu y_{\nu+1}, y_{\nu+1} y_{\nu+2}) \neq \xi(y_{\nu+1} y_{\nu+2}, y_{\nu+2} y_{\nu+3})$ for $\nu < s-3$. This is for instance true for $s = 3$ and arbitrary $\{y_0, y_1, y_2\}_<$. We assert that then $s \leqq 5$. For suppose $s \geqq 6$. Then we can choose $z_4,, z_r$ with $y_4 < z_4 < < z_r$. Then there is $\lambda < 2$ with

$\{y_0,, y_3, z_4,, z_{r-1}\} \in P_{\lambda,1-\lambda}$; $\{y_1,, y_4, z_5,, z_r\} \in P_{1-\lambda,\lambda}$. Hence both (a) and (b) are false, and $r \geqq 5$. Now there is $\mu < 2$ with

$$\{y_0,, y_4, z_5,, z_{r-1}\} \in P_{\mu,1-\mu,\mu}; \{y_1,, y_5, z_6,, z_r\} \in P_{1-\mu,\mu,1-\mu},$$

so that both (c) and (d) are false as well. This is impossible. Hence $3 \leqq s \leqq 5$. We now suppose that $s$ has its largest possible value for the given $X$. Put

$$y_{s-3} = u;\ \ y_{s-2} = v;\ \ y_{s-1} = w;\ \ X' = \{y:\ y > w\}.$$

Then $X' \in [X]^b$. Let $\{w_0, w_1, w_2\}_< \subset X'$.

*Case* 1a. $uv > vw$. Then, by the maximality of $s$, whenever $u < u_0 < u_1 < u_2$, then not all the relations $uu_0 > u_0u_1 < u_1u_2$ hold. By using this repeatedly we conclude: $uv > vw > ww_0$; $vw_0 = ww_0$; $uv > vw_0 > w_0w_1$; $vw_1 = w_0w_1$; $uv > vw_1 > w_1w_2$; $w_0w_1 > w_1w_2$. But since $b \geqq \aleph_0$ we can choose $\{w, w_0,, \hat{w}_\omega)_<$. Then, by what has just been proved, we have $w_\nu w_{\nu+1} > w_{\nu+1}w_{\nu+2}$ for $\nu < \omega$ which is impossible.

*Case* 1b. $uv < vw$. Then, by the maximality of $s$, whenever $u < u_0 < u_1 < u_2$, not all the relations $uu_0 < u_0u_1 > u_1u_2$ hold. By using this repeatedly we conclude: $uv < vw < ww_0$; $vw_0 = vw$; $uv < vw_0 < w_0w_1$; $uw_0 = uv$; $uw_0 < w_0w_1 < w_1w_2$. Hence $[X']^r \subset P_{0,,0}$.

*Case* 2. $t = 1$. Then the same proof as in the Case 1 applies except that every inequality $x < y$ between elements of $X$ is reversed. This completes the proof.

**13. 12.** If $r \geqq 3$ and $[X]^r \subset P_{1,,1}$, then $|X| < \aleph_0$.

PROOF. If $|X| \geqq \aleph_0$, then we can choose $\{x_0,, \hat{x}_\omega\}_< \subset X$. Then, by definition of $P_{1,,1}$, $x_\nu x_{\nu+1} > x_{\nu+1}x_{\nu+2}$ for $\nu < \omega$ which is impossible.

**13. 13.** Let $r \geqq 5$; $X \subset V$; $|X| \geqq r+1$; $s < 2$;

$$[X]^r \subset K_{s,,s}(P + P_{010}).$$

Then there is $t < 2$ such that $[X]^r \subset P_{t,,t}$.

PROOF. Assume there is no such $t$. Then there are elements $x_0,, x_{r-1}, y_0,, y_{r-1} \in X$ such that $\{x_0,, x_{r-1}\}_< \notin P_{0,,0}$ and $\{y_0,, y_{r-1}\}_< \notin P_{1,,1}$. We can choose $x_r \in X - \{x_0,, x_{r-1}\}$. Let $\{x_0,, x_r, y_0,, y_{r-1}\} = \{z_0,, z_{n-1}\}_<$. Then $n \geqq r+1$; $x_\varrho = z_{\alpha(\varrho)}$; $y_\varrho = z_{\beta(\varrho)}$; $\alpha(\varrho), \beta(\varrho) < n$ for $\varrho < r$. We have either $z_0 << z_{n-1}$ or $z_0 >> z_{n-1}$. We have exactly one of the following three cases:

*Case* 1. $\{z_0,, z_{r-1}\} \in P_{0,,0}$. Then $z_0z_1 << z_{r-2}z_{r-1}$. Since $\{z_\nu,, z_{\nu+r-1}\} \in P_{0,,0} + P_{1,,1} + P_{010}$ for $\nu \leqq n-r$, it follows for $\nu = 1,, n-r$, in this order, that $\{z_\nu,, z_{\nu+r-1}\} \in P_{0,,0}$. Hence $z_0z_1 << z_{n-2}z_{n-1}$; $x_\varrho x_{\varrho+1} = z_{\alpha(\varrho)}z_{\alpha(\varrho+1)} = z_{\alpha(\varrho)}z_{\alpha(\varrho)+1} < z_{\alpha(\varrho+1)}z_{\alpha(\varrho+1)+1} = x_{\varrho+1}x_{\varrho+2}$ for $\varrho \leqq r-3$; $\{x_0,, x_{r-1}\} \in P_{0,,0}$ which is a contradiction.

*Case* 2. $\{z_0,, z_{r-1}\} \in P_{1,,1}$. Then $z_0z_1 >> z_{r-2}z_{r-1}$; $\{z_\nu,, z_{\nu+r-1}\} \in P_{1,,1}$ for

$$\nu \leqq n-r;\ \ z_0z_1 >> z_{n-2}z_{n-1};\ \ y_\varrho y_{\varrho+1} = z_{\beta(\varrho)}z_{\beta(\varrho+1)} = z_{\beta(\varrho+1)-1}\, z_{\beta(\varrho+1)} >$$

$$> z_{\beta(\varrho+2)-1}\, z_{\beta(\varrho+2)} = y_{\varrho+1}y_{\varrho+2}$$

for $\varrho \leqq r-3$; $\{y_0,, y_{r-1}\} \in P_{1,,1}$ which is a contradiction.

*Case* 3. $\{z_0,, z_{r-1}\} \in P_{010}$. Then $z_0z_1 < z_1z_2 > z_2z_3 < z_3z_4$ and, clearly, $\{z_1,, z_r\} \notin P + P_{010}$ which is a contradiction. The assertion is therefore proved.

**13. 14.** Let $X \subset V$; $r \geqq 3$; $[X]^r \subset P_{s,,s}$; $|X| \geqq r$. Then there is $t<2$ with $[X]^r \subset K_{t,,t}$. Also, for $\{x, y, z\}_\prec \subset X$, if $s=0$ then $xy = xz < yz$, and if $s=1$ then $xy > xz = yz$.

PROOF. By 13. 6. there is $t<2$ with $[X]^r \subset K_{t,,t}$. There is $\{x_0,, x_{r-1}\}_\prec \subset X$ such that

$$x = x_\mu; \quad y = x_\nu; \quad z = x_\lambda \quad \text{for some} \quad \mu < \nu < \lambda < r.$$

Then either $x_0 << x_{r-1}$ or $x_0 >> x_{r-1}$. Let $\beta < \gamma < r$. If $s=0$, then $x_0 x_1 << \; << x_{r-2} x_{r-1}$ and hence

$$x_\beta x_\gamma = \min (x_\beta x_{\beta+1}, x_{\gamma-1} x_\gamma) = x_\beta x_{\beta+1}.$$

If $s=1$, then $x_0 x_1 >> x_{r-2} x_{r-1}$ and hence

$$x_\beta x_\gamma = \min (x_\beta x_{\beta+1}, x_{\gamma-1} x_\gamma) = x_{\gamma-1} x_\gamma.$$

These relations imply the assertion.

## 14. COUNTER EXAMPLES FOR $r \geqq 3$. LEMMAS

We begin with two negative relations.

THEOREM 11. $2^a \nrightarrow (a^+, r+1)^r$ *for* $a \geqq \aleph_0$; $r \geqq 3$.

PROOF. Let $a = \aleph_\alpha$. Then $[V]^r = I_0 +' I_1$, where $I_1 = K_{01}$.

1. Let $X \subset V$; $[X]^r \subset I_0$; $|X| \geqq \aleph_0$. Then, by 13. 7, there is $X' \subset X$ such that $|X'| = |X|$ and $[X']^r \subset K_{t,,t}$ for some $t$. Then $|X| = |X'| \leqq a$ by 11. 5 (i).

2. Let $X \subset V$ and $[X]^r \subset I_1$. Then, by 13. 5 (a), $|X| < r+1$. Since $|V| = 2^a$, Theorem 11 follows.

THEOREM 12. $a \nrightarrow (a, r+1)^r$, *if* $a > a'$ *and* $r \geqq 3$.

PROOF. Let $n = \omega(a')$; $|S| = a$; $S = \Sigma'(\nu < n) S_\nu$; $|S_\nu| < a$ for $\nu < n$. Then $[S]^r = \; = I_0 +' I_1$, where

$$I_1 = [S]^r \{A : (\exists \mu, \nu)(\mu < \nu < \eta \wedge |AS_\mu| = r-1 \wedge |AS_\nu| = 1)\}.$$

Then $X \in [S]^a$ implies $[X]^r I_1 \neq \emptyset$, and $Y \in [S]^{r+1}$ implies $[Y]^r I_0 \neq \emptyset$. This proves Theorem 12.

($*$) COROLLARY 5. *If* $r \geqq 3$; $a \geqq \aleph_0$, *and if a is not inaccessible then* $a \nrightarrow (a, r+1)^r$.

This follows from Theorem 11 and 12.

We are now going to discuss the various converses of Lemma 2 mentioned in 13. 1.

Throughout the proofs of Lemmas 5A—5F we put

$$\bar{I}_\nu = P_{0,,0} \{\{x_0,, x_{r-1}\}_\prec : \delta'(x_0,, x_{r-1}) \in I_\nu^*\} \text{ for } \nu < m.$$

In every case the sets $I_\nu^*$ will have been defined.

LEMMA 5A. *Let $a \geqq \aleph_0$, and let $b_0,, \hat{b}_m$ be cardinals such that*

(A) $\begin{cases} r \geqq 3; \text{ there are } \mu < \nu < m \text{ such that } b_\mu, b_\nu \geqq \aleph_0, \text{ and at least one of } b_\mu, b_\nu \\ \text{is regular.} \end{cases}$

*Let $a \nrightarrow (b_\nu)^{r-1}_{\nu<m}$. Then $2^a \nrightarrow (b_\nu + 1)^r_{\nu<m}$.*

PROOF. We may suppose $b_0,, \hat{b}_m \geqq r$. By 4. 1 we may assume $b_0 = b_0'$ and $b_1 \geqq \aleph_0$. Let $a = \aleph_\alpha$; $n = \omega_\alpha$; $S = [0, n)$. Then there is a partition $[S]^{r-1} = \Sigma'(\nu < m) I_\nu^*$ such that

(1) $$b_\nu \notin [I_\nu^*]_{r-1} \text{ for } \nu < m.$$

Then $[V]^r = \Sigma'(\nu < m) I_\nu$, where

(2) $$I_1 = K_{01} + \bar{I}_1; \quad I_\nu = \bar{I}_\nu \qquad (2 \leqq \nu < m).$$

Let $X \in [V]^{b_0}$; $[X]^r \subset I_0$. Then $[X]^r K_{01} = \emptyset$ and, by 13. 7 (a), there are a set $X' \in [X]^{b_0}$ and a number $t$ such that $[X']^r \subset K_{t,,t}$. By 13. 8 and $b_0 = b_0'$ there is $X'' \in [X']^{b_0}$ such that $[X'']^r \subset P_{0,,0}$. Then

(3) $$[X'']^r \subset P_{0,,0} I_0.$$

Next, let $Y \in [V]^{b_1}$; $[Y]^r \subset I_1$. Then $[Y]^r K_{10} = \emptyset$ and, by 13. 7 (b), there is $Y' \in [Y]^{b_1}$ such that $[Y']^r \subset K$. Then, by (2),

(4) $$[Y']^r \subset P_{0,,0} I_1.$$

It follows from (2), (3) and (4) that it is sufficient to prove that

(5) $$\text{if } X \subset V;\ \nu < m;\ [X]^r \subset P_{0,,0} I_\nu, \text{ then } |X| < b_\nu + 1.$$

Let us therefore assume that

$$X = \{x_0,, \hat{x}_k\}_< \subset V; \quad \nu < m; \quad [X]^r \subset P_{0,,0} I_\nu.$$

For future applications we note that the in rest of this proof we do not make any use of any hypothesis not mentioned in (5). Put $D = \{x_\varkappa x_{\varkappa+1} : \varkappa + 1 < k\}$. Then $x_\varkappa x_{\varkappa+1} < x_{\varkappa+1} x_{\varkappa+2}$ for $\varkappa + 2 < k$ and hence $|D| = |k \dot{-} 1|$. Let $E \in [D]^{r-1}$. Then there are $\varkappa_0 < < \varkappa_{r-1} < k$ such that $E = \{x_{\varkappa_\varrho} x_{\varkappa_\varrho+1} : \varrho < r-1\}$. Then, by 13. 14 and (2), $E = \{x_{\varkappa_\varrho} x_{\varkappa_\varrho+1} : \varrho < r-1\} = \delta'(x_{\varkappa_0},, x_{\varkappa_{r-1}}) \in I_\nu^*$. Hence $[D]^{r-1} \subset I_\nu^*$ and, by (1), $|k \dot{-} 1| = |D| < b_\nu$; $|X| = |k| < b_\nu + 1$. This proves Lemma 5A.

LEMMA 5B. *Let $a \geqq \aleph_0$; $m \geqq 2$, and let $b_0,, \hat{b}_m$ be such that*

(B) $$r \geqq 4; \text{ there is } \nu < m \text{ with } b_\nu = b_\nu'.$$

*Let $a \nrightarrow (b_\nu)^{r-1}_{\nu<m}$. Then $2^a \nrightarrow (b_\nu + 1)^r_{\nu-m}$.*

REMARK. For $r \geqq 4$ Lemma 5A follows from Lemma 5B.

PROOF. We may assume $b_0 = b_0'$ and $b_0,, \hat{b}_m \geqq r$. Just as in the proof of Lemma 5A, let $a = \aleph_\alpha$; $n = \omega_\alpha$; $S = [0, n)$. Then there is a partition $[S]^{r-1} = \Sigma'(\nu < m) I_\nu^*$ such that

(6) $$b_\nu \notin [I_\nu^*]_{r-1} \text{ for } \nu < m.$$

Then $[V]^r = \Sigma'(\nu < m) I_\nu$, where

(7) $$I_1 = K_{010} + \bar{I}_1; \quad I_\nu = \bar{I}_\nu \quad (2 \leqq \nu < m).$$

Let $X \in [V]^{b_0}$; $[X]^r \subset I_0$. Then $[X]^r K_{010} = \varnothing$ and, by 13. 7 (c), there are $X' \in [X]^{b_0}$ and $t$ such that $[X']^r \subset K_{t,,t}$. By 13. 8 and $b_0 = b_0'$ there is $X'' \in [X']^{b_0}$ such that $[X'']^r \subset P_{0,,0}$. Then (3) holds. Next, let $Y \in [V]^{b_1+1}$; $[Y]^r \subset I_1$. Then, by (7),

$$[Y]^r \subset K_{010} + K$$

and hence, by 13. 6 and $|Y| = b_1 + 1 \geqq r+1$, we have $[Y]^r \subset K$, and therefore (4) holds for $Y' = Y$. It follows from (7), (3) and (4) that it suffices to prove (5). This proof was given in the proof of Lemma 5A, so that Lemma 5B is established.

LEMMA 5C. *Let $a \geqq \aleph_0$; $m \geqq 2$, and let $b_0,, \hat{b}_m$ be such that*

(C) $$r \geqq 5; \text{ there is } v < m \text{ with } b_v \geqq \aleph_0.$$

*Let $a \nrightarrow (b_v)^{r-1}_{v<m}$. Then $2^a \nrightarrow (b_v+1)^r_{v<m}$.*

PROOF. We may assume $b_0 \geqq \aleph_0$ and $b_0,, \hat{b}_m \geqq r$. Define $\alpha, n, S, I_v^*$ as in the proof of Lemma 5A so that (6) holds. Then $[V]^r = \Sigma'(v<m) I_v$, where

(8) $$I_1 = K_{010} + P_{010} + \bar{I}_1; \quad I_v = \bar{I}_v \qquad (2 \leqq v < m).$$

Let $X \in [V]^{b_0}$; $[X]^r \subset I_0$. Then $[X]^r K_{010} = \varnothing$ and, as in the proof of Lemma 5B, there is $X'' \in [X]^{b_0}$ such that (3) holds. Next, let $Y \in [V]^{b_1+1}$; $[Y]^r \subset I_1$. Then, by (8), $[Y]^r \subset K_{010} + K$ and hence, by 13. 6 and $b_1 + 1 \geqq r+1$, $[Y]^r \subset K$. It now follows from (8) that $[Y]^r \subset P_{010} + P_{0,,0}$. We deduce from 13. 6 (a) (i) and 13. 13 that $[Y]^r \subset P_{0,,0}$. Therefore, again, (4) holds for $Y' = Y$. It follows from (8), (3) and (4) that we only need to prove (5), and the proof of Lemma 5A applies. This proves Lemma 5C.

LEMMA 5D. *Let $r \geqq 4$ and $a \geqq \aleph_0$. Let $b_0,, \hat{b}_m$ be cardinals, and put*

$$c_r = 2^{r-1} + 2^{r-2} - 4.$$

*Let $a \nrightarrow (b_v)^{r-1}_{v<m}$. Then $2^a \nrightarrow ((r+1)_{c_r}, (b_v+1)_{v<m})^r$.*

PROOF. Define $\alpha, n, S, I_v^*$ as in the proof of Lemma 5A so that (6) holds. Let $2 \leqq p < \omega$. Denote by $\varepsilon_p(0),, \varepsilon_p(2^p-3)$ all systems $(\sigma_0,, \sigma_{p-1})$ with $\sigma_0,, \sigma_{p-1} < 2$ except the two systems $(0,, 0)$, $(1,, 1)$. Then

$$[V]^r = \Sigma'(i < 2^{r-1}-2) I_i' +' \Sigma'(j < 2^{r-2}-2) I_j'' +' \Sigma'(v<m) I_v,$$

where the $I_i', I_j'', I_v$ are defined by the following rules. Let $A = \{x_0,, x_{r-1}\}_< \subset V$.

(9) If $A \notin K$, so that $\eta(x_0,, x_{r-1}) = \varepsilon_{r-1}(i)$ for some $i < 2^{r-1}-2$, then $A \in I_i'$.

(10) If $A \in K - P$, so that $\xi(\delta(x_0,, x_{r-1})) = \varepsilon_{r-2}(j)$ for some $j < 2^{r-2}-2$, then $A \in I_j''$.

(11) If $A \in P$, so that $\delta'(x_0,, x_{r-1}) \in I_v^*$ for some $v < m$, then $A \in I_v$.

Now let $X' \subset V$; $i < 2^{r-1}-2$; $[X']^r \subset I_i'$. Then $[X']^r \subset K_{\varepsilon_{r-1}(i)}$ by (9). Hence, by 13. 5 (c),

(12) $$|X'| < r+1.$$

Next, let $X'' \subset V$; $j < 2^{r-2}-2$; $[X'']^r \subset I''_j$. Then $[X'']^r \subset P_{\varepsilon_{r-2}(j)}$ by (10) and hence, by 13. 9,

(13) $$|X''| < r+1.$$

Finally, let $X \in [V]^{b_\nu+1}$; $[X]^r \subset I_\nu$; $\nu < m$. Then, by (11), $[X]^r \subset P$ and hence, by 13. 6 (b), there are $s, t < 2$ such that

(14) $$[X]^r \subset K_{s,,s} P_{t,,t}.$$

It follows from (12), (13) and (14) that it suffices, again, to prove (5), and this proof was given in the proof of Lemma 5A. This establieshes Lemma 5D.

LEMMA 5D′. *Let* $|S| = a \geqq \aleph_0$; $|T| = 2^a$; $r \geqq 4$;

$$[S]^{r-1} = \Sigma'(\nu < m) I^*_\nu.$$

*Then there is a partition* $[T]^r = \Sigma'(\nu < m) I_\nu$ *such that*

$$[I_\nu]_r \subset [I^*_\nu]_{r-1} + [0, \omega) \qquad \textit{for} \quad \nu < m.$$

COROLLARY. *If* $r \geqq 4$; $m \geqq 2$; $a, b_0,, \hat{b}_m \geqq \aleph_0$; $a \nrightarrow (b_\nu)^{r-1}_{\nu<m}$, *then*

$$2^a \nrightarrow (b_\nu)^r_{\nu<m}.$$

PROOF. Let $a = \aleph_\alpha$. We may assume $S = [0, \omega_\alpha)$ and $T = V$. Put

$$I_1 = K_{01} + P_{01} + \bar{I}_1; \quad I_\nu = \bar{I}_\nu \qquad (2 \leqq \nu < m).$$

Let $b \geqq \aleph_0$; $\nu < m$; $b \in [I_\nu]_r$. Then there is $X \in [V]^b$ with $[X]^r \subset I_\nu$. Then there is $s < 2$ with $[X]^r K_{s,1-s} = \emptyset$. By 13. 7 there is $X' \in [X]^b$ with $[X']^r \subset K$. There is $t < 2$ with $[X']^r P_{t,1-t} = \emptyset$. By 13. 11 there is $X'' = \{x_0,, \hat{x}_k\}_< \in [X']^b$ with $[X'']^r \subset P_{0,,0}$. Then $k \geqq \omega(b)$, and $x_\lambda x_{\lambda+1} < x_\mu x_{\mu+1}$ for $\lambda < \mu < \omega(b)$. Put $D = \{x_\lambda x_{\lambda+1} : \lambda < \omega(b)\}$. Let $L \in [D]^{r-1}$. Then there are $\lambda_0 < \dots < \lambda_{r-2} < \omega(b)$ with $L = \{x_{\lambda_\varrho} x_{\lambda_\varrho+1} : \varrho < r-1\}$. Put $\lambda_{r-1} = \lambda_{r-2}+1$. Then, by 13. 14, $x_{\lambda_\varrho} x_{\lambda_\varrho+1} = x_{\lambda_\varrho} x_{\lambda_{\varrho+1}}$ for $\varrho < r-1$ and, by definition of $I_\nu$, we have

$$L = \{x_{\lambda_\varrho} x_{\lambda_{\varrho+1}} : \varrho < r-1\} = \delta'(x_{\lambda_0},, x_{\lambda_{r-1}}) \in I^*_\nu,$$

$$[D]^{r-1} \subset I^*_\nu; \quad b = |D| \in [I^*_\nu]_{r-1}.$$

This proves Lemma 5D′.

The corollary follows since by Lemma 5D we can choose the $I^*_\nu$ such that $b_\nu \notin [I^*_\nu]_{r-1}$ for $\nu < m$. Then $b_\nu \notin [I_\nu]_r$ for $\nu < m$ which proves the assertion.

LEMMA 5E. *Let* $a \geqq \aleph_0$; $m \geqq 1$, *and let* $b_0,, \hat{b}_m$ *be such that* $b_\nu = b'_\nu$ *for at least one* $\nu < m$. *Let* $a \nrightarrow (b_\nu)^2_{\nu<m}$. *Then* $2^a \nrightarrow (4, (b_\nu+1)_{\nu<m})^3$.

PROOF. We may assume $b_0 = b'_0$. Let $r = 3$. Define $\alpha, n, S, I^*_\nu$ as in the proof of Lemma 5A so that (6) holds. Then $[V]^3 = I +' \Sigma'(\nu < m) I_\nu$, where

(15) $$I = K_{01}; \quad I_\nu = \bar{I}_\nu \qquad (1 \leqq \nu < m).$$

Let $X \subset V$; $[X]^3 \subset I$. Then, by (15), $[X]^3 \subset K_{01}$ and hence, by 13. 5 (a), $|X| < 4$. Next, let $Y \in [V]^{b_0}$; $[Y]^3 \subset I_0$. Then, by (15), $[Y]^3 K_{01} = \emptyset$ and hence, by 13. 7 (a) and 13. 8, there is $Y' \in [Y]^{b_0}$ with $[Y']^3 \subset P_0$. Let $Y' = \{y_0,, \hat{y}_k\}_<$; $D = \{y_\varkappa y_{\varkappa+1} : \varkappa+1 < k\}$. Then, by 13. 14 and (15), $[D]^{r-1} \subset I^*_0$ and hence, by (6), $b_0 = |Y'| =$

$= |k \dot{-} 1| = |D| < b_0$ which is a contradiction. Finally, let $Z \in [V]^{b_\nu+1}$; $[Z]^3 \subset I_\nu$; $1 \leqq \nu < m$. Then $[Z]^3 \subset P_0 I_\nu$. Therefore, again, it suffices to prove (5), and Lemma 5E follows.

LEMMA 5F. *Let* $a \geqq \aleph_0$, *and* $a \nrightarrow (b_\nu)^2_{\nu<m}$. *Then*

$$2^a \nrightarrow (4, 4, (b_\nu+1)_{\nu<m2})^3.$$

REMARK. This result is an analogue of Lemma 5D for the case $r=3$, where $c_r=2$. In this case, however, we cannot prove the stronger statement $2^a \nrightarrow (4, 4, (b_\nu+1)_{\nu<m})^3$, which would formally be the case $r=3$ of Lemma 5D, since we needed 13. 6 (b), and this latter proposition is false for $r=3$.

PROOF. Let $r=3$, and define $\alpha, n, S, I^*_\nu$ as in the proof of Lemma 5A so that (6) holds. Then

$$[V]^3 = I_0 +' I_1 +' \Sigma'(\nu<m) I_{0\nu} +' \Sigma(\nu<m) I_{1\nu},$$

where the $I_t$ and $I_{t\nu}$ are defined as follows.

$$I_t = K_{t,1-t} \text{ for } t<2;$$

$$I_{t\nu} = P_t\{\{x, y, z\}_\prec : \delta'(x, y, z) \in I^*_\nu\} \text{ for } t<2 \text{ and } \nu<m.$$

We are using here the fact that $[V]^3 = K_{01} +' K_{10} +' K = P_0 +' P_1$. If $[X]^3 \subset I_t$, then $[X]^3 \subset K_{t,1-t}$ and by 13. 5, $|X|<4$. If $[Y]^3 \subset I_{t\nu}$, then $[Y]^3 \subset P_t$, and it follows, just as in the proof of Lemma 5E, that if $Y = \{y_0,, \hat{y}_k\}_\prec$, then

$$[\{y_\varkappa y_{\varkappa+1} : \varkappa+1<k\}]^2 \subset I^*_\nu; \quad |k \dot{-} 1| < b_\nu;$$

$$|Y| = |k| < b_\nu + 1.$$

This proves Lemma 5F.

Now we are going to discuss two theorems which we cannot prove without using $(*)$.

$(*)$ THEOREM 13. *Let* $a > a'$. *Then* $a^+ \nrightarrow (a, 5)^4$.

PROOF. Let $a = \aleph_\alpha$; $r=4$; $n=\omega_\alpha$; $l=\omega(a')$;

$$a_0 < \cdot < \hat{a}_l < a = \sup(\lambda<l) a_\lambda,$$

and put

$$\lambda(\nu) = \min(a_\lambda > |\nu|)\lambda \quad \text{for} \quad \nu<n.$$

If $A = \{x_0,, x_3\}_\prec \subset V$, then we put $\Lambda(A) = (\lambda_0(A),, \lambda_2(A))$, where $\lambda_\nu(A) = \lambda(x_\nu x_{\nu+1})$ for $\nu<3$. Then $[V]^4 = I_0 +' I_1$, where

$$I_1 = K_{010} + K\{A : \lambda_0(A) \geqq \lambda_1(A) < \lambda_2(A)\}. \tag{16}$$

Let us suppose that $a \in [I_0]_4$. Then there is $X \in [V]^a$ with $[X]^4 \subset I_0$ and, by (16), $[X]^4 K_{010} = \emptyset$. Hence, by 13. 7, there is $X' \in [X]^a$ such that $[X']^4 \subset K_{sss} \subset K$ for some $s$. Let $X' = \{x_0,, \hat{x}_k\}_\prec$. Then $k \geqq n$.

*Case* 1. $\lambda(x_\mu x_\nu) < \lambda(x_\nu x_\tau)$ for $\mu<\nu<\tau<n$. Put $\lambda_\mu = \lambda(x_\mu x_{\mu+1})$ for $\mu<n$. Then

$$\lambda_\mu = \lambda(x_\mu x_{\mu+1}) < \lambda(x_{\mu+1} x_\nu) < \lambda(x_\nu x_{\nu+1}) = \lambda_\nu \text{ for } \mu+1<\nu<n,$$

and

$$\lambda_\mu = \lambda(x_\mu x_{\mu+1}) < \lambda(x_\nu x_{\nu+1}) = \lambda_\nu \text{ for } \mu+1 = \nu<n.$$

Hence $\lambda_0 << \hat{\lambda}_n < l$; $a = |n| \leqq |l| = a'$ which is a contradiction.

*Case* 2. There are numbers $\mu < \nu < \tau < n$ such that

$$\varrho = \lambda(x_\mu x_\nu) \geqq \lambda(x_\nu x_\tau).$$

Then $[\{x_\mu, x_\nu, x_\tau, x_\sigma, x_\varphi\}]^4 \subset I_0$ for $\tau < \sigma < \varphi < n$ and hence, by (16),

$$\varrho = \lambda(x_\mu x_\nu) \geqq \lambda(x_\nu x_\tau) \geqq \lambda(x_\tau x_\sigma) \geqq \lambda(x_\sigma x_\varphi) \text{ for } \tau < \sigma < \varphi < n. \tag{17}$$

Put $X'' = \{x_{\tau+1},, \hat{x}_n\}$. Then $|X''| = a$. Let $x_\sigma = (x_\sigma(0),, \hat{x}_\sigma(n))$ for $\tau < \sigma < n$. We define elements $y_\sigma$ of $V$ by putting $y_\sigma = (y_\sigma(0),, \hat{y}_\sigma(n))$ where, for $\nu < n$ and $\tau < \sigma < n$,

$$y_\sigma(\nu) = x_\sigma(\nu) \text{ if } \lambda(\nu) \leqq \varrho, \text{ and } y_\sigma(\nu) = 0 \text{ if } \lambda(\nu) > \varrho. \tag{18}$$

Put $Y = \{y_{\tau+1},, \hat{y}_n\}$. Let $\tau < \sigma < \varphi < n$. Then, by (17), $\lambda(x_\sigma x_\varphi) \leqq \varrho$. Hence, by (18), we have, for $\nu_0 = x_\sigma x_\varphi$, $y_\sigma(\nu_0) = x_\sigma(\nu_0) \neq x_\varphi(\nu_0) = y_\varphi(\nu_0)$. Therefore $y_\sigma \neq y_\varphi$, and $|Y| = a$. On the other hand, by (18), $|Y| \leqq 2^{a_\varrho} = a_\varrho^+ < a$ which is a contradiction. Hence $a \notin [I_0]_4$.

Next, suppose that $5 \in [I_1]_4$. Then there is $Z \in [V]^5$ with $[Z]^4 \subset I_1$. Then, by (16), $[Z]^4 \subset K_{010} + K$ and hence, by 13. 6, $[Z]^4 \subset K_{sss}$ for some $s$. Let $Z = \{z_0,, z_4\}_<$; $\lambda_\nu = \lambda(z_\nu z_{\nu+1})$ for $\nu < 4$. Then, by (16), $\lambda_0 \geqq \lambda_1 < \lambda_2$ and, at the same time, $\lambda_1 \geqq \lambda_2 < \lambda_3$ which is impossible. Hence $5 \notin [I_1]_4$, and Theorem 13 follows.

($*$) THEOREM 14. *Let $a > a'$ and $c = a'^-$. Then $a^+ \nrightarrow (a, (4)_c)^3$.*

PROOF. Define $\alpha, n, l, a_0,, \hat{a}_l, \lambda(\nu)$ as in the proof of Theorem 13 except that now $r = 3$. If $A = \{x_0,, x_2\}_< \subset V$, we put $\lambda_\nu(A) = \lambda(x_\nu x_{\nu+1})$ for $\nu < 2$. By Theorem 8, we have $2^c \nrightarrow (3)_c^2$. Hence there is a partition

$$[[0, l)]^2 = \Sigma'(\nu < \omega(c)) I_\nu^*$$

such that $3 \notin [I_\nu^*]_2$ for $\nu < \omega(c)$. Then

$$[V]^3 = I_0 +' I_1 +' \Sigma'(\nu < \omega(c) \wedge t < 2) I_{\nu t},$$

where

$$\left\{ \begin{array}{ll} I_0 = K_{01} + K\{A : \lambda_0(A) = \lambda_1(A)\}; & I_1 = K_{10}; \\ I_{\nu t} = P_t\{A : \{\lambda_0(A), \lambda_1(A)\} \in I_\nu^*\} & \text{for } \nu < \omega(c); \quad t < 2. \end{array} \right. \tag{19}$$

Let $X \in [V]^a$; $[X]^3 \subset I_0$. Then, by (19), $[X]^3 K_{10} = \emptyset$ and, by 13. 7, there is $X' \in [X]^a$ with $[X']^3 \subset K_{ss}$ for some $s$. Let $X' = \{x_0,, \hat{x}_k\}_<$. Then $k \geqq n$, and either $x_0 << \hat{x}_k$ or $x_0 >> \hat{x}_k$. Put $\lambda(x_0, x_1) = \varrho$. Then, by (19), $\lambda(x_\mu x_{\mu+1}) = \varrho$ for $\mu < n$ and, more generally, $\lambda(x_\mu x_\nu) = \varrho$ for $\mu < \nu < n$. We now argue as in case 2 of the proof of Theorem 13. Let $x_\sigma = (x_\sigma(0),, \hat{x}_\sigma(n))$ for $\sigma < n$. Put $y_\sigma(\nu) = x_\sigma(\nu)$ if $\lambda(\nu) \leqq \varrho$, and $y_\sigma(\nu) = 0$ if $\lambda(\nu) > \varrho$, for $\sigma, \nu < n$. Put $y_\sigma = (y_\sigma(0),, \hat{y}_\sigma(n))$ for $\sigma < n$, and $Y = \{y_0,, \hat{y}_n\}$. If $\sigma < \varphi < n$, then $\lambda(x_\sigma x_\varphi) = \varrho$ and hence, for $\nu_0 = x_\sigma x_\varphi$, $y_\sigma(\nu_0) = x_\sigma(\nu_0) \neq x_\varphi(\nu_0) = y_\varphi(\nu_0)$. Thus $y_\sigma \neq y_\varphi$; $|Y| = a$. On the other hand, $|Y| \leqq 2^{a_\varrho} = a_\varrho^+ < a$ which is a contradiction. Hence $a \notin [I_0]$. Next, let $[Z]^3 \subset I_1$. Then, by (19), $[Z]^3 K_{01} = \emptyset$ and, by 13. 5, we have $|Z| < 4$. Hence $4 \notin [I_1]_3$. Finally, let $Z_0 \in [V]^4$; $[Z_0]^3 \subset I_{\nu t}$ for some $\nu < \omega(c)$ and $t < 2$. Then, by (19), $[Z_0]^3 \subset P_t$. Let $Z_0 = \{z_0,, z_3\}_<$. Then either $z_0 << z_3$ or $z_0 >> z_3$. Put $D = \{\lambda(z_0 z_1),, \lambda(z_2 z_3)\}$. Then, by (19) and 11. 6 (ii), $[D]^2 \subset I_\nu^*$. But this is false since $|D| = 3$, and Theorem 14 is proved.

## 15. DISCUSSION OF RELATION I. MAIN THEOREMS

**15. 1. A definition.** We define a function $\mathrm{cr}(\beta)$ by putting, for every ordinal $\beta$,

$$\mathrm{cr}\,(\beta) = \mathrm{cf}\,(\mathrm{cf}\,(\beta) \dot{-} 1).$$

This means that if $\aleph_\beta = b$, then $\aleph_{\mathrm{cr}\,(\beta)} = b'^{-'}$. We call $\mathrm{cr}\,(\beta)$ the *critical number* belonging to $\beta$. Explicitly, it is given by the following rule.

*Case 1.* $\beta$ is of the first kind. Then $\beta = \alpha + 1$, and $\mathrm{cr}\,(\beta) = \mathrm{cf}(\alpha)$.

*Case 2.* $\beta$ is of the second kind.

*Case 2a.* $\aleph'_\beta$ is weakly inaccessible. Then $\mathrm{cr}\,(\beta) = \mathrm{cf}(\beta)$.

*Case* 2b. $\aleph'_\beta$ is not weakly inaccessible. Then $\mathrm{cf}\,(\beta) = \gamma + 1$, and $\mathrm{cr}\,(\beta) = \mathrm{cf}(\gamma)$.

We note that if $\aleph_\beta$ is weakly inaccessible, then $\mathrm{cr}\,(\beta) = \beta$, and that $\mathrm{cr}\,(\beta) < \beta$ otherwise.

The following Main Theorem I gives a summary of the results obtained so far for $r = 2$ as well as the essential part of the results for $r \geqq 3$. Its proof consists of a rather lengthy discussion of cases. During this discussion we shall require some new corollaries of earlier results.

**15. 2. The first Main Theorem.** $(*)$ THEOREM I. *Let* $n \geqq 2$, *and let*

$$2 \leqq r < b_0,, \hat{b}_n \leqq \aleph_\beta .$$

*Consider the relation*

$$\text{(R)} \qquad \aleph_{\beta+(r-2)} \to (b_0,, \hat{b}_n)^r$$

*and the following conditions:*

$$\text{(IA)} \qquad b_0 = \aleph_\beta,$$

$$\text{(IB)} \qquad b_0,, \hat{b}_n < \aleph_\beta,$$

$$\text{(CA)} \qquad b_1 \ldots \hat{b}_n \leqq \aleph_{\mathrm{cr}(\beta)},$$

$$\text{(CB)} \qquad b_0 \ldots \hat{b}_n < \aleph_\beta .$$

*Then we have the following results:*

(i) *If* (IA) *holds, then* (CA) *is necessary for the truth of* (R) *except possibly when*

$$(1) \qquad r \geqq 3; \quad \beta > \mathrm{cf}(\beta) > \mathrm{cf}(\beta) \dot{-} 1 > \mathrm{cr}\,(\beta); \quad b_1,, \hat{b}_n < \aleph_0 .$$

(ii) *If* (IA) *holds and* $b_1 \geqq \aleph_0$, *then* (CA) *is necessary for the truth of* (R).

(iii) *If* (IA) *holds, then* (CA) *is sufficient for the truth of* (R) *except possibly when* $\aleph'_\beta$ *is inaccessible and greater than* $\aleph_0$.

(iv) *If* (IA) *holds and* $b_1,, \hat{b}_n < \aleph'_\beta$, *then* (CA) *is sufficient for the truth of* (R).

(v) *If* (IB), *holds, then* (CB) *is necessary and sufficient for the truth of* (R).

Clearly, (ii) follows from (i). Since frequent use will be made of these results in the case $n < \omega$ we state explicitly what the theorem asserts in this case. We consider the relation

$$\aleph_{\beta+(r-2)} \to (b_0,, b_{n-1})^r,$$

where $2 \leqq n < \omega$ and

$$\aleph_\beta \geqq b_0 \geqq \geqq b_{n-1} > r \geqq 2.$$

For any $\beta$, we have:

(2) $$\aleph_{\beta+(r-2)} \nrightarrow (\aleph_\beta, \aleph^{+}_{\mathrm{cr}(\beta)})^r;$$

(3) $$\aleph_{\beta+(r-2)} \rightarrow \big(\aleph_\beta, (\aleph_{\mathrm{cr}(\beta)})_{n-1}\big)^r$$

except possibly when $\mathrm{cf}(\beta) = \mathrm{cf}(\beta) \dot- 1 > 0$.

(4) $$\begin{cases} \aleph_{\beta+(r-2)} \rightarrow \big(\aleph_\beta, (b)_{n-1}\big)^r & \text{if} \quad b < \aleph_{\mathrm{cr}(\beta)}; \\ \aleph_{\beta+(r-2)} \rightarrow (b)^r_n & \text{if} \quad b < \aleph_\beta. \end{cases}$$

Theorem I leaves some questions on I-relations unanswered, such as that of the truth of

(5) $$\aleph_{\beta+1} \rightarrow \big(\aleph_\beta, (4)_{\aleph_{\mathrm{cr}(\beta)}}\big)^3$$

when

(6) $$\beta > \mathrm{cf}(\beta) > \mathrm{cf}(\beta) \dot- 1 > \mathrm{cr}(\beta).$$

We conjecture that (5) is false when (6) holds. It seems that to prove this only a slight refinement of the methods used in section 14 would be sufficient; however we have not been able to settle this question. Here the simplest unsolved problem is

($*$) PROBLEM 2.
$$? \aleph_{\omega_{\omega+1}+1} \rightarrow \big(\aleph_{\omega_{\omega+1}}, (4)_{\aleph_0}\big)^3.$$

This corresponds to the smallest $\beta$ which satisfies (6).

**15. 3. Proof of Theorem I.** Consider the following conditions:

(C*A) $$n < \omega_{\mathrm{cr}(\beta)}; \quad b_1 ,, \hat{b}_n \leqq \aleph_{\mathrm{cr}(\beta)}.$$

(C*B) $$\begin{cases} n < \omega_{\beta \dot- 1}; \quad b_0 ,, \hat{b}_n < \aleph_\beta; \\ \text{there is } c < \aleph_{\beta \dot- 1} \text{ with } |\{\nu: b_\nu \geqq c\}| < \aleph_{\mathrm{cf}(\beta \dot- 1)}. \end{cases}$$

**15. 4.** The conditions $(CA)$ and $(C^*A)$ are equivalent.

PROOF. If (CA) holds, then $b_1 ,, \hat{b}_n \leqq b_1 \dots \hat{b}_n \leqq \aleph_{\mathrm{cr}(\beta)}$;

$$|n \dot- 1| < 3^{|n \dot- 1|} \leqq b_1 \dots \hat{b}_n \leqq \aleph_{\mathrm{cr}(\beta)};$$

$$n < \omega_{\mathrm{cr}(\beta)},$$

and (C*A) follows. Vice versa, if (C*A) holds then, since $\aleph_{\mathrm{cr}(\beta)}$ is regular,

$$b_1 \dots \hat{b}_n \leqq (\aleph_{\mathrm{cr}(\beta)})^{|n \dot- 1|} \leqq \aleph_{\mathrm{cr}(\beta)},$$

and (CA) holds.

**15. 5.** The conditions $(CB)$ and $(C^*B)$ are equivalent.

PROOF. 1. Let (CB) hold. Then $|n| < 3^{|n|} \leqq b_0 \dots \hat{b}_n < \aleph_\beta$; $n < \omega_{\beta \dot- 1}$. Let us assume that (C*B) is false. Then $|\{\nu: b_\nu \geqq c\}| \geqq \aleph_{\mathrm{cf}(\beta \dot- 1)}$ for $c < \aleph_{\beta \dot- 1}$. Then we can construct inductively a sequence $\nu_0 < \ldots < \hat{\nu}_l < n$, where $l = \omega_{\mathrm{cf}(\beta \dot- 1)}$, such that $\sup(\lambda < l) b_{\nu_\lambda} = \aleph_{\beta \dot- 1}$. If there is a number $m < l$ such that $b_{\nu_\lambda} = \aleph_{\beta \dot- 1}$ for $m \leqq \lambda < l$, then we obtain the contradiction

$$\aleph_\beta > b_0 \ldots \hat{b}_n \geqq b_{\nu_m} \ldots \hat{b}_{\nu_l} = (\aleph_{\beta \dot- 1})^{\aleph_{\mathrm{cf}(\beta \dot- 1)}} > \aleph_{\beta \dot- 1}.$$

Hence there is no such $m$, and we can impose on the $\nu_\lambda$ the additional condition $b_{\nu_0} < \ldots < \hat{b}_{\nu_l}$. Then, by Kőnig's theorem, $\aleph_\beta > b_0 \ldots \hat{b}_n \geqq b_{\nu_1} \ldots \hat{b}_{\nu_l} > b_{\nu_0} + \ldots + \hat{b}_{\nu_l} \geqq \aleph_{\beta \dot- 1}$ which is the desired contradiction. Hence (C*B) is true.

2. Let (C*B) hold for some $c$. Put

$$N_0 = \{\nu : b_\nu \geqq c\}; \quad N_1 = \{\nu : b_\nu < c\}.$$

*Case* 1. $\beta = 0$. Then $n < \omega$; $b_0,, \hat{b}_n < \aleph_0$, and (CB) holds.

*Case* 2. $\beta \dot- 1 = \beta > 0$. Then there is $\gamma < \beta$ such that

$$\Pi(\nu \in N_0) b_\nu \leqq 2^{\Sigma(\nu \in N_0) b_\nu} \leqq 2^{\aleph_\gamma} = \aleph_{\gamma+1} < \aleph_\beta,$$

$$\Pi(\nu \in N_1) b_\nu \leqq c^{|n|} \leqq 2^{c|n|} \leqq 2^{\aleph_\gamma} < \aleph_\beta.$$

Hence (CB) holds.

*Case* 3. $\beta \dot- 1 < \beta$. Then $\beta = \delta + 1$, and $|N_0| < \aleph'_\delta$;

$$\Pi(\nu \in N_0) b_\nu \leqq \aleph_\delta^{|N_0|} < \aleph_\delta,$$

$$\Pi(\nu \in N_1) b_\nu \leqq c^{|n|} \leqq 2^{c|n|} < \aleph_\beta,$$

and (CB) follows.

**15. 6. Proof of Theorem I** (i). We assume (IA) and (R) and want to deduce that either (CA) or (1) holds. Suppose that (CA) is false. Then, by 15. 4, (C*A) is false. It is sufficient to prove the following three propositions:

($\ast$) COROLLARY 6.

$$\aleph_{\beta+(r-2)} \nrightarrow (\aleph_\beta, \aleph^+_{\mathrm{cr}(\beta)})^r \quad \textit{for} \quad r \geqq 2.$$

($\ast$) COROLLARY 7.

$$\aleph_{\beta+(r-2)} \nrightarrow (\aleph_\beta, (r+1)_{\aleph_{\mathrm{cr}'(\beta)}})^r \quad \textit{for} \ r \geqq 2$$

*except possibly when*

(7) $$r \geqq 3; \quad \beta > \mathrm{cf}(\beta) > \mathrm{cf}(\beta) \dot- 1 > \mathrm{cr}(\beta).$$

($\ast$) COROLLARY 8.

$$\aleph_{\beta+(r-2)} \nrightarrow (\aleph_\beta, \aleph_0, (r+1)_{\aleph_{\mathrm{cr}(\beta)}})^r \quad \textit{for} \quad r \geqq 2.$$

For let us assume these three propositions proved. Our aim is to prove: Let (IA) and (R) be true, and $n \geqq 2$; $2 \leqq r < b_0,, \hat{b}_n \leqq \aleph_\beta$. Let both (C*A) and (1) be false. Then a contradiction follows. To exhibit this contradiction we consider the cases:

*Case* 1. $n < \omega_{\mathrm{cr}(\beta)}$. Then, since (C*A) is false, there is $\nu < n$ with $b_\nu > \aleph_{\mathrm{cr}(\beta)}$. Then, by (IA) and (R), $\aleph_{\beta+(r-2)} \to (\aleph_\beta, \aleph^+_{\mathrm{cr}(\beta)})^+$ which contradicts Corollary 6.

*Case* 2. $n \geqq \omega_{\mathrm{cr}(\beta)}$. Then, by (IA) and (R), $\aleph_{\beta+(r-2)} \to (\aleph_\beta, (r+1)_{\aleph_{\mathrm{cr}(\beta)}})^r$. By Corollary 7 this implies that (7) holds. Since (1) is false, we conclude that there is

$\nu \in [1, n)$ with $b_\nu \geqq \aleph_0$. Then, by (IA) and (R),

$$\aleph_{\beta+(r-2)} \to (\aleph_\beta, \aleph_0, (r+1)_{\aleph_{\mathrm{cr}(\beta)}})^r$$

which contradicts Corollary 8. All this shows that in order to prove Theorem I (i) it suffices to prove Corollaries 6, 7, 8.

PROOF OF COROLLARY 6. For $r=2$ we have to prove

(8) $$\aleph_\beta \not\to (\aleph_\beta, \aleph^+_{\mathrm{cr}(\beta)})^2.$$

*Case* 1. $\beta = \alpha+1$. Then $\mathrm{cr}(\beta) = \mathrm{cf}(\alpha)$, and (8) follows from Theorem 7.

*Case* 2. $\beta = \beta \dot{-} 1 = \mathrm{cf}(\beta)$. Then $\mathrm{cr}(\beta) = \beta$, and (8) is trivial.

*Case* 3. $\beta = \beta \dot{-} 1 > \mathrm{cf}(\beta)$. Put $\mathrm{cf}(\beta) = \gamma$. Then, by Lemma 4, (8) is equivalent to

(9) $$\aleph_\gamma \not\to (\aleph_\gamma, \aleph^+_{\mathrm{cr}(\beta)})^2.$$

Now $\mathrm{cf}(\gamma) = \gamma$, and $\mathrm{cr}(\gamma) = \mathrm{cf}(\gamma \dot{-} 1) = \mathrm{cr}(\beta)$. Hence (9) follows from case 1 or case 2, when applied to $\gamma$ in place of $\beta$. Thus we have proved Corollary 6 for $r=2$. To complete the proof assume that, for some $r \geqq 3$, we have

$$\aleph_{\beta+(r-3)} \not\to (\aleph_\beta, \aleph^+_{\mathrm{cr}(\beta)})^{r-1}.$$

Then, since $\aleph^+_{\mathrm{cr}(\beta)}$ is regular, it follows from Lemma 5A that

$$\aleph_{\beta+(r-2)} \not\to (\aleph_\beta + 1, \aleph^+_{\mathrm{cr}(\beta)} + 1)^r,$$

and Corollary 6 is established by induction over $r$.

PROOF OF COROLLARY 7. For $r=2$ we have to prove that if (7) is false then

(10) $$\aleph_\beta \not\to (\aleph_\beta, (3)_{\aleph_{\mathrm{cr}(\beta)}})^2.$$

*Case* 1. $\beta = \alpha+1$. Then $\mathrm{cr}(\beta) = \mathrm{cf}(\alpha)$, and (10) means

(11) $$\aleph_{\alpha+1} \not\to (\aleph_{\alpha+1}, (3)_{\aleph_{\mathrm{cf}(\alpha)}})^2.$$

*Case* 1a. $\alpha = \mathrm{cf}(\alpha)$. Then (11) follows from $(*)$ and Theorem 8.

*Case* 1b. $\alpha > \mathrm{cf}(\alpha)$. Then (11) follows from Theorem 10.

*Case* 2. $\beta = \beta \dot{-} 1 = \mathrm{cf}(\beta)$. Then $\mathrm{cr}(\beta) = \beta$, and (10) is trivial.

*Case* 3. $\beta = \beta \dot{-} 1 > \mathrm{cf}(\beta)$. Put $\mathrm{cf}(\beta) = \gamma$. Then $\mathrm{cr}(\gamma) = \mathrm{cr}(\beta)$ and, by Lemma 4, (10) is equivalent to

(12) $$\aleph_\gamma \not\to (\aleph_\gamma, (3)_{\aleph_{\mathrm{cr}(\gamma)}})^2.$$

But (12) is true by case 1 or 2, when applied to $\gamma$ instead of $\beta$. This proves Corollary 7 for $r=2$. We now prove it for $r=3$ when the assertion is

(13) $$\aleph_{\beta+1} \not\to (\aleph_\beta, (4)_{\aleph_{\mathrm{cr}(\beta)}})^3.$$

*Case* 1. $\beta = \mathrm{cf}(\beta)$. Then by (10) and Lemma 5E we have

$$\aleph_{\beta+1} \not\to (4, \aleph_\beta, (4)_{\aleph_{\mathrm{cr}(\beta)}})^3$$

which is the same as (13).

*Case* 2. $\beta > \mathrm{cf}(\beta)$. Then, by Theorem 14,

$$\aleph_{\beta+1} \nrightarrow (\aleph_\beta, (4)_{\aleph_{\mathrm{cf}(\beta)\dot{-}1}})^3. \tag{14}$$

*Case* 2a. $\mathrm{cf}(\beta) = \gamma+1$. If $\gamma > \mathrm{cf}(\gamma)$, then

$$\beta > \mathrm{cf}(\beta) > \mathrm{cf}(\beta)\dot{-}1 = \gamma > \mathrm{cf}(\gamma) = \mathrm{cr}\,(\beta)$$

so that (7) would be true, contrary to the hypothesis. Thus $\gamma = \mathrm{cf}(\gamma)$; $\mathrm{cr}\,(\beta) = \mathrm{cf}(\beta)\dot{-}1$, and (13) follows from (14).

*Case* 2b. $\mathrm{cf}(\beta)\dot{-}1 = \mathrm{cf}(\beta)$. Then $\mathrm{cr}\,(\beta) = \mathrm{cf}(\beta)\dot{-}1$, and (13) follows from (14).

We have shown that the assertion of Corollary 7 is true for $r=2$ and for $r=3$. Now suppose that, for some $r \geqq 4$,

$$\aleph_{\beta+(r-3)} \nrightarrow (\aleph_\beta, (r)_{\aleph_{\mathrm{cr}(\beta)}})^{r-1}.$$

Then, by Lemma 5D, there is $c_r < \aleph_0$ such that

$$\aleph_{\beta+(r-2)} \nrightarrow ((r+1)_{c_r}, \aleph_\beta+1, (r+1)_{\aleph_{\mathrm{cr}(\beta)}})^r.$$

This is the same as the assertion of Corollary 7 and so establishes Corollary 7 by induction over $r$.

Proof of Corollary 8. If $r=2$, then the assertion of Corollary 8 follows from Corollary 7. Suppose that, for some $r \geqq 3$,

$$\aleph_{\beta+(r-3)} \nrightarrow (\aleph_\beta, \aleph_0, (r)_{\aleph_{\mathrm{cr}(\beta)}})^{r-1}.$$

Then, by Lemma 5A,

$$\aleph_{\beta+(r-2)} \nrightarrow (\aleph_\beta+1, \aleph_0+1, (r+1)_{\aleph_{\mathrm{cr}(\beta)}})^r.$$

This proves Corollary 8 by induction over $r$ and concludes the proof of Theorem I (i). As has already been pointed out, part (ii) of Theorem I follows from (i).

**15. 7. Proof of Theorem** I (iii). We have to prove: If (IA) and (CA) hold then either $(R)$ holds or $\mathrm{cf}(\beta) = \mathrm{cf}(\beta)\dot{-}| > 0$.

It suffices to prove:

($*$) Corollary 9.

$$\aleph_{\beta+(r-2)} \to (\aleph_\beta, (\aleph_{\mathrm{cr}(\beta)})_c)^r \tag{15}$$

*for* $r \geqq 2$ *and* $c < \aleph_{\mathrm{cr}(\beta)}$ *except possibly when*

$$\mathrm{cf}(\beta) = \mathrm{cf}(\beta)\dot{-}1 > 0, \tag{16}$$

*i. e. when* $\aleph'_\beta$ *is inaccessible and greather than* $\aleph_0$.

For, to deduce (iii) from Corollary 9 let us assume that (IA) and (CA) are true and that the condition (16) is not satisfied. We have to deduce (R). In fact, (C*A) follows from (CA), and hence $|n| < \aleph_{\mathrm{cr}(\beta)}$. Now, by Corollary 9, the relation (15) holds with $c = |n|$. Then (R) follows from (C*A) and (IA).

Proof of Corollary 9. We assume that (16) is false. We begin by proving the relation (15) for $r=2$ when it states

$$\aleph_\beta \to (\aleph_\beta, (\aleph_{\mathrm{cr}(\beta)})_c)^2.$$

By Lemma 4 this follows from

(17) $$\aleph'_{\beta} \to (\aleph'_\beta, (\aleph_{\mathrm{cr}(\beta)})_c)^2.$$

*Case* 1. $\mathrm{cf}(\beta) = \gamma+1$. Then $\mathrm{cr}\,(\beta) = \mathrm{cf}(\gamma)$, and (17) follows from Theorem 1.

*Case* 2. $\mathrm{cf}(\beta) = \mathrm{cf}(\beta) \dot- 1$. Then, since (16) is false, we have $\mathrm{cf}(\beta)=0$ and $c<\aleph_0$, and (17) follows from Ramsey's theorem. This proves (15) for $r=2$. Now let $r \geqq 3$, and suppose

$$\aleph_{\beta+(r-3)} \to (\aleph_\beta, (\aleph_{\mathrm{cr}(\beta)})_c)^{r-1}.$$

Then, by Lemma 2, we obtain (15), and Corollary 9 follows by induction over $r$. This completes the proof of (iii).

**15. 8. Proof of Theorem** I (iv). We shall use the equivalence of (CA) and, (C*A). We have to prove: If $r \geqq 2$; $2 \leqq n < \omega_{\mathrm{cr}(\beta)}$; $b_1,, \hat{b}_n \leqq \aleph_{\mathrm{cr}(\beta)}$; $b_1,, b_n < \aleph'_\beta$, then

(18) $$\aleph_{\beta+(r-2)} \to (\aleph_\beta, b_1,, \hat{b}_n)^r.$$

By Lemma 2 we need only consider the case $r=2$, and for $r=2$, (18) is, by Lemma 4, equivalent to

$$\aleph'_\beta \to (\aleph'_\beta, b_1,, \hat{b}_n)^2.$$

Hence, putting $\mathrm{cf}(\beta)=\gamma$, we see that it suffices to prove:

($*$) Corollary 10. *Let* $\gamma = \mathrm{cf}(\gamma)$; $2 \leqq n < \omega_{\mathrm{cf}(\gamma \dot- 1)}$; $b_1,, \hat{b}_n < \aleph_\gamma$; $b_1,, \hat{b}_n \leqq$ $\leqq \aleph_{\mathrm{cf}(\gamma \dot- 1)}$. *Then*

(19) $$\aleph_\gamma \to (\aleph_\gamma, b_1,, \hat{b}_n)^2.$$

Proof of Corollary 10.

*Case* 1. $\gamma = \delta+1$. Then (19) follows from the proposition:

$$\aleph_{\delta+1} \to (\aleph_{\delta+1}, (\aleph'_\delta)_c)^2 \quad \text{for} \quad c < \aleph'_\delta,$$

and this proposition follows from Corollary 1.

*Case* 2. $\gamma = \gamma \dot- 1$. Then $\aleph_\gamma$ is inaccessible, and (19) follows from Theorem 5. We now prove the two parts of Theorem I (v).

**15. 9.** If (IB) and (CB) hold, then (R) follows.

Proof. By 15. 5 the condition (C*B) holds, so that $n < \omega_{\beta \dot- 1}$; $b_0,, \hat{b}_n < \aleph_\beta$, and there is $c < \aleph_{\beta \dot- 1}$ with

$$|\{v : b_v \geqq c\}| < \aleph'_{\beta \dot- 1}.$$

We want to deduce (R).

*Case* 1. $\beta \dot- 1 = \beta$. Then it suffices to prove:

(20) $$\begin{cases} \aleph_{\beta+(r-2)} \to (c_0,, \hat{c}_m, (c)_e)^r \\ \text{if } r \geqq 2;\ c < \aleph_\beta;\ m < \omega_{\mathrm{cf}(\beta)};\ c_0,, \hat{c}_m < \aleph_\beta;\ e < \aleph_\beta. \end{cases}$$

By definition of $\mathrm{cf}(\beta)$, (20) is the same as:

$$\aleph_{\beta+(r-2)} \to (c)^r_e \text{ if } r \geqq 2 \text{ and } c, e < \aleph_\beta.$$

By Lemma 2 we need only consider the case $r=2$, i. e.

(21) $$\aleph_\beta \to (c)^2_e \text{ for } c, e < \aleph_\beta.$$

*Case* 1a. $\beta=0$. Then (21) follows from Ramsey's theorem.

*Case* 1b. $\beta>0$. Then $\beta = \beta \dot{-} 1 > 0$, and there is $\gamma<\beta$ with $c, e < \aleph_\gamma$. Then, by Corollary 1,

(22) $$\aleph_{\gamma+2} \to (\aleph_{\gamma+2}, (\aleph_{\gamma+1})_{\aleph_\gamma})^2.$$

Now (21) follows from (22).

*Case* 2. $\beta = \alpha+1$. Then it is sufficient to prove:

(23) $$\begin{cases} \aleph_{\alpha+(r-1)} \to ((\aleph_\alpha)_d, (c)_e)^r \\ \text{if } r \geqq 2;\ c<\aleph_\alpha;\ d<\aleph'_\alpha;\ e<\aleph_\alpha. \end{cases}$$

By Lemma 2 we need only consider the case $r=2$ when (23) states:

(24) $$\begin{cases} \aleph_{\alpha+1} \to ((\aleph_\alpha)_d, (c)_e)^2 \\ \text{if } c, e<\aleph_\alpha \text{ and } d<\aleph'_\alpha. \end{cases}$$

*Case* 2a. $\alpha = \mathrm{cf}(\alpha)$. Then (24) follows from Corollary 1.

*Case* 2b. $\alpha > \mathrm{cf}(\alpha)$. Then, by Theorem 2, $\aleph_{\alpha+1} \to ((\aleph_\alpha)_d, \aleph_\alpha)^2$. Hence it suffices to prove $\aleph_\alpha \to (c)^2_e$, and this follows from (21) since $\alpha \dot{-} 1 = \alpha$. This shows that if (IB) and (CB) hold, then (R) follows.

**15. 10.** If (IB) and (R) hold, then (CB) holds.

PROOF. We assume that (IB) holds and that (CB), and so (C*B), is false. We have to deduce that (R) is false. Thus we are given: $n \geqq 2;\ 2 \leqq r < b_0,, \hat{b}_n < \aleph_\beta$; $b_0 \ldots \hat{b}_n \geqq \aleph_\beta$, and either (i) $n \geqq \omega_{\beta \dot{-} 1}$, or (ii) $n < \omega_{\beta \dot{-} 1}$ and $|\{\nu: b_\nu \geqq c\}| \geqq \aleph'_{\beta \dot{-} 1}$ for $c < \aleph_{\beta \dot{-} 1}$. We have to deduce

$$\aleph_{\beta+(r-2)} \nrightarrow (b_\nu)^r_{\nu<n}.$$

*Case* 1. (i) holds. Then it suffices to prove

($*$) COROLLARY 11.

$$\aleph_{\beta+(r-2)} \nrightarrow (r+1)_{\aleph_{\beta \dot{-} 1}}.$$

*Case* 2. (ii) holds. Put $m = \omega_{\mathrm{cf}(\beta \dot{-} 1)}$. Then $\beta>0$; $n \geqq \omega$;

$$\aleph_{\beta \dot{-} 1} > |n| \geqq \aleph'_{\beta \dot{-} 1} \geqq |m|.$$

*Case* 2a. $\beta = \beta \dot{-} 1$. Then

(25) $$|\{\nu: b_\nu \geqq c\}| \geqq \aleph'_\beta \text{ for } c<\aleph_\beta.$$

Let $\aleph_0 \leqq c_0 << \hat{c}_m < \aleph_\beta = \sup(\mu<m) c_\mu$. Then, by (25), there are $\nu_0 << \hat{\nu}_m < n$ with $b_{\nu_\mu} \geqq c^+_\mu$ for $\mu<m$. Then $\Pi(\mu<m) c^+_\mu \geqq \aleph_\beta$, and it is sufficient to prove

$$\aleph_{\beta+(r-2)} \nrightarrow (c^+_\mu)^r_{\mu<m}.$$

*Case* 2b. $\beta = \alpha+1$. Then $b_0,, \hat{b}_n \leqq \aleph_\alpha$; $b_0 \dots \hat{b}_n > \aleph_\alpha$. Let $N = \{\nu: b_\nu \geqq \aleph_0\}$. By putting $c = \aleph_0$ in (ii) we find $|N| \geqq \aleph_\alpha'$. We may assume $0,1 \in N$. Put $c_\nu = b_\nu - r + 2$ if $\nu \notin N$, and $c_\nu = b_\nu$ if $\nu \in N$. Then $c_0,, \hat{c}_n \geqq 3$. Also, for $\nu \notin N$,

$$b_\nu = c_\nu + (r-2) < c_\nu + 2^{r-2} < \tfrac{1}{2}c_\nu^r + \tfrac{1}{2}c_\nu^r = c_\nu^r,$$

$$\aleph_\alpha < b_0 \dots \hat{b}_n \leqq (c_0 \dots \hat{c}_n)^r; \quad c_0 \dots \hat{c}_n > \aleph_\alpha; \quad c_2 \dots \hat{c}_n > \aleph_\alpha.$$

It suffices to prove that

$$\aleph_{\beta+(r-2)} \nrightarrow \big(\aleph_0, \aleph_0, (c_\nu + (r-2))_{2 \leqq \nu < n}\big)^r$$

if $r \geqq 2$; $c_2 \dots \hat{c}_n \geqq \aleph_\beta$. It now follows that to settle case 2 it suffices to prove

($*$) COROLLARY 12.

$$\aleph_{\beta+(r-2)} \nrightarrow \big(d_\nu + (r-2)\big)^r_{\nu<n}$$

*if*

$r, n \geqq 2$; $d_0 = d_0'$; $d_1 \geqq \aleph_0$; $d_0 \dots \hat{d}_n \geqq \aleph_\beta$; $n < \omega_{\beta \dot{-} 1}$; $3 \leqq d_0,, \hat{d}_n < \aleph_\beta$.

PROOF OF COROLLARY 11.

*Case* 1. $\beta = \beta \dot{-} 1$. If $r = 2$, then the assertion is $\aleph_\beta \nrightarrow (3)^2_{\aleph_\beta}$, and this is true. Now Lemma 5F gives $\aleph_{\beta+1} \nrightarrow (4)^3_{\aleph_\beta}$, and repeated application of Lemma 5D proves the assertion for $r \geqq 3$.

*Case* 2. $\beta = \alpha+1$. If $r = 2$, then we have to prove $\aleph_{\alpha+1} \nrightarrow (3)^2_{\aleph\alpha}$. This is true by Theorem 8. Now Lemmas 5F and 5D establish the assertion for $r \geqq 3$, and Corollary 11 follows.

PROOF OF COROLLARY 12.

*Case* 1. $r = 2$.

*Case* 1a. $\beta = \alpha+1$. Then $d_0,, \hat{d}_n \leqq \aleph_\alpha < d_0 \dots \hat{d}_n \leqq (\aleph_\alpha)^{|n|}$; $\omega_{\mathrm{cf}(\alpha)} \leqq n < \omega_\alpha$. Hence, by Theorem 9, $\aleph_{\alpha+1} \nrightarrow (d_0,, \hat{d}_n)^2$ which is the assertion.

*Case* 1b. $\beta = \beta \dot{-} 1$. If $\beta = \mathrm{cf}(\beta) > 0$, then $d_0 \dots \hat{d}_n \leqq 2^{d_0 + + \hat{d}_n} < \aleph_\beta$ which is a contradiction, and if $\beta = 0$, then $d_0 \dots \hat{d}_n < \aleph_0$ which is a contradiction. Hence $\beta > \mathrm{cf}(\beta)$. If $\sup(\nu < n) d_\nu = e < \aleph_\beta$, then $\aleph_\beta \leqq d_0 \dots \hat{d}_n \leqq e^{|n|} < \aleph_\beta$ which is false. Hence $\sup(\nu < n) d_\nu = \aleph_\beta$, and if $m = \omega_{\mathrm{cf}(\beta)}$, then there are $\nu_0 < < \hat{\nu}_m < n$ such that $d_{\nu_0} < < \hat{d}_{\nu_m}$ and $\sup(\mu < m) d_{\nu_\mu} = \aleph_\beta$. Then

$$\aleph_\beta \leqq \Sigma(\mu < m) d_{\nu_\mu} < \Pi(\mu < m) d_{\nu_\mu}.$$

Hence, by Theorem 9, $\aleph_{\beta+1} \nrightarrow (d_{\nu_\mu})^2_{\mu<m}$ and therefore $\aleph_\beta \nrightarrow (d_\nu)^2_{\nu<n}$.

*Case* 2. $r \geqq 3$. We may assume $\aleph_{\beta+(r-3)} \nrightarrow (d_\nu + (r-3))^{r-1}_{\nu<n}$. Then, by Lemma 5A, $\aleph_{\beta+(r-2)} \nrightarrow (d_\nu + (r-2))^r_{\nu<n}$, and Corollary 12 is proved. This completes the proof of Theorem I.

**15. 11. The second Main Theorem.** We shall now discuss the I-relation in the most general case. It will be convenient to introduce the remainder function $\varrho(\alpha)$ which is defined by the relations $\alpha = \omega\gamma(\alpha) + \varrho(\alpha)$; $\varrho(\alpha) < \omega$.

($\ast$) THEOREM II. *Let $n \geqq 2$, and suppose that $2 \leqq r < b_0,, \hat{b}_n \leqq \aleph_\alpha$. Consider the relations:*

(R*) $$\aleph_\alpha \to (b_0,, \hat{b}_n)^r;$$

(IIA) $$\varrho(\alpha) \geqq r-2, \text{ and } b_0,, \hat{b}_n \leqq \aleph_{\alpha \dot{-} (r-2)};$$

(IIB) $$b_0 > \aleph_{\alpha \dot{-} (r-2)};$$

(IIC) $$\varrho(\alpha) < r-2, \text{ and } b_0,, \hat{b}_n \leqq \aleph_{\alpha \dot{-} (r-2)};$$

(IIC1) $$b_0,, \hat{b}_n < \aleph_{\alpha \dot{-} (r-2)};$$

(IIC2) $$b_0 = \aleph_{\alpha \dot{-} (r-2)}.$$

*Then we have the following results:*

(i) *If* (IIA) *holds, then the truth of* (R*) *has been discussed in Theorem I.*

(ii) *If* (IIB) *holds, then* (R*) *is false.*

(iii) *If* (IIC) *and* (IIC1) *hold, then a necessary and sufficient condition for* (R*) *is $b_0 \ldots \hat{b}_n < \aleph_{\alpha \dot{-} (r-2)}$.*

(iv) *If* (IIC) *and* (IIC2) *hold, then a necessary condition for* (R*) *is that $\aleph_{\alpha \dot{-} (r-2)}$ be inaccessible.*

We note that (IIB) implies $\aleph_{\alpha \dot{-} (r-2)} < b_0 \leqq \aleph_\alpha$ and hence $r \geqq 3$.

PROOF OF THEOREM II. Put $\gamma = \alpha \dot{-} (r-2)$.

PROOF OF (ii). It suffices to prove that

$$\aleph_{\gamma+(r-2)} \nrightarrow (\aleph_{\gamma+1}, r+1)^r \text{ for } r \geqq 3.$$

We shall in fact prove a stronger result ($\delta = \gamma + 1$):

($\ast$) COROLLARY 13. *If $r \geqq 3$ and $\aleph_\delta$ is not inaccessible, then*

(26) $$\aleph_{\delta+(r-3)} \nrightarrow (\aleph_\delta, r+1)^r.$$

PROOF. We use induction over $r$. If $r=3$, we have to prove $\aleph_\delta \nrightarrow (\aleph_\delta, 4)^3$, and this follows from Corollary 5. If $\delta = \mathrm{cf}(\delta)$, then this leads, by Lemma 5B, to $\aleph_{\delta+1} \nrightarrow (\aleph_\delta, 5)^4$, and if $\delta > \mathrm{cf}(\delta)$, then this last relation follows from Theorem 13. Thus Corollary 13 is proved for $r < 5$. Suppose now that, for some $r \geqq 5$ and some $\delta$, we have $\aleph_{\delta+(r-4)} \nrightarrow (\aleph_\delta, r)^{r-1}$. Then, by Lemma 5C, we deduce (26). This proves Corollary 13 and so part (ii).

PROOF OF (iii). We have to prove the following two results:

(27) If (IIC) and (IIC1) hold and $b_0 \ldots \hat{b}_n < \aleph_{\alpha \dot{-} (r-2)}$, then (R*) follows.

(28) If (IIC), (IIC1) and (R*) hold, then $b_0 \ldots \hat{b}_n < \aleph_{\alpha \dot{-} (r-2)}$.

PROOF OF (27). Since $\varrho(\alpha) < r-2$, $\gamma$ is of the second kind, and $r \geqq 3$. If $\gamma = 0$, then $n < \omega$ and $b_0,, \hat{b}_n < \omega$. In this case (R*) follows from Ramsey's theorem. Now let $\gamma > 0$. By Lemma 2 it suffices to prove ($s = r - \varrho(\alpha)$).

(∗) COROLLARY 14. *If* $s \geqq 3$; $n \geqq 2$; $\gamma = \gamma \dot{-} 1 > 0$; $\aleph_0 \leqq d_0,, \hat{d}_n < \aleph_\gamma$; $d_0 \ldots \hat{d}_n < \aleph_\gamma$, *then*

(29) $$\aleph_\gamma \to (d_0,, \hat{d}_n)^s.$$

PROOF OF COROLLARY 14. $|n| < 2^{|n|} \leqq d_0 \ldots \hat{d}_n = d < \aleph_\gamma$;

$$d_0,, \hat{d}_n \leqq d.$$

There is $\beta < \gamma$ with $d < \aleph_\beta$. Then, by Theorem 3,

$$\aleph_{(\beta+1)+(s-1)} \to (\aleph_{\beta+2}, (\aleph_{\beta+1})_n)^s,$$

and hence (29) follows.

PROOF OF (28). If we assume $b_0 \ldots \hat{b}_n \geqq \aleph_\gamma$ then, by Theorem I (v),

$$\aleph_{\gamma+(r-2)} \nrightarrow (b_0,, \hat{b}_n)^r.$$

But $\gamma + (r-2) > \alpha$. Hence (R*) is false which is the desired contradiction. This establishes (iii).

PROOF OF (iv). It suffices to prove the following statement.

(30) If $\varrho(\alpha) < r-2$, and if $\aleph_\gamma$ is not inaccessible, then $\aleph_\alpha \nrightarrow (\aleph_\gamma, r+1)^r$.

PROOF OF (30). We have $r \geqq 3$. Hence, by Corollary 13,

$$\aleph_{\gamma+(r-3)} \nrightarrow (\aleph_\gamma, r+1)^r.$$

As $\gamma + (r-3) \geqq \alpha$, the conclusion follows. This concludes the proof of Theorem II.

## 16. THE RELATION I IN THE CASE OF A FINITE NUMBER OF FINITE CARDINALS

We shall now investigate relations $a \to (b_0,, \hat{b}_n)^r$ in the special case when $2 \leqq n < \omega$; $r \geqq 2$; $b_0,, \hat{b}_n < \aleph_0$. By Ramsey's theorem there is a least finite number $a$ such that $a \to (b_\nu)^r_{\nu<n}$, and we shall denote this number by $f(b_0,, b_{n-1}, r)$. In contrast to the infinite case we cannot yet find the exact value of this function $f$ but we can give some estimates of its value. We restrict ourselves to the case $n=2$, and we put $f(b, b, r) = g(b, r)$. We assume that $b_0, b_1 \geqq 1$. The main results known so far are:

**16. 1.** $f(b_0, b_1, 2) \leqq \binom{b_0+b_1-2}{b_0-1}$ (see [8]).

**16. 2.** $g(b, 2) \geqq 2^{\frac{1}{2}b}$ (see [9]).

P. ERDŐS and R. RADO [3] have obtained the following upper estimate for $g(b, r)$. Put $a * b = a^b$ and, generally, $a_0 * a_1 * * a_m = a_0 * (a_1 * * a_m)$ for $2 \leqq m < \omega$.

**16. 3.** If $2 \leqq r \leqq b < \omega$, then

$$g(b, r) \leqq 2 * (2^{r-1}) * (2^{r-2}) * * (2^2) * (2b - 2r + 1)$$

and hence $g(b, r) \leqq 2 * 2 * * 2 * (k_r b)$ ($r$ "factors" in all), where the positive real number $k_r$ depends on $r$ only. In the case $r=2$ the estimates 16. 1 and 16. 2 show that neither is very far from best possible. For fixed $r \geqq 3$, say $r=3$, it was not known at the time [3] was written whether the order of magnitude in 16. 3 was approximately best possible. P. ERDŐS proved a result in such a direction:

**16. 4.** There is a positive real number $c$ such that $g(b, 3) \geqq 2^{cb^2}$ for all $b$. This is stated, without detailed proof, in [9].

It is reasonable to conjecture that, in fact,

$$g(b, 3) \geqq 2^{2^{c_3 b}}$$

for some absolute real constant $c_3 > 0$ and that, more generally,

$$g(b, r) \geqq 2 * 2 * * 2 * (c_r b) \qquad (r \text{ "factors"}) \tag{1}$$

for some real positive $c_r$ which is independent of $b$. By means of the methods of section 14 we can prove the following "stepping-up" lemma.

LEMMA 6. *There is a real number $c \geqq 1/10$ such that*

$$g(b, r) \geqq 2 * (cg(b, r-1)) \qquad \textit{for } r > 4.$$

Using this lemma we can deduce from 16. 2 that for $r \geqq 3$

$$g(b, r) \geqq 2 * 2 * * 2 * (\tfrac{1}{2} c^{r-3} b) \qquad (r-1 \text{ "factors"}).$$

This result approaches the conjecture (1) but a big gap still exists in the case $r=3$ between the conjecture and the established estimate. Since these results are obviously not final we omit the proofs. Further special results concerning the function $f$ are discussed in [10], [11], [12], [13].

## 17. DISCUSSION OF THE RELATION II

In this section we shall prove some negative results of the form

$$a \nrightarrow (b)_c^{<\aleph_0}.$$

The connection of such results with the abstract measure problem was mentioned in 8. 2.

We need some preliminary results. We shall use the definitions of 3. 2.

**17. 1.** (a) If $a < b$, then $a \nrightarrow (b)_2^{<\aleph_0}$.

(b) If $c \geqq \aleph_0$, then $c \nrightarrow (\aleph_0)_c^{<\aleph_0}$.

(c) $\aleph_0 \nrightarrow (\aleph_0)_2^{<\aleph_0}$.

PROOF OF (a). This follows from $a \nrightarrow (b)_2^r$ $(r \geqq 1)$.

PROOF OF (b). If $|S| = c$ then there is, for every $r$, a partition

$$[S]^r = \Sigma'(v < \omega(c)) I(r, v)$$

such that $|I(r, v)| \leqq 1$ for all $r$ and $v$. This proves (b).

PROOF OF (c). This result is essentially [3], p. 435, example 2. Let $S=[0,\omega)$ and $r\geqq 1$. Then $[S]^r = I(r,0)+'I(r,1)$, where $I(r,0) = [S]^r\{\{x_0,, x_{r-1}\}_<: x_0<r\}$. If $X=\{x_0,, \hat{x}_\omega\}_< \subset S$ then, for every $r>x_0$, we have $\{x_0,, x_{r-1}\}\in I(r,0)$ and $\{x_r,, x_{2r-1}\}\in I(r,1)$ which implies the assertion.

**17. 2.** Let $a, b\geqq\aleph_0$; $c\geqq 2$; $a\not\to(b)_c^{<\aleph_0}$. Then

$$2^a \not\to (b)_c^{<\aleph_0}.$$

PROOF. 1. Let $a=\aleph_\alpha$; $m=\omega(c)$; $S=[0,\omega_\alpha)$. Then, for $r\geqq 1$, there is a partition

$$[S]^{r-1} = \Sigma'(\nu<m)I^*(r,\nu)$$

such that, given $D\in[S]^b$, there are infinitely many $r$ which satisfy

$$[D]^{r-1}\not\subset I^*(r,\nu) \quad \text{for} \quad \nu<m.$$

Then there is for $r\geqq 4$ a partition

$$[V]^r = \Sigma'(\nu<m)I(r,\nu)$$

such that, in the notation of section 14,

$$I(r,1) = K_{01}+P_{01}+\bar{I}_1;\quad \bar{I}(r,\nu)=\bar{I}_\nu \qquad (2\leqq\nu<m).$$

2. Let $X\in[V]^b$. We want to show that there are infinitely many $r$ such that

$$[X]^r\not\subset I(r,\nu) \qquad \text{for} \quad \nu<m.$$

We may assume that there are a number $r\geqq 4$ and a number $\nu<m$ such that $[X]^r\subset$ $\subset I(r,\nu)$. Then $[X]^rK_{s,1-s}=\varnothing$ for some $s$, and by 13. 7 there is $X'\in[X]^b$ with $[X']^r\subset K$. Now, $[X']^rP_{t,1-t}=\varnothing$ for some $t$, and by 13. 11 there is $X''=\{x_0,, \hat{x}_k\}_<\in$ $\in[X']^b$ with $[X'']^r\subset P_{0,,0}$. Then $k\geqq\omega(b)$ and $x_\lambda x_{\lambda+1}<x_\mu x_{\mu+1}$ for $\lambda<\mu<k\dot{-}1$. Put $D=\{x_\lambda x_{\lambda+1}: \lambda<\omega(b)\}$. Then $|D|=b$, and hence there is an infinite set $R\subset[4,\omega)$ such that $[D]^{r-1}\not\subset I^*(r,\nu)$ for $r\in R$ and $\nu<m$.

3. Now let $r\in R$ and $\nu<m$. Let us suppose that $[X]^r\subset I(r,\nu)$. Let $L\in[D]^{r-1}$. Then $L=\{x_{\lambda_\varrho}x_{\lambda_\varrho+1}: \varrho<r-1\}$, where $\lambda_0<\,<\lambda_{r-2}<\omega(b)$. Put $\lambda_{r-1}=\lambda_{r-2}+1$. Then, by 13. 14, $x_{\lambda_\varrho}x_{\lambda_\varrho+1}=x_{\lambda_\varrho}x_{\lambda_{\varrho+1}}$ for $\varrho<r-1$ and so, by definition of $I(r,\nu)$,

$$L = \{x_{\lambda_\varrho}x_{\lambda_{\varrho+1}}: \varrho<r-1\} = \delta'(x_{\lambda_0},, x_{\lambda_{r-1}})\in I^*(r,\nu);$$

$$[D]^{r-1}\subset I^*(r,\nu)$$

which is a contradiction. This proves that

$$[X]^r\not\subset I(r,\nu) \quad \text{for } r\in R \text{ and } \nu<m$$

and so completes the proof of 17. 2.

**17. 3.** Let $a>a'$; $b\geqq\aleph_0$; $c\geqq 2$. Suppose that $a_0\not\to(b)_c^{<\aleph_0}$ for every $a_0<a$. Then $a\not\to(b)_c^{<\aleph_0}$.

PROOF. Put $\omega(a')=k$; $\omega(c)=m$. Let $S=\Sigma'(\varkappa<k)S_\varkappa$ and $|S_\varkappa|<|S|=a$ for $\varkappa<k$. Then there are partitions

$$[S_\varkappa]^r = \Sigma'(\nu<m)I(\varkappa,r,\nu) \quad (\text{partition } \Delta_{\varkappa r})$$

for $\varkappa < k$ and $r < \omega$, and partitions

$$[[0, k)]^r = \Sigma'(v < m) I^*(r, v) \quad \text{(partition } \Delta_r^*)$$

for $r < \omega$ such that the following conditions hold.

$$(1) \qquad \begin{cases} \text{If } \varkappa < k \text{ and } X \in [S_\varkappa]^b, \text{ then there are infinitely many } r \text{ such that} \\ \qquad |\Delta_{\varkappa r}| > 1 \text{ in } [X]^r. \end{cases}$$

$$(2) \qquad \begin{cases} \text{If } M \in [[0, k)]^b, \text{ then there are infinitely many } r \text{ such that} \\ \qquad |\Delta_r^*| > 1 \text{ in } [M]^r. \end{cases}$$

Let $r \geqq 3$. Then there is a partition

$$[S]^r = \Sigma'(v < m) I(r, v) \quad \text{(partition } \Delta_r)$$

such that the following rules are satisfied. Let $A = \{x_0,, x_{r-1}\}_{\neq} \subset S$; $\varkappa_0 \leqq \leqq \varkappa_{r-1} < < k$; $x_\varrho \in S_{\varkappa_\varrho}$ for $\varrho < r$.

If $\varkappa_0 = \varkappa_{r-1}$ and $A \in I(\varkappa_0, r, v)$, then $A \in I(r, v)$. If $\varkappa_0 < < \varkappa_{r-1}$ and $\{\varkappa_0,, \varkappa_{r-1}\} \in \in I^*(r, v)$, then $A \in I(r, v)$. If $\varkappa_0 < \varkappa_1 = \varkappa_{r-1}$, then $A \in I(r, 0)$. If $\varkappa_0 = \varkappa_{r-2} < \varkappa_{r-1}$, then $A \in I(r, 1)$.

Now let $X \in [S]^b$. We want to find infinitely many $r$ such that

$$(3) \qquad [X]^r \not\subset I(r, v) \quad \text{for} \quad v < m.$$

We may assume that $[X]^p \subset I(p, v_0)$ for some $p \in [3, \omega)$ and some $v_0 < m$. If there are numbers $\varkappa_0 < \varkappa_1 < k$ with $|XS_{\varkappa_0}|, |XS_{\varkappa_1}| \geqq p-1$, then we can choose $v_1 \in [0, 2) - \{v_0\}$ and obtain $[X]^p I(p, v_1) \neq \emptyset$ which contradicts $[X]^p \subset I(p, v_0)$. Hence there is at most one $\varkappa < k$ with $|XS_\varkappa| \geqq p-1$. Put $M = \{\varkappa : XS_\varkappa \neq \emptyset\}$.

*Case* 1. $|M| = b$. Then we choose $x_\varkappa \in XS_\varkappa$ for $\varkappa \in M$ and put $X' = \{x_\varkappa : \varkappa \in M\}$. Then by (2) there is an infinite set $R \subset [0, \omega)$ such that, for $r \in R$, $|\Delta_r^*| > 1$ in $[M]^r$. Then, for $r \in R$, $|\Delta_r| > 1$ in $[X']^r$, and hence also in $[X]^r$. This proves (3).

*Case* 2. $|M| < b$. Then there is $\varkappa_0 < k$ such that $|XS_\varkappa| < p-1$ for $\varkappa \neq \varkappa_0$. Then $b = |X| \leqq |XS_{\varkappa_0}| + (p-2)|M|$; $|XS_{\varkappa_0}| = b$, and we put $X' = XS_{\varkappa_0}$. Then $X' \subset S_{\varkappa_0}$; $|X'| = b$, and by (1) there is an infinite set $R \subset [0, \omega)$ such that, for $r \in R$, $|\Delta_{\varkappa_0 r}| > 1$ in $[X']^r$. Then, for $r \in R$, $|\Delta_r| > 1$ in $[X']^r$ and hence also in $[X]^r$. This proves 17. 3.

**17. 4.** If $a, b, c$ are such that $a_0 \nrightarrow (b)_c^{<\aleph_0}$ for every $a_0 < a$, then

$$a \nrightarrow (b^+)_c^{<\aleph_0}.$$

Proof. Put $\omega(a) = n$; $\omega(b) = k$; $\omega(c) = m$. If $r < \omega$ and $n_0 < n$, then there is a partition

$$[[0, n_0)]^r = \Sigma'(v < m) I(n_0, r, v) \qquad \text{(partition } \Delta_{n_0 r})$$

such that whenever $Y \in [[0, n_0)]^b$, then there are infinitely many $r$ such that $|\Delta_{n_0 r}| > 1$ in $[Y]^r$. Then for every $r$ there is a partition

$$[[0, n)]^{r+1} = \Sigma'(v < m) I(r, v) \qquad \text{(partition } \Delta_r)$$

such that

$$I(r, v) = \{\{x_0,, x_r\}_< : \{x_0,, \hat{x}_r\} \in I(x_r, r, v)\} \qquad \text{for } v < m.$$

Now let $X \in [[0, n)]^{b+}$. Then there is a set $\{x_0,, x_k\}_< \subset X$. Then $Y = \{x_0,, \hat{x}_k\} \in \in [[0, x_k)]^b$, and hence there is an infinite set $R \subset [0, \omega)$ such that $|\Delta_{x_k r}| > 1$ in $[Y]^r$ for every $r \in R$. Then, for $r \in R$, $|\Delta_r| > 1$ in $[Y + \{x_k\}]^{r+1}$ and hence in $[X]^{r+1}$. This proves 17. 4.

We now come to the main result of this section.

THEOREM 15. *Let* $\alpha \geqq 0$, *and let* $\{d_0,, \hat{d}_k\}_<$ *be the set of all strongly inaccessible cardinals not exceeding* $\aleph_\alpha$. *Then* $k \geqq 1$, *and*

$$\aleph_\alpha \nrightarrow (\aleph_{k-1})_2^{<\aleph_0} \quad \text{if} \quad k < \omega,$$
$$\aleph_\alpha \nrightarrow (\aleph_k)_2^{<\aleph_0} \quad \text{if} \quad k \geqq \omega.$$

PROOF. We use induction over $\alpha$. If $\alpha = 0$ then $k = 1$, since $\aleph_0$ is strongly inaccessible, and the assertion holds by 17. 1 (c). Let $\beta > 0$, and suppose that the assertion holds for all $\alpha < \beta$. Let $\{d_0,, \hat{d}_l\}_<$ be the set of all strongly inaccessible cardinals not exceeding $\aleph_\beta$. Put $n = l-1$ if $l < \omega$, and $n = l$ if $l \geqq \omega$. Then, by induction hypothesis, we have for $\alpha < \beta$

$$\aleph_\alpha \nrightarrow (\aleph_n)_2^{<\aleph_0}. \tag{4}$$

*Case* 1. $\beta > \mathrm{cf}(\beta)$. Then (4) implies, by 17. 3, the desired relation

$$\aleph_\beta \nrightarrow (\aleph_n)_2^{<\aleph_0}. \tag{5}$$

*Case* 2. $\beta = \mathrm{cf}(\beta)$.

*Case* 2a. $2^{\aleph_\gamma} \geqq \aleph_\beta$ for some $\gamma < \beta$. Then (4) holds for $\alpha = \gamma$ and hence, by 17.2,

$$2^{\aleph_\gamma} \nrightarrow (\aleph_n)_2^{<\aleph_0}.$$

This implies (5).

*Case* 2b. $2^{\aleph_\gamma} < \aleph_\beta$ for every $\gamma < \beta$. Then $\aleph_\beta$ is strongly inaccessible; $l = \lambda + 1$; $d_\lambda = \aleph_\beta$. Then $\aleph_0 < \aleph_\beta = d_\lambda$, and hence $\lambda \geqq 1$.

*Case* 2b1. $\lambda < \omega$. Then, by induction hypothesis,

$$\aleph_\alpha \nrightarrow (\aleph_{\lambda-1})_2^{<\aleph_0} \qquad \text{for } \alpha < \beta$$

and hence, by 17. 4,

$$\aleph_\beta \nrightarrow (\aleph_\lambda)_2^{<\aleph_0}.$$

But $\lambda = l-1 = n$ so that (5) holds.

*Case* 2b2. $\lambda \geqq \omega$. Then, similarly,

$$\aleph_\alpha \nrightarrow (\aleph_\lambda)_2^{<\aleph_0} \qquad \text{for } \alpha < \beta$$

and hence $\aleph_\beta \nrightarrow (\aleph_{\lambda+1})_2^{<\aleph_0}$, where $\lambda + 1 = l = n$. This proves Theorem 15.

Let $\{d_0,, \hat{d}_n\}_<$ be the set of all strongly inaccessible cardinals below some given cardinal. Then Theorem 15 yields the following results.

COROLLARY 15. (i) $d_\nu \nrightarrow (\aleph_\nu)_2^{<\aleph_0}$ *for* $\nu < \min(n, \omega)$;

(ii) $d_\nu \nrightarrow (\aleph_{\nu+1})_2^{<\aleph_0}$ *for* $\nu < n$;

(iii) $d_\nu \nrightarrow (d_\nu)_2^{<\aleph_0}$ *if* $\nu < n$ *and* $d_\nu > \aleph_\nu$;

(iv) *if all strongly inaccessible cardinals d are less than some fixed cardinal,* *and if* $\{d_0,, \hat{d}_m\}_<$ *is the set of all such* $d$, *then*

$$\aleph_\alpha \nrightarrow (\aleph_m)_2^{<\aleph_0} \quad \textit{for all } \alpha.$$

Let $c \geqq \aleph_0$. Then, by employing at the start of the induction argument the relation 17. 1 (b) instead of 17. 1 (c) we obtain by the same method as was used in the proof of Theorem 15 the following result.

THEOREM 16. *Let* $\aleph_\alpha \geqq c \geqq \aleph_0$. *Denote by* $\{\hat{d}_0(c),, \hat{d}_k(c)\}_<$ *the set of all strongly inaccessible cardinals* $d$ *such that* $c < d \leqq \aleph_\alpha$. *Then* $k \geqq 1$, *and*

$$\aleph_\alpha \nrightarrow (\aleph_{k-1})_c^{<\aleph_0} \qquad \textit{if} \quad k < \omega;$$

$$\aleph_\alpha \nrightarrow (\aleph_k)_c^{<\aleph_0} \qquad \textit{if} \quad k \geqq \omega.$$

The proof, as well as the analogue to Corollary 15, may be omitted.

## 18. THE RELATIONS IV, V AND VI. PROBLEMS

In this section we shall deal with some generalizations of the partition relations I and II. Let $a, r, n, b_0,, \hat{b}_n$ be given.

**18. 1.** DEFINITION. The relation *(partition relation IV)*

(1) $$a \to [b_0,, \hat{b}_n]^r,$$

also written in the form $a \to [b_\nu]^r_{\nu<n}$, denotes the following statement. Whenever $|S| = a$ and $[S]^r = \Sigma'(\nu < n) I_\nu$, then there are a set $X \subset S$ and a number $\nu < n$ such that $|X| = b_\nu$ and $[X]^r I_\nu = \emptyset$.

**18. 2.** DEFINITION. If $b_0 = = \hat{b}_n = b$ and $|n| = c$, then (1) is also written in the form

(2) $$a \to [b]^r_c.$$

**18. 3.** The relation *(partition relation V)*

$$a \to [b]^r_{c,d}$$

denotes the following statement. Whenever $|S| = a$ and $[S]^r = \Sigma'(\nu < \omega(c)) I_\nu$, then there are a set $X \subset S$ and a set $D \subset [0, \omega(c))$ such that $|X| = b$; $|D| \leqq d$, and

$$[X]^r \subset \Sigma(\nu \in D) I_\nu.$$

The relation (1) is only of interest if $r \geqq 2$; $a \geqq \aleph_0$; $n \geqq 2$; $r < b_0,, \hat{b}_n \leqq a$. In this section we shall mainly investigate the relations IV and V as well as problems suggested by them.

The following propositions follow immediately from our definitions.

**18. 4.** The relations $a \to (b_0, b_1)^r$ and $a \to [b_0, b_1]^r$ are equivalent. The relations $a \to (b)^r_c$ and $a \to [b]^r_{c,1}$ are equivalent. If $1 \leqq c < \omega$, then the relations $a \to [b]^r_{c,c-1}$ and $a \to [b]^r_c$ are equivalent.

For $r \geqq 3$ our results concerning the relations IV and V are rather incomplete. The positive relations will be deduced by means of the general methods used in the discussion of I-relations but to obtain our negative relations we shall employ new ideas for the construction of suitable counter examples. Interesting problems of a new type will arise.

Just like the relations I, II and III so the relations IV and V possess obvious monotonicity and other properties whose proof is left to the reader. We mention:

**18. 5.** Let $m \geqq 2$ and $v_0 < \ldots < \hat{v}_m < n$. Then, for any cardinals $b_0, \ldots, \hat{b}_n$, $a \to (b_{v_0}, \ldots, \hat{b}_{v_m})^r$ implies $a \to [b_0, \ldots, \hat{b}_n]^r$. In particular, if $\mu < v < n$ and $a \to (b_\mu, b_v)^r$, then $a \to [b_0, \ldots, \hat{b}_n]^r$.

**18. 6.** Let the cardinals $a, b, c, d, e$ satisfy $c < d \leqq e$. Then $a \to [b]^r_{d,c}$ implies $a \to [b]^r_e$.

**18. 7.** If $a \geqq \aleph_0$, then $a \to [a]^r_{c,a}$ for all $c, r$. If $c \leqq d$, then $a \to [b]^r_{c,d}$ for all $b \leqq a$ and all $r$. If $c > b \geqq \aleph_0$, then $a \to [b]^r_c$ for $a \geqq b$ and all $r$. If $c \geqq e$, then $a \to [b]^r_{c,d}$ implies $a \to [b]^r_{e,d}$.

Up to 18. 10 we shall mainly be concerned with IV-relations. We shall then consider V-relations and in particular mention some of the unsolved problems. Finally, VI-relations will be briefly considered in 18. 19.

By Theorem I we have the following results:

(i) $\aleph_\alpha \to (\aleph_\beta, \aleph_\gamma)^2$ holds if $\beta, \gamma < \alpha$.

(ii) A necessary condition for $\aleph_\alpha \to (\aleph_\alpha, \aleph_\gamma)^2$ is $\gamma \leqq \mathrm{cr}(\alpha)$.
This condition is at the same time sufficient except possibly when $\gamma = \mathrm{cf}(\alpha)$ and $\aleph'_\alpha$ is inaccessible.

Taking into consideration 18. 4—18. 6 we see that the principal problems about IV-relations for $r = 2$ concern the truth of relations of theform $a \to [b_v]^2_{v<n}$, where $a = \aleph_\gamma$;

$$\aleph_{\mathrm{cr}(\gamma)} < b_0 \leqq \aleph_\gamma; \quad b_1 = \ldots = \hat{b}_n = a; \quad 2 \leqq n \leqq \omega_\gamma.$$

If $a'$ is inaccessible, then there remain unsolved problems (see Problem 1). The complete discussion of the cases when $a'$ is not inaccessible will be given by Theorems 17, 18, 20, 21, 22. For $r \geqq 4$ we have only isolated results.

Consider first the case $a = \aleph_{\alpha+1}$ when $\mathrm{cr}(\alpha+1) = \mathrm{cf}(\alpha)$. Here we have the best possible negative result:

$$\aleph_{\alpha+1} \nrightarrow [b_v]^2_{v<\omega_{\alpha+1}}$$

if $b_0 \geqq \aleph'^{+}_\alpha$; $b_1, \ldots, \hat{b}_{\omega_{\alpha+1}} = \aleph_\alpha$. The proof differs slightly according to whether $\alpha = \mathrm{cf}(\alpha)$ or $\alpha > \mathrm{cf}(\alpha)$. We first prove

($*$) THEOREM 17. *$\aleph_{\alpha+1} \nrightarrow [\aleph_{\alpha+1}]^2_{\aleph_{\alpha+1}}$ for every $\alpha$.*

Instead of Theorem 17 we shall prove the more general Theorem 17A which will have an application in the discussion of the polarized relation.

($*$) THEOREM 17A. *Let $|S| = \aleph_{\alpha+1}$. Then there is a partition*

$$[S]^2 = \Sigma'(v < \omega_{\alpha+1}) I_v \tag{3}$$

*which has the following property. Whenever $A, B \subset S$; $|A| = \aleph_\alpha$; $|B| = \aleph_{\alpha+1}$, and $v_0 < \omega_{\alpha+1}$, then there are elements $x \in A$ and $y \in B$ such that $\{x, y\} \in I_{v_0}$.*

A suggestive notation for the assertion of this theorem would be

$$\aleph_{\alpha+1} \nrightarrow \begin{bmatrix} \aleph_\alpha \\ \aleph_{\alpha+1} \end{bmatrix}^{1,1}_{\aleph_{\alpha+1}}.$$

PROOF. Put $a=\aleph_\alpha$; $m=\omega_\alpha$; $n=\omega_{\alpha+1}$. Let $S=\{y_0,, \hat{y}_n\}_{\neq}$. Then we can write $[S]^a=\{X_0,, \hat{X}_n\}_{\neq}$.

1. Let $\varrho<n$. Put $S_\varrho=\{y_0,, \hat{y}_\varrho\}$ and $Z_\varrho=\{X_\pi: \pi<\varrho \wedge X_\pi \subset S_\varrho\}$. We can write

$$S_\varrho = \{x(\varrho, \mu): \mu<m_\varrho\}_{\neq}$$

and

$$Z_\varrho \times [0, \varrho) = \{(X(\varrho, \mu), v(\varrho, \mu)): \mu<n_\varrho\},$$

where $m_\varrho$ and $n_\varrho$ are initial ordinals not exceeding $m$. We can find inductively numbers $\sigma(\varrho, \lambda)<m_\varrho$, for $\lambda<n_\varrho$, such that $x(\varrho, \sigma(\varrho, \mu)) \in X(\varrho, \mu)-\{x(\varrho, \sigma(\varrho, \lambda)): \lambda<\mu\}$ for $\mu<n_\varrho$. Then $x(\varrho, \sigma(\varrho, \mu)) \in X(\varrho, \mu)$ for $\mu<n_\varrho$; $\sigma(\varrho, \mu) \neq \sigma(\varrho, \lambda)$ for $\lambda<\mu<n_\varrho$.

2. Let $\pi<\varrho<n$. Then there is exactly one $\tau(\varrho, \pi)<m_\varrho$ with $x(\varrho, \tau(\varrho, \pi))=y_\pi$. There is a partition (3) such that

$$\{y_\pi, y_\varrho\} \in I_{v(\varrho,\mu)}, \text{ if } \mu<n_\varrho \text{ and } \tau(\varrho, \pi)=\sigma(\varrho, \mu).$$

This condition does not define the partition since the $\{y_\pi, y_\varrho\}$ with $\tau(\varrho, \pi) \notin \{\sigma(\varrho, \mu): \mu<n_\varrho\}$ may be placed into any arbitrary class $I_v$.

Now let $A, B \subset S$; $|A|=a$; $|B|=a^+$; $v_0<n$. Then there is $\varrho_0<n$ with $A \in Z_{\varrho_0}$. Next, we can find $y_\varrho \in B$ with $v_0, \varrho_0<\varrho<n$. Then $(A, v_0) \in Z_\varrho \times [0, \varrho)$, and hence there is $\mu<n_\varrho$ with $(A, v_0)=(X(\varrho, \mu), v(\varrho, \mu))$. Then $x(\varrho, \sigma(\varrho, \mu))=y_\pi$ for some $\pi<\varrho$, and we have

$$x(\varrho, \sigma(\varrho, \mu))=y_\pi=x(\varrho, \tau(\varrho, \pi)); \quad \tau(\varrho, \pi)=\sigma(\varrho, \mu);$$

$$\{y_\pi, y_\varrho\} \in I_{v(\varrho,\mu)}=I_{v_0}; \quad y_\pi=x(\varrho, \sigma(\varrho, \mu)) \in X(\varrho, \mu)=A; \quad y_\varrho \in B.$$

This proves that the partition (3) has the required property, and Theorem 17A follows.

We are now going to sharpen Theorem 17.

($*$) THEOREM 18.

$$\aleph_{\alpha+1} \nrightarrow [\aleph_\alpha'^+, (\aleph_{\alpha+1})_{\aleph_{\alpha+1}}]^2 \quad \textit{for all } \alpha.$$

PROOF. If $\alpha=\mathrm{cf}(\alpha)$, then the assertion follows from Theorem 17. Now let $\alpha>\mathrm{cf}(\alpha)$. We consider the set $V'(\alpha)$ introduced in 11. 3, together with its two orders $x<y$ and $x \prec y$. Put $a=\aleph_\alpha$; $m=\omega_\alpha$; $n=\omega_{\alpha+1}$; $S=V'(\alpha)$. Then we can write $S=\{y_0,, \hat{y}_n\}_{\neq}$. For $P, Q \subset S$ the relation $P<Q$ means that whenever $x \in P$ and $y \in Q$ then $x<y$. If $P=\{x\}$ then we write $x<Q$ instead of $\{x\}<Q$. Similarly we use $P \prec Q$. We write $[S]^a=\{X_0,, \hat{X}_n\}_{\neq}$.

1. Let $\varrho<n$; $S_\varrho=\{y_0,, \hat{y}_\varrho\}$; $Z_\varrho=\{X_\pi: \pi<\varrho \wedge X_\pi \prec y_\varrho<X_\pi\}$. We can write

$$S_\varrho = \{x(\varrho, \mu): \mu<m_\varrho\}_{\neq};$$

$$Z_\varrho \times [0, \varrho) = \{(X(\varrho, \mu), v(\varrho, \mu)): \mu<n_\varrho\},$$

where $m_\varrho$ and $n_\varrho$ are initial ordinals not exceeding $m$. We can find inductively numbers

$\sigma(\varrho, \lambda) < m_\varrho$ for $\lambda < n_\varrho$ such that

$$x(\varrho, \sigma(\varrho, \mu)) \in X(\varrho, \mu) - \{x(\varrho, \sigma(\varrho, \lambda)) \colon \lambda < \mu\} \quad \text{for } \mu < n_\varrho.$$

Then

$$x(\varrho, \sigma(\varrho, \mu)) \in X(\varrho, \mu) \quad \text{for} \quad \mu < n_\varrho;$$

$$\sigma(\varrho, \mu) \neq \sigma(\varrho, \lambda) \quad \text{for} \quad \lambda < \mu < n_\varrho.$$

2. Let $\pi < \varrho < n$. Then there is exactly one $\tau(\varrho, \pi) < m_\varrho$ with $x(\varrho, \tau(\varrho, \pi)) = y_\pi$. There is a partition

$$[S]^2 = I + '\Sigma'(\nu < n) I_\nu$$

defined by the rules:

$$\{y_\pi, y_\varrho\} \in I_{\nu(\varrho,\mu)} \quad \text{if} \quad \tau(\varrho, \pi) = \sigma(\varrho, \mu);$$

$$\{y_\pi, y_\varrho\} \in I \quad \text{if} \quad \tau(\varrho, \pi) \notin \{\sigma(\varrho, \mu) \colon \mu < n_\varrho\}.$$

It now suffices to prove that this partition has the following properties:

(i) If $X \subset S$ and $[X]^2 I = \emptyset$, then $|X| \leqq a'$.

(ii) If $X \subset S$; $\nu_0 < n$; $[X]^2 I_{\nu_0} = \emptyset$, then $|X| \leqq a$.

3. Proof of (i). Let $X \subset S$ and $[X]^2 I = \emptyset$. Then we can write

$$X = \{y_{\varrho_0}, , \hat{y}_{\varrho_k}\}_<.$$

Let $\{\pi, \varrho\}_< \subset \{\varrho_\varkappa \colon \varkappa < k\}$. Then there is $\mu < n_\varrho$ such that $\{y_\pi, y_\varrho\} \in I_{\nu(\varrho,\mu)}$; $\tau(\varrho, \pi) = \sigma(\varrho, \mu)$;

$$y_\pi = x(\varrho, \tau(\varrho, \pi)) = x(\varrho, \sigma(\varrho, \mu)) \in X_{\varrho\mu} \in Z_\varrho.$$

By definition of $Z_\varrho$ we have $X_{\varrho\mu} \prec y_\varrho < X_{\varrho\mu}$; $y_\pi \prec y_\varrho < y_\pi$. Hence $\mathrm{tp}\,(X, <) = k^*$, and by 11. 5 (ii), $|X| = |k| \leqq a'$. This proves (i).

4. Proof of (ii). Let $X \subset S$; $\nu_0 < n$; $|X| = a^+$. We want to find $x, y \in X$ with $\{x, y\} \in I_{\nu_0}$. It is well known that there are sets $A, B \in [X]^{a^+}$ with $A < B$. This follows for instance from 11. 5 (ii) and [1], p. 446, Lemma 1. There is $\varrho_0 < n$ with $X_{\varrho_0} \subset B$. There is $\lambda < n$ with $[0, \varrho_0) + \Sigma(y_\pi \in X_{\varrho_0})[0, \pi) = [0, \lambda)$. There is $y_\varrho \in A$ with $\lambda, \nu_0 < \varrho < n$. Then $(X_{\varrho_0}, \nu_0) \in Z_\varrho \times [0, \varrho)$, and hence there is $\mu < n_\varrho$ with $(X_{\varrho_0}, \nu_0) = (X(\varrho, \mu), \nu(\varrho, \mu))$. There is $\pi < \varrho$ with $x(\varrho, \sigma(\varrho, \mu)) = y_\pi = x(\varrho, \tau(\varrho, \pi))$. Then $\tau(\varrho, \pi) = \sigma(\varrho, \mu)$; $\{y_\pi, y_\varrho\} \in$ $\in I_{\nu(\varrho,\mu)} = I_{\nu_0}$;

$$y_\pi = x(\varrho, \sigma(\varrho, \mu)) \in X(\varrho, \mu) = X_{\varrho_0} \subset B \subset X; \quad y_\varrho \in A \subset X.$$

This proves (ii) and completes the proof of Theorem 18.

Let us now consider the case $r \geqq 3$. By Theorem 11 we have $(\ast)$ $\aleph_{\alpha+1} \nrightarrow$ $\nrightarrow [\aleph_{\alpha+1}, r+1]^r$. The following theorem gives a best possible generalization of this result.

$(\ast)$ Theorem 19. *For $r \geqq 3$ and every $\alpha$,*

$$\aleph_{\alpha+1} \nrightarrow [r+1, (\aleph_{\alpha+1})_{\aleph_{\alpha+1}}]^r.$$

This theorem will follow as immediate corollary of Theorem 28 which will be proved in 19. 2. It is clear that Theorem 19 does not settle the problem of deciding whether for any given cardinals $b_0,, \hat{b}_n < \aleph_{\alpha+1}$ we have $\aleph_{\alpha+1} \rightarrow [b_0,, \hat{b}_n]^r$. We shall consider such problems later, in 18. 9.

Let us now turn to relations of the form $\aleph_\alpha \rightarrow [b_0,, \hat{b}_n]^r$ when $\alpha > \mathrm{cf}(\alpha)$. Here our results are almost complete for every $r \geqq 2$. Unsolved problems remain only when $\aleph'_\alpha$ is inaccessible and greater than $\aleph_0$.

Consider first the case when $\alpha > \mathrm{cf}(\alpha) = \beta + 1$ so that $\mathrm{cr}(\alpha) = \mathrm{cf}(\beta)$.

*Case* 1. $r = 2$. We have to discuss the relation $\aleph_\alpha \rightarrow [b, (\aleph_\alpha)_c]^2$ when $\aleph'_\beta < b \leqq \aleph_\alpha$ and $2 \leqq c \leqq \aleph_\alpha$.

*Case* 2. $r \geqq 3$. Then we have $\aleph_\alpha \rightarrow (b_0, b_1)^r$ for every $b_0, b_1 < \aleph_\alpha$. For there is $\beta_0 < \alpha$ with $b_0, b_1 < \aleph_{\beta_0}$, and then, by Theorem I,

$$\aleph_\alpha > \aleph_{\beta_0+(r-2)} \rightarrow (b_0, b_1)^r.$$

Also, by Theorem 12, $\aleph_\alpha \nrightarrow (\aleph_\alpha, r+1)^r$. Thus there only remains to discuss the relation $\aleph_\alpha \rightarrow [b, (\aleph_\alpha)_c]^r$ when $r < b \leqq \aleph_\alpha$ and $2 \leqq c \leqq \aleph_\alpha$. The following best possible theorem settles all these questions for $c > \aleph'_\alpha$ when $\aleph_\alpha$ is singular.

($*$) THEOREM 20. *If* $r \geqq 2$; $\alpha > \mathrm{cf}(\alpha)$; $c > \aleph'_\alpha$, *then*

$$\aleph_\alpha \rightarrow [\aleph_\alpha]^r_c.$$

Instead of Theorem 20 we shall prove the following stronger theorem involving a V-relation.

($*$) THEOEREM 20A. *If* $r \geqq 2$ *and* $\aleph'_\alpha < c < \aleph_\alpha$, *then*

$$\aleph_\alpha \rightarrow [\aleph_\alpha]^r_{c, \aleph'_\alpha}.$$

It is clear from 18. 6 that Theorem 20A implies Theorem 20.

PROOF. Let $|S| = \aleph_\alpha$; $[S]^r = \Sigma'(\nu < \omega(c)) I_\nu$ (partition $\Delta$). Then, by Lemma 3, there is a set $T = \Sigma'(\mu < m) S_\mu \subset S$ such that $m = \omega_{\mathrm{cf}(\alpha)}$; $|T| = \aleph_\alpha$, and $\Delta$ is canonical in $(S_0,, \hat{S}_m)$. Let $r_0 + + \hat{r}_m = r$. Then there is $f(r_0,, \hat{r}_m) < \omega(c)$ such that whenever $X \subset T$ and $|XS_\mu| = r_\mu$ for $\mu < m$, then $X \in I_{f(r_0,, \hat{r}_m)}$. Hence

$$[T]^r \subset \Sigma(r_0 + + \hat{r}_m = r) I_{f(r_0,, \hat{r}_m)},$$

and the assertion follows from

$$|\{(r_0,, \hat{r}_m) : r_0 + + \hat{r}_m = r\}| \leqq |m|^r = \aleph'_\alpha.$$

The following best possible theorem settles the case when $c \leqq \aleph'_\alpha$ and $\alpha > \mathrm{cf}(\alpha) = \beta + 1$.

($*$) THEOREM 21. *Let* $\alpha > \mathrm{cf}(\alpha) = \beta + 1$ *and* $c \leqq \aleph_{\beta+1}$. *Then*

(i) $\aleph_\alpha \nrightarrow [\aleph'^{+}_\beta, (\aleph_\alpha)_c]^2$;

(ii) $\aleph_\alpha \nrightarrow [r+1, (\aleph_\alpha)_c]^r$ *for* $r \geqq 3$.

PROOF. We may assume $c = \aleph_{\beta+1}$. Let $\omega_{\beta+1} = n$; $S = \Sigma'(\nu < n) S_\nu$; $|S_\nu| < |S| = \aleph_\alpha$ for $\nu < n$.

PROOF OF (i). By Theorem 18, there is a partition

$$[[0, n)]^2 = \Sigma'(\nu < n) I_\nu$$

which brings into evidence the relation

$$\aleph_{\beta+1} \nrightarrow [\aleph_\beta'^{+}, (\aleph_{\beta+1})_{\aleph_{\beta+1}}]^2.$$

This means that if $D, E \subset [0, n)$ then $[D]^2 I_0^* = \varnothing$ implies $|D| \leqq \aleph_\beta'$, and, for $1 \leqq \nu < n$, $[E]^2 I_\nu^* = \varnothing$ implies $|E| \leqq \aleph_\beta$. Then

$$[S]^2 = \Sigma'(\nu < n) I_\nu,$$

where

$$I_\nu = \Sigma(\{\lambda, \mu\} \in I_\nu^*)[S_\lambda, S_\mu]^{1,1} \qquad \text{for } 1 \leqq \nu < n.$$

Put $D(X) = \{\nu : XS_\nu \neq \varnothing\}$ for $X \subset S$. Then, if $X \subset S$ and $[X]^2 I_0 = \varnothing$, we have $[D(X)]^2 I_0^* = \varnothing$ and hence $|X| = |D(X)| \leqq \aleph_\beta'$. If $Y \subset S$; $1 \leqq \nu < n$; $[Y]^2 I_\nu = \varnothing$; $|Y| = \aleph_\alpha$, then $|D(Y)| = \aleph_{\beta+1}$; $[D(Y)]^2 I_\nu^* = \varnothing$ which is a contradiction. This proves (i).

Proof of (ii). Let $r \geqq 3$. Then, by Theorem 19, there is a partition

$$[[0, n)]^r = \Sigma'(\nu < n) I_\nu^*$$

which brings into evidence the relation

$$\aleph_{\beta+1} \nrightarrow [\gamma + 1, (\aleph_{\beta+1})_{\aleph_{\beta+1}}]^r.$$

Then

$$[S]^r = \Sigma'(\nu < n) I_\nu,$$

where

$$I_\nu = \Sigma(\{\nu_0,, \nu_{r-1}\} \in I_\nu^*)[S_{\nu_0},, S_{\nu_{r-1}}]^{1,,1} \quad \text{for} \quad 1 \leqq \nu < n.$$

Then exactly as in the proof of (i), we obtain the desired properties of this last partition, and Theorem 21 follows.

The relation of Theorem 20 is valid for every singular $\aleph_\alpha$, and Theorem 21 shows that this relation is best possible provided that $\aleph_\alpha'$ is not inaccessible. If, on the other hand, $\aleph_\alpha'$ is inaccessible then we have the following result which is in some ways stronger.

($*$) Theorem 22. *Let* $r \geqq 2$ *and* $\alpha > \mathrm{cf}(\alpha)$. *Then*

(i) $\qquad \aleph_\alpha \rightarrow [\aleph_\alpha]^r_{2^{r-1}}$.

*If, in addition, either* $\mathrm{cf}(\alpha) = 0$, *or* $\mathrm{cf}(\alpha) > 0$ *and* $\aleph_\alpha'$ *is measurable,* then*

(ii) $\qquad \aleph_\alpha \nrightarrow [\aleph_\alpha]^r_c$ *for* $c > 2^{r-1}$;

(iii) $\qquad \aleph_\alpha \rightarrow [\aleph_\alpha]^r_{c, 2^{r-1}}$ *for* $c < \aleph_\alpha$.

Proof of (i). Let $n = \omega_{\mathrm{cf}(\alpha)}$; $S = \Sigma'(\nu < n) S_\nu$; $|S_\nu| < |S| = \aleph_\alpha$ for $\nu < n$. Then there is a partition

$$[S]^r = \Sigma'(p \leqq r \wedge r_0,, r_{p-1} \geqq 1 \wedge r_0 + + r_{p-1} = r) I(r_0,, r_{p-1}),$$

where $I(r_0,, r_{p-1})$ is the set of all $A \in [S]^r$ such that, for some $\nu_0 < < \nu_{p-1} < n$, we have $|AS_{\nu_\lambda}| = r_\lambda$ for $\lambda < p$. It follows that if $X \in [S]^{\aleph_\alpha}$ then $|XS_\nu| \geqq r$ for at least $r$ values of $\nu$. But then $[X]^r I(r_0,, r_{p-1}) \neq \varnothing$ for every choice of $(r_0,, r_{p-1})$, and since there are exactly $2^{r-1}$ such systems the assertion follows.

* i. e., does not possess property $P_3$ of [24]. See 8. 2.

PROOF OF (iii). Let $|S| = \aleph_\alpha$; $n = \omega_{\mathrm{cf}(\alpha)}$;

$$[S]^r = \Sigma'(\mu < \omega(c)) I_\mu \qquad \text{(partition } \Delta).$$

Then, by Lemma 3B, there is a set $T = \Sigma'(\nu < n) S_\nu \subset S$ such that $\Delta$ is super-canonical in $(S_0,, \hat{S}_n)$, and $|S_\nu| < |T| = \aleph_\alpha$ for $\nu < n$. Put $D = \{\mu : [T]^r I_\mu \neq \emptyset\}$. Then, by definition of super-canonicity, $|D| \leqq 2^{r-1}$, and the assertion follows.

PROOF OF (ii). This relation follows from (iii) by 18. 6. This proves Theorem 22.

REMARK. In some sense (iii) is best possible. For we have

(iv) $$\aleph_\alpha \nrightarrow [\aleph_\alpha]^r_{\aleph'_\alpha, d} \quad \text{if} \quad d < \aleph'_\alpha < \aleph_\alpha \quad \text{and} \quad r \geqq 2.$$

To prove this, let $n, S, S_\nu$ be as in the proof of (i). Put $D(X) = \{\nu : XS_\nu \neq \emptyset\}$ for $X \subset S$. Then

$$[S]^r = \Sigma'(E \in [[0, n)]^{\leqq r}) I(E) \quad \text{(partition } \Delta),$$

where

$$I(E) = [S]^r \{X : D(X) = E\}.$$

We have $|\Delta| \leqq \aleph'_\alpha$. Now let $X \subset S$ and $|X| = \aleph_\alpha$. Then $|D(X)| = \aleph'_\alpha$ and hence

$$|\{E : [X]^r I(E) \neq \emptyset\}| = \aleph'_\alpha$$

which proves (iv), in fact by means of the same partition for all $d$.

Up to this point the following problem remains open. Let $\alpha > \mathrm{cf}(\alpha)$ and let $\aleph'_\alpha$ be inaccessible. Is then

$$\aleph_\alpha \rightarrow [b, (\aleph_\alpha)_{c-1}]^r$$

true for $3 \leqq r < b < \aleph_\alpha$; $2 \leqq c \leqq 2^{r-1}$? We shall show that this problem can be reduced to a finite combinatorial problem provided either $\aleph'_\alpha = \aleph_0$ or $\aleph'_\alpha$ is measurable.

**18. 8.** DEFINITION. For $r \geqq 1$ and every $m$ denote by $F_{rm}$ the set of all functions $f$ which are defined on the set $V_r$ of all systems $(r_0,, r_{p-1})$ with $r_0,, r_{p-1} \geqq 1$ and $r_0 + + r_{p-1} = r$, and whose values lie in $[0, m)$. We have $|V_r| = 2^{r-1}$.

($*$) THEOREM 23. *Let* $r \geqq 1$. *Suppose that* $a > a' = \aleph_0$ *or, more generally,* $a > a'$ *and* $a' \rightarrow (a', a')^2$. *Let* $b_0,, \hat{b}_m \leqq a$. *Then the relation*

(4) $$a \rightarrow [b_0,, \hat{b}_m]^r$$

*holds if and only if the following finite combinatorial condition is satisfied: Either* (a) $m > 2^{r-1}$ *or* (b) $m \leqq 2^{r-1}$ *and, given any function* $f \in F_{rm}$, *there always exists a number* $\nu = \nu(f) < m$ *such that at least one of the following four conditions* (i) – (iv) *holds:*

(i) $b_\nu \leqq (r-1)^2$; $b_\nu = c_0 + + \hat{c}_k$ *for some* $k$ *and some* $c_0,, \hat{c}_k \geqq 1$ *such that whenever* $\varkappa_0 < < \hat{\varkappa}_p < k$, *and* $r_\pi \leqq c_{\varkappa_\pi}$ *for* $\pi < p$ *and* $(r_0,, \hat{r}_p) \in V_r$, *then* $f(r_0,, \hat{r}_p) \neq \nu$.

(ii) $(r-1)^2 < b_\nu \leqq a'$; $f(1,, 1) \neq \nu$.

(iii) $(r-1)^2 < b_\nu < a$; $f(r) \neq \nu$.

(iv) $b_\nu = a$; $f(r_0,, \hat{r}_p) \neq \nu$ *for all* $(r_0,, \hat{r}_p) \in V_r$.

REMARKS. 1. The theorem implies that if $m \leqq 2^{r-1}$ and $a > a'$; $b_0,, \hat{b}_m \leqq a$; $a' \to (a', a')^r$, then (4) is equivalent to the relation

(5) $$a \to [b_{\lambda_0},, \hat{b}_{\lambda_{m_0}}, (a')_{m_1}, (a'^+)_{m_2}, (a)_{m_3}]^r,$$

where

$$\{\lambda_0,, \hat{\lambda}_{m_0}\}_< = \{v: b_v \leqq (r-1)^2\}; \quad m_1 = |\{v: (r-1)^2 < b_v \leqq a'\}|;$$

$$m_2 = |\{v: a' < b_v < a\}|; \quad m_3 = |\{v: b_v = a\}|.$$

Also, if (5) holds for one such $a$ it holds for all such $a$.

2. The theorem implies that (4) holds whenever $a > a'$; $a' \to (a', a')^r$; $b_0,, \hat{b}_m < a$, and $(r-1)^2 < b_v < a$ for at least two values of $v < m$. For in this case either (a) or (b) (iii) holds.

3. The hypothesis $a' \to (a', a')^r$ is only required for showing that (b) implies (4).

PROOF. Put $l = \omega(a')$. I. Suppose that (4) holds for some $r, a, m, b_0,, \hat{b}_m$ such that $a > a'$ and $r \geqq 1$. Let $m \leqq 2^{r-1}$, and $S = \Sigma'(\lambda < l)\, S_\lambda$, where $|S_\lambda| < |S| = a$ for $\lambda < l$. Let $f \in F_{rm}$. Then a partition

$$[S]^r = \Sigma'(v < m)\, I(v)$$

is defined by the following rule. If $A \in [S]^r$; $\lambda_0 < < \hat{\lambda}_p < l$; $|AS_{\lambda_\pi}| = r_\pi$ for $\pi < p$; $(r_0,, \hat{r}_p) \in V_r$, then $A \in I(f(r_0,, \hat{r}_p))$. By (4) there are $v < m$ and $X \in [S]^{b_v}$ such that $[X]^r I(v) = \emptyset$. Then there are numbers $\mu_0 < < \hat{\mu}_k < l$ such that $X \subset \Sigma(\varkappa < k)\, S_{\mu_\varkappa}$ and $|XS_{\mu_\varkappa}| = c_\varkappa \geqq 1$ for $\varkappa < k$. Then $c_0 + + \hat{c}_k = b_v$. Let $\varkappa_0 < < \hat{\varkappa}_p < k$; $r_\pi \leqq c_{\varkappa_\pi}$ for $\pi < p$, and $(r_0,, \hat{r}_p) \in V_r$. Then there is $A \in [X]^r$ such that $|AS_{\mu_{\varkappa_\pi}}| = r_\pi$ for $\pi < p$. Then $A \in I(f(r_0,, \hat{r}_p))$. On the other hand, $A \in [X]^r$ and hence $A \notin I(v)$. Therefore $f(r_0,, \hat{r}_p) \neq v$.

*Case* 1. $b_v \leqq (r-1)^2$. Then (i) holds.

*Case* 2. $(r-1)^2 < b_v \leqq a'$.

*Case* 2a. $c_\varkappa \geqq r$ for some $\varkappa < k$. Then we may take above $p = 1$ and $r_0 = r$, and (iii) follows.

*Case* 2b. $c_\varkappa < r$ for all $\varkappa < k$. Then $k \geqq r$, and we may take $p = r$ and $r_0 = = \hat{r}_p = 1$. Then (ii) holds.

*Case* 3. $a' < b_v < a$. Then $c_\varkappa \geqq r$ for some $\varkappa < k$, and as in case 2a we deduce that (iii) holds.

*Case* 4. $b_v = a$. Then $c_\varkappa \geqq r$ for at least $r$ values of $\varkappa$, and (iv) holds since in this case any system $(r_0,, \hat{r}_p) \in V_r$ may be taken.

II. Let either (a) or (b) hold. Put $n = \min(m, 2^{r-1}+1)$. Let $|S| = a$; $[S]^r = \Sigma'(\mu < m)\, I(\mu)$. Then $[S]^r = \Sigma'(v < n)\, I'(v)$ (partition $\Delta$), where

$$I'(v) = I(v) \qquad \text{for } 1 \leqq v < n.$$

Then, by Lemma 3B, there is a set $T = \Sigma'(\lambda < l)\, S_\lambda \subset S$ such that $\Delta$ is super-canonical in $(S_0,, \hat{S}_l)$: $|S_\lambda| = a_\lambda < a = |T|$ for $\lambda < l$, and $r \leqq a_0 < < \hat{a}_l$. This means that there is a function $f \in F_{rn}$ such that $A \in I'(f(r_0,, \hat{r}_p))$ whenever $A \in [T]^r$; $|AS_{\lambda_\pi}| = r_\pi$ for $\pi < p$; $\lambda_0 < < \hat{\lambda}_p < l$; $(r_0,, \hat{r}_p) \in V_r$.

*Case* 1. (a) holds. Then $n > 2^{r-1}$, and there is $\nu < n$ such that $f(r_0,, \hat{r}_p) \neq \nu$ for all $(r_0,, \hat{r}_p) \in V_r$. Then

$$[T]^r I(\nu) \subset [T]^r I'(\nu) = \emptyset; \quad |T| = a \geqq b_\nu,$$

so that the requirement for (4) is satisfied.

*Case* 2. (b) holds. Then $n = m$, and $I'(\nu) = I(\nu)$ for $\nu < m$. There is $l_0 < l$ such that for every $\nu < m$ we have either $b_\nu = a$ or $b_\nu \leqq a_{l_0}$. If (i) holds then we can choose $X_\varkappa \in [S_{l_0+\varkappa}]^{c_\varkappa}$ for $\varkappa < k$. Put $X = \Sigma(\varkappa < k) X_\varkappa$. Then $|X| = b_\nu$ and $[X]^r I(\nu) = \emptyset$ as required by (4). If (ii) holds then we choose $x_\lambda \in S_\lambda$ for $\lambda < l$ and put $X = \{x_0,, \hat{x}_l\}$. Then $|X| = a' \geqq b_\nu$; $[X]^r I(\nu) = \emptyset$. If (iii) holds then $|S_{l_0}| \geqq b_\nu$ and $[S_{l_0}]^r I(\nu) = \emptyset$. If (iv) holds then $|T| = a = b_\nu$ and $[T]^r I(\nu) = \emptyset$. This proves Theorem 23.

**18. 9.** Let $r \geqq 3$; $1 \leqq c < 2^{r-1}$. By Theorem 23 there is a least finite number $b = f^*(r, c)$ such that whenever $a > a' = \aleph_0$, then $a \nrightarrow [b, (a)_c]^r$. The choice of $a$ is irrelevant. The value of $f^*(r, c)$ can be found by solving a finite combinatorial problem which we are unable to do. We have only very incomplete results which we do not propose to discuss in this paper.

**18. 10.** Let us now return to partition relations whose left hand side is a cardinal of the first kind. As has already been pointed out our results here are rather incomplete. First a "stepping up" proposition.

($*$) THEOREM 24. *Let*

$$r \geqq 1; \quad a \geqq \aleph_0; \quad r < b_0,, \hat{b}_m \leqq a. \tag{6}$$

*Then* $a \rightarrow [b_\nu]^r_{\nu<m}$ *implies* $a^+ \rightarrow [b_\nu + 1]^{r+1}_{\nu<m}$.

The proof is parallel to that of Lemma 2 and is omitted. One might conjecture that, in analogy to Lemma 5, under the hypothesis (6) and some other fairly wide assumptions

$$a \nrightarrow [b_\nu]^r_{\nu<m} \quad \text{implies} \quad a^+ \nrightarrow [b_\nu + 1]^{r+1}_{\nu<m} \tag{7}$$

but we have only been able to prove this in very special cases.

A best possible result is given by the following theorem.

($*$) THEOREM 25.

$$\aleph_2 \rightarrow [\aleph_0, \aleph_1, \aleph_1]^3, \tag{8}$$

$$\aleph_2 \nrightarrow [\aleph_1, \aleph_1, \aleph_1]^3. \tag{9}$$

PROOF OF (8). We have $\aleph_1 \rightarrow (\aleph_0, \aleph_1)^2$ and hence $\aleph_1 \rightarrow [\aleph_0, \aleph_1]^2$, $\aleph_1 \rightarrow$ $\rightarrow [\aleph_0, \aleph_1, \aleph_1]^2$, and (8) follows from Theorem 24. We omit the proof of (9) since it employs a rather special method.

By Theorem 17 we have $\aleph_1 \nrightarrow [\aleph_1]^2_c$ for $2 \leqq c \leqq \aleph_1$. Hence the conjecture (7) would imply that

$$\aleph_2 \nrightarrow [\aleph_1]^3_c \qquad \text{for } 2 \leqq c \leqq \aleph_1,$$

but we are unable to prove this relation. Thus the simplest unsolved problem here is

($*$) PROBLEM 3.

$$? \ \aleph_2 \nrightarrow [\aleph_1]^3_4.$$

We cannot even prove the weaker relation

$$\aleph_2 \nrightarrow [\aleph_2, \aleph_1, \aleph_1, \aleph_1]^3.$$

We mention that the proof of the conjecture (7) would not settle all the problems arising in the present context. Thus we have, by Theorem 22, $\aleph_\omega \to [\aleph_\omega]^2_3$. Hence, by Theorem 24, $\aleph_{\omega+1} \to [\aleph_\omega]^3_3$. Also, by Theorem I, $\aleph_{\omega+1} \to (\aleph_0, \aleph_\omega)^3$ i. e. $\aleph_{\omega+1} \to$ $\to [\aleph_0, \aleph_\omega]^3$. Therefore, trivially, $\aleph_{\omega+1} \to [\aleph_0, \aleph_\omega, b]^3$ for any $b$. Hence one might conjecture that a best possible negative result is

(10) $$\aleph_{\omega+1} \nrightarrow [\aleph_1, \aleph_\omega, \aleph_{\omega+1}]^3.$$

This problem remains unsolved, and it can certainly not be settled by means of the conjecture (7). Instead of (10) we can only prove the weaker relation

$$\aleph_{\omega+1} \nrightarrow [\aleph_2, \aleph_\omega, \aleph_{\omega+1}]^3$$

which, in fact, follows from our next theorem.

($*$) THEOREM 26. (i) *Let* $a > a' = \aleph_0$. *Then*

$$a^+ \nrightarrow [\aleph_2, a, a^+]^3.$$

(ii) *Let* $a > a' > \aleph_0$ *and suppose that* $a' \to (a', a')^3$. *Then*

$$a^+ \nrightarrow [a'^{++}, a, a^+]^3.$$

We shall prove Theorem 26 in section 19. Many further problems could be stated here but we are not even able to give a complete discussion of the unsolved problems. Now we turn to the relation V.

**18. 11.** Let $|S| = a \geqq \aleph_0$ and $r \geqq 1$. Then there is a partition

$$[S]^r = \Sigma'(\nu < \omega(a)) I_\nu$$

such that $|I_\nu| = 1$ for $\nu < \omega(a)$. Hence we have: If $a \geqq \aleph_0$ and $r \geqq 1$, then the relation

$$a \to [b]^r_{a,d}$$

holds if and only if, either (i) $b < \aleph_0$ and $d \geqq \binom{b}{r}$, or (ii) $b \geqq \aleph_0$ and $d \geqq b$. Therefore in studying relations $a \to [b]^r_{c,d}$ it suffices to consider the case $c < a$.

($*$) **18. 12.** Consider first the case $a > a'$. Then, by Theorem I, $a \to (b)^r_c$ holds for $b, c < a$, and therefore $a \to [b]^r_{c,1}$. Hence we need only consider relations of the form

(11) $$a \to [a]^r_{c,d}$$

where $r \geqq 1$ and $d < c < a$. Theorems 20A, 21 and 22 give an almost complete discussion of the relation (11). We have, assuming ($*$): $a \to [a]^r_{c,d}$ if either (i) $a' \leqq d \leqq c < a$ or (ii) $a' = \aleph_0$ and $2^{r-1} \leqq d \leqq c < \aleph_0$. $a \nrightarrow [a]^r_{c,d}$ if either (iii) $d < a' \leqq c$ or (iv) $d < a' = e^+$ and $d < c$, for some $e$, or (v) $d < c, 2^{r-1}$.

These statements follow immediately from the theorems quoted, except that relating to (iii). Let us, therefore, assume that $d < a' = c < a$. Let $n = \omega(a')$; $S =$

$=\Sigma'(\nu<n)S_\nu$; $|S_\nu|<a=|S|$ for $\nu<n$. Then there is a partition $[S]^r=\Sigma'(\nu<n)I_\nu$ where, for $A\in[S]^r$, we have $A\in I_{\nu_0}$ whenever $\nu_0=\min(AS_\nu\neq\emptyset)\nu$. Now let $X\in[S]_a$. Then $|\{\nu: XS_\nu\neq\emptyset\}|=a'$ and hence $|\{\nu: [X]^r I_\nu\neq\emptyset\}|=a'>d$. This proves $a\nrightarrow[a]^r_{c,d}$.

If the truth of the relation $a\rightarrow[a]^r_{c,d}$, where $a>a'$, is not decided by any of the results relating to (i)—(v) above, then we have

$$2^{r-1}\leqq d<c<a'=a'^{-}>\aleph_0$$

so that $a'$ is inaccessible and greater than $\aleph_0$. Thus we do not know whether for such an $a$ the relation $a\rightarrow[a]^2_{3,2}$ is true or false.

**18. 13.** We now consider the case when $a=a'$ and $r=2$. If $a$ is inaccessible then, of course, every problem remains open. Suppose now that $a=e^+$. By 18. 11 we need only discuss the relation $e^+\rightarrow[b]^2_{c,d}$ where $b\leqq e^+$ and $e\geqq c>d$. It follows from Theorem 17 that $e^+\nrightarrow[e^+]^2_{c,d}$ for $d<c\leqq e$. On the other hand we have, by Theorem 2, $e^+\rightarrow(e)^2_c$ for $c<e'$ and, by Theorem I, $e^+\rightarrow(b)^2_c$ if $b<e$ and $c<e$. Hence we have

(*) **18. 14.** Let $c\geqq\aleph_0$. Then

$$e^+\rightarrow[e]^2_{c,1}\quad\text{for}\quad c<e',$$
$$e^+\rightarrow[b]^2_{c,1}\quad\text{for}\quad b,c<e.$$

Furthermore, $c^+\nrightarrow(3)^2_c$ by Theorem 8, and $d^{++}\rightarrow(3)^2_d$ for $d\geqq\aleph_0$, by Theorem 2. Hence, by an obvious transitivity property of our relations,

(*) **18. 15.** $c^+\nrightarrow[d^{++}]^2_{c,d}$ for $c>d\geqq\aleph_0$.

We can also say something about the case $d<\aleph_0$. By Ramsey's theorem, there is a least number $f(d)<\aleph_0$ such that $f(d)\rightarrow(3)^2_d$. Then, by the same transitivity property, $c^+\nrightarrow[f(d)]^2_{c,d}$ for $c\geqq\aleph_0$. Hence, given $d<\aleph_0\leqq c$, there exists a least number $g_0(c,d)<\aleph_0$ such that $c^+\nrightarrow[g_0(c,d)]^2_{c,d}$, and we have $g_0(c,d)\leqq f(d)$. As a corollary we obtain $c^+\nrightarrow[\aleph_0]^2_{c,d}$ for $d<\aleph_0\leqq c$. In fact it can be proved that $g_0(c,d)=d+2$. This follows easily if instead of Theorem 8 we use a result of P. ERDŐS and J. TUKEY * which states that the complete graph of power $c^+$ can be decomposed into the union of $c$ trees, if $c\geqq\aleph_0$.

By comparing the results proved in 18. 13, 18. 14 and 18. 15 we see that in the case $a=a'=e^+$ the following problem remains open:

$$?\ c^+\rightarrow[d^+]^2_{c,d}\quad\text{for}\quad c>d\geqq\aleph_0.$$

The simplest unsolved cases are:

(*) PROBLEM 3. 1. (a) $?\ \aleph_2\rightarrow[\aleph_1]^2_{\aleph_1,\aleph_0}$;

(b) $?\ \aleph_3\rightarrow[\aleph_2]^2_{\aleph_2,\aleph_1}$;

$?\ \aleph_3\rightarrow[\aleph_1]^2_{\aleph_2,\aleph_0}$.

Problem 3. 1 (a) was known to us before we introduced the relation V. We came

* This result, for $c=\aleph_0$, was first proved by ERDŐS and TUKEY but not published. Their proof is published in [23].

to it when considering a problem of ULAM. It seems to be the most difficult and interesting unsolved problem on the relation V.

*Added in proof (23. III. 1965).* It has been recently proved by F. ROWBOTTOM that Gödel's axiom of constructibility implies

$$\aleph_2 \nrightarrow [\aleph_1]^2_{\aleph_1, \aleph_0}.$$

**18. 16.** We now consider relations of the form $a^+ \to [b]^2_{c,d}$ where $a > a'$. It can be seen from 18. 13—18. 15 that the only case with $b = a$ which still needs discussing is

(12) $$?\ a^+ \to [a]^2_{c,d} \quad \text{for} \quad a' \leqq c < a;\ d \geqq 1.$$

Also, the only case with $c = a$ is

(13) $$?\ a^+ \to [d^+]^2_{a,d} \quad \text{for} \quad a > d \geqq \aleph_0.$$

About (12) we now prove:

(*) **18. 17.**

(14) $$a^+ \nrightarrow [a]^2_{a',d} \quad \text{if} \quad a > a' > d;$$

(15) $$a^+ \to [a]^2_{\bullet, a'} \quad \text{if} \quad a > a' \text{ and } c < a.$$

PROOF OF (14). Let $a = \aleph_\alpha$. We use the definitions and notation of 11. 3. Let $S = V'(\alpha)$ and $m = \omega(a')$. Then there is a partition

$$[S]^2 = \Sigma'(\nu < m) I_\nu$$

where $I_\nu = [S]^2\{\{x, y\}_< : xy = \nu\}$ for $\nu < m$. If $X \subset S$; $D \subset [0, m)$; $|D| \leqq d$; $[X]^2 \subset \subset \Sigma(\nu \in D) I_\nu$, then there is $\nu < m$ with $D \subset [0, \nu)$. Then $\{x, y\}_{\neq} \subset X$ implies $xy < \nu$, and

$$|X| \leqq \Pi(\mu < \nu)\, \aleph_{\alpha_\mu} \leqq (\aleph_{\alpha_\nu})^{|\nu|} < \aleph_\alpha$$

as required for a proof of (14).

PROOF OF (15). By Theorem 20A, $a \to [a]^2_{c, a'}$ which implies (15). About (13) we prove:

**18. 18.** $a^+ \nrightarrow [d^+]^2_{a,d}$ if $a' < d' \leqq d < a$.

PROOF. 1. Let $a = \aleph_\alpha$; $S = V'(\alpha)$; $m = \omega(a')$. We can write

$$\{(\varrho, \sigma, \lambda) : \varrho < \sigma < \omega_{\alpha_\lambda} \wedge \lambda < m\} = \{(\varrho_\nu, \sigma_\nu, \lambda_\nu) : \nu < \omega_\alpha\}_{\neq}.$$

Then $[S]^2 = \Sigma'(\nu < \omega_\alpha) I_\nu$, where the $I_\nu$ are defined as follows. If $\{x, y\}_< \subset S$; $xy = \lambda$; $x = (x_0,, \hat{x}_m)$; $y = (y_0,, \hat{y}_m)$; $(x_\lambda, y_\lambda, \lambda) = (\varrho_\nu, \sigma_\nu, \lambda_\nu)$, then $\{x, y\} \in I_\nu$.

2. Let $X \subset S$; $[X]^2 \subset \Sigma(\nu \in D) I_\nu$; $D \subset [0, \omega_\alpha)$; $|D| \leqq d$. Put $D_\lambda = \{\tau : \tau < \omega_{\alpha_\lambda} \wedge \wedge (\exists \nu)(\nu \in D \wedge \tau \in \{\varrho_\nu, \sigma_\nu\})\}$ for $\lambda < m$. Then $|D_\lambda| \leqq 2|D| \leqq d$. If $\{x, y\}_{\neq} \subset X$; $x = (x_0,, \hat{x}_m)$; $y = (y_0,, \hat{y}_m)$, then there is $\nu \in D$ with $\{x, y\} \in I_\nu$. Put $xy = \lambda$. Then we have: If $x < y$, then $(x_\lambda, y_\lambda, \lambda) = (\varrho_\nu, \sigma_\nu, \lambda_\nu)$; and if $x > y$, then $(y_\lambda, x_\lambda, \lambda) = (\varrho_\nu, \sigma_\nu, \lambda_\nu)$. Hence in any case,

$$x_\lambda, y_\lambda \in \{\varrho_\nu, \sigma_\nu\}; \quad x_\lambda, y_\lambda \in D_\lambda; \quad x_\lambda \neq y_\lambda.$$

3. Let $x=(x_0,, \hat{x}_m)\in X$. Put $f(x)=(f_0(x),, \hat{f}_m(x))$ where, for $\lambda<m$, $f_\lambda(x)=x_\lambda$ if $x_\lambda\in D_\lambda$, and $f_\lambda(x)=0$ otherwise. It follows from 2 that $\{x, y\}_{\neq}\subset X$ implies $f(x)\neq f(y)$. Hence, since $a'<d'$,

$$|X|=|\{f(x): x\in X\}|\leqq \Pi(\lambda<m)(|D_\lambda|+1)\leqq d^{a'}=d.$$

This proves 18. 18.

We do not know what happens when the condition $a'<d'$ is replaced by $a'\geqq d'$. The simplest unsolved problem here is

($*$) PROBLEM 3. 2.

$$?\ \aleph_{\omega 2+1}\to[\aleph_{\omega+1}]^2_{\aleph_{\omega 2}, \aleph_\omega}.$$

REMARKS. Our proof gives in fact more than $a^+\nrightarrow[d^+]^2_{a,d}$ for $a'<d'\leqq d<a$. For the partition which put this relation into evidence is independent of $d$ so that it has the required property for all $d$ simultaneously. One could ask quite generally whether whenever it is known that $a\nrightarrow[q(d)]^2_{c,d}$ for some fixed $a$, $c$ and every member $d$ of a set $M$, it is then always possible to find a single partition which has the required property simultaneously for all $d\in M$.

We wish to remark that one can obtain new problems of the Ramsey type relating to the V-relation in the case of finite sets, i. e. when $a<\aleph_0$, but we do not investigate this.

Having just discussed a generalization of the ordinary partition relation I we are now going to introduce a similar generalisation of the relation II.

**18. 19.** DEFINITION. The relation

$$a\to[b]^{<\aleph_0}_{c_0,, \hat{c}_\omega}\quad\text{(relation VI)}$$

expresses the following condition. Whenever $|S|=a$ and

$$[S]^r=\Sigma'(v<\omega(c_r))I(r, v)\quad\text{for}\quad r<\omega,$$

then there are a set $X\in[S]^b$ and numbers $r_0$ and $v_0(r)<\omega(c_r)$ such that

$$[X]^r I(r, v_0(r))=\emptyset\quad\text{for}\quad r\geqq r_0.$$

Clearly, by 3. 2 the relations $a\to(b)^{<\aleph_0}_2$ and $a\to[b]^{<\aleph_0}_{2,, \hat{2}}$ are equivalent. Also, if $c_r\leqq d_r$ for $r<\omega$, then the relation

$$a\to[b]^{\aleph_0}_{c_0,, \hat{c}_\omega}$$

implies

$$a\to[b]^{<\aleph_0}_{d_0,, \hat{d}_\omega}.$$

This shows that the relation VI leads to new problems only in cases when $a\nrightarrow(b)^{<\aleph_0}_2$, and here there are interesting and perhaps difficult questions. With our present methods we cannot solve even the simplest problems. We now state some of the simplest unsolved problems.

PROBLEM 4. Is it true that, either when $c_r=\aleph_0$ for all $r$ or merely when $\sup(r<\omega)c_r=\omega$, we have

$$\text{either}\quad\aleph_0\to[\aleph_0]^{<\aleph_0}_{c_0,, \hat{c}_\omega}\quad\text{or}\quad 2^{\aleph_0}\to[\aleph_0]^{<\aleph_0}_{c_0,, \hat{c}_\omega}\ ?$$

We can only prove the following simple result.

THEOREM 27. *If* $2 \leqq m < \omega$ *then* $2^{\aleph_0} \nrightarrow [\aleph_0]^{<\aleph_0}_{m,, \hat{m}}$.

PROOF. Let $S$ be the set of all real numbers $x$ in $0 \leqq x \leqq 1$. For $r \geqq 1$ and $A = \{a_0,, a_{r-1}\}_< \subset S$ put

$$f(A) = f(a_0,, a_{r-1}) = |\{\varrho : \varrho < r-1 \wedge a_{\varrho+1} - a_\varrho \geqq 1/r\}|.$$

Then there is a partition

$$[S]^r = \Sigma'(\nu < m) I(r, \nu)$$

such that, for $A \in [S]^r$, we have

$$A \in I(r, \nu) \quad \text{if} \quad \nu = \min(f(A), m-1).$$

Now let $X \in [S]^{\aleph_0}$. We shall find $r_0$ such that* $[X]^r I(r, \nu) \neq \emptyset$ for $r \geqq r_0$ and $\nu < m$. We proceed as follows. We may assume that $X = \{x_0,, \hat{x}_\omega\}_<$. There is $r_0 \geqq m$ such that $x_{\mu+1} - x_\mu \geqq 1/r_0$ for all $\mu < m-1$. Let $r \geqq r_0$ and $\nu < m$. Then, for all sufficiently large $n < \omega$, $f(x_0,, \hat{x}_\nu, x_{n+\nu},, \hat{x}_{n+r}) = \nu$, and Theorem 27 is proved. We have in fact proved somewhat more than is stated in the theorem since our partition is in a certain sense independent of $m$. If $m$ increases by 1 then one class splits into two classes while the other classes remain unchanged. An obvious modification of the proof shows that, more generally, if $|S| = 2^{\aleph_0}$ then there are partitions

$$[S]^r = \Sigma'(\nu < r) I(r, \nu),$$

for $r < \omega$, such that, given any set $X \in [S]^{\aleph_0}$ and any number $\nu < \omega$, there is a number $r_0(X, \nu) < \omega$ such that

$$[X]^r I(r, \nu) \neq \emptyset \quad \text{for} \quad r \geqq r_0(X, \nu).$$

Since our stepping-up method does not seem to work in problems of this kind we do not know whether

$$2^{2^{\aleph_0}} \nrightarrow [\aleph_0]^{<\aleph_0}_{3,, \hat{3}}.$$

## 19. FURTHER REFINEMENTS OF RELATIONS I AND IV

Corollary 7, with $\beta = 0$ and $r = 3$, gives $\aleph_1 \nrightarrow (\aleph_1, 4)^3$. Thus if $|S| = \aleph_1$ then there is a partition $[S]^3 = I_0 + I_1$ such that whenever $X \subset S$ and $[X]^3 \subset I_0$, then $|X| < \aleph_1$, and if $Y \subset S$ and $[Y]^3 \subset I_1$, then $|Y| < 4$. The following problem arises: let $|S| = \aleph_1$ and $[S]^3 = I_0 + I_1$. Suppose that whenever $X \subset S$ and $[X]^3 \subset I_0$, then $|X| < \aleph_1$. Does this imply that there always is $Y \subset S$ with $|Y| = 4$ and $|[Y]^3 I_1| \geqq 2$ or perhaps even $|[Y]^3 I_1| \geqq 3$? If the answer is in the affirmative then we denote this fact by the relations

$$\aleph_1 \to \left(\aleph_1, \begin{bmatrix}4\\2\end{bmatrix}\right)^3 \quad \text{or} \quad \aleph_1 \to \left(\aleph_1, \begin{bmatrix}4\\3\end{bmatrix}\right)^3$$

respectively.

* It will be seen that $r_0$, for the purpose of this proof, need only satisfy a condition which is weaker than what follows.

Generally we introduce the following extension of the I-relation $a \to (b_0,, \hat{b}_m)^r$ and IV-relation $a \to [b_0,, \hat{b}_m]^r$.

**19. 1.** DEFINITION. Let, for each $\nu < m$, the symbol $\Gamma_\nu$ denote either a cardinal $b_\nu$ or a pair $\begin{bmatrix} i_\nu \\ j_\nu \end{bmatrix}$ of finite cardinals. Then the relation

$$a \to (\Gamma_0,, \hat{\Gamma}_m)^r$$

is said to hold if the following condition is satisfied. Whenever $|S| = a$ and $[S]^r = \Sigma(\nu < m) I_\nu$, then there always exist a set $X \subset S$ and a number $\nu < m$ such that either $\Gamma_\nu = b_\nu$; $|X| = b_\nu$; $[X]^r \subset I_\nu$ or $\Gamma_\nu = \begin{bmatrix} i_\nu \\ j_\nu \end{bmatrix}$; $|X| = i_\nu$; $|[X]^r I_\nu| \geqq j_\nu$.

**19. 2.** DEFINITION. Let $\Gamma_\nu$ be as in definition 19. 1. Then the relation

$$a \to [\Gamma_0,, \hat{\Gamma}_m]^r$$

is said to hold if the following condition is satisfied. Whenever $|S| = a$ and $[S]^r = \Sigma'(\nu < m) I_\nu$, then there always exist a set $X \subset S$ and a number $\nu < m$ such that either $\Gamma_\nu = b_\nu$; $|X| = b_\nu$; $[X]^r I_\nu = \emptyset$ or $\Gamma_\nu = \begin{bmatrix} i_\nu \\ j_\nu \end{bmatrix}$; $|X| = i_\nu$; $|[X]^r \Sigma(\mu \neq \nu) I_\mu| \geqq j_\nu$.
Our new relations coincide with the I-relations and the IV-relations if all $\Gamma_\nu = b_\nu$. Just as in the case of the relations I and IV we shall use the obvious abbreviations when a number of $\Gamma_\nu$ are equal. Clearly the genuinely new cases in which the new relations have to be studied are of the following form. Suppose that we know that $a \nrightarrow (b_0,, \hat{b}_m)^r$ and that some of the $b_\nu$ are finite. Then we replace some of these $b_\nu$ by a symbol $\begin{bmatrix} b_\nu \\ j_\nu \end{bmatrix}$ and can then ask whether the new relation, now of the form $a \to (\Gamma_0,, \hat{\Gamma}_m)^r$, is true. We cannot give a systematic discussion. We are going to prove only some isolated results relating to cases where either interesting new phenomena arise of where these results help us in deciding the truth of some of the original relations I or IV.

($*$) THEOREM 28. *If* $a \geqq \aleph_0$ *and* $r \geqq 3$, *then*

$$a^+ \nrightarrow \left[\begin{bmatrix} r+1 \\ 3 \end{bmatrix}, (a^+)_{a^+}\right]^r.$$

PROOF. Put $n = \omega(a^+)$ and $S = [0, n)$. Then we can write $[S]^a = \{A_0,, \hat{A}_n\}_{\neq}$. For $\nu < n$ put

$$Z_\nu = \{A_\mu : \mu < \nu \wedge A_\mu \subset [0, \nu)\}.$$

Then $Z_\nu = \{A_{\nu\varrho} : \varrho < \varphi_\nu\}_{\neq}$ for some $\varphi_\nu \leqq \nu$. For fixed $\nu < n$ we can find, by transfinite construction, sets

$$X(\nu, \varrho, \sigma) \in [S]^{r-1} \quad \text{for} \quad \varrho < \varphi_\nu \quad \text{and} \quad \sigma < \nu$$

such that $X(\nu, \varrho, \sigma) \subset A_{\nu\varrho}$ for $\varrho < \varphi_\nu$ and $\sigma < \nu$, and

$$X(\nu, \varrho_0, \sigma_0) X(\nu, \varrho_1, \sigma_1) = \emptyset \quad \text{if} \quad (\varrho_0, \sigma_0) \neq (\varrho_1, \sigma_1). \tag{1}$$

Then there is a partition

$$[S]^r = \Sigma'(\sigma < n) I_\sigma$$

such that, for $A=\{v_0,, v_{r-1}\}_< \subset S$ and $1 \leqq \sigma < n$, we have $A \in I_\sigma$ if and only if

$$\{v_0,, v_{r-2}\} = X(v_{r-1}, \varrho, \sigma) \quad \text{for some} \quad \varrho < q_{v_{r-1}}.$$

This partition has the desired properties. For:

1. Let $B=\{v_0,, v_r\}_< \subset S$; $|[B]^r(I_1 + \dots + I_n)| \geqq 3$. Then there are sets $X_0, X_1$ such that $\{X_0, X_1\}_{\neq} \subset [B]^r$; $v_r \in X_0 X_1$; $X_0 \in I_{\sigma_0}$; $X_1 \in I_{\sigma_1}$; $1 \leqq \sigma_0 \leqq \sigma_1 < n$. Then $X_0 - \{v_r\} = X(v_r, \varrho_0, \sigma_0)$; $X_1 - \{v_r\} = X(v_r, \varrho_1, \sigma_1)$; $(\varrho_0, \sigma_0) \neq (\varrho_1, \sigma_1)$. Then, by (1), $(X_0 - \{v_r\})(X_1 - \{v_r\}) = \emptyset$; $2(r-1) \leqq |B - \{v_r\}| = r$ which is a contradiction.

2. Let $S' \subset S$; $|S'| = a^+$; $1 \leqq \sigma_0 < n$. Choose $A \in [S']^a$. Then $A = A_{v_0}$ for some $v_0 < n$. Then there is $v$ with $v_0, \sigma_0 < v < n$; $v \in S'$; $A_{v_0} \subset [0, v)$. Then $A_{v_0} \in Z_v$ and hence $A_{v_0} = A_{v\varrho_0}$ for some $\varrho_0 < q_v$. Then

$$X(v, \varrho_0, \sigma_0) \subset A_{v\varrho_0} = A_{v_0} \subset [0, v)$$

and therefore $X(v, \varrho_0, \sigma_0) + \{v\} \in [S']^r I_{\sigma_0}$. Hence $[S']^r I_{\sigma_0} \neq \emptyset$, and Theorem 28 follows.

REMARKS. 1. Theorem 19 is a corollary of Theorem 28.

2. By arguments similar to those used in the proof of Theorem 28 it would be easy to determine the least $j(s) < \omega$ such that

$$a^+ \nrightarrow \left[\begin{bmatrix} r+s \\ j(s) \end{bmatrix}, (a^+)_{a^+}\right]^r$$

but we omit this.

In some sense Theorem 28 is best possible. For we have

(∗) THEOREM 29. *If* $a \geqq \aleph_0$ *and* $r \geqq 3$, *then*

$$a^+ \rightarrow \left[\begin{bmatrix} r+1 \\ 2 \end{bmatrix}, a^+\right]^r. \tag{2}$$

We note that (2) is the same as $a^+ \rightarrow \left(\begin{bmatrix} r+1 \\ 2 \end{bmatrix}, a^+\right)^r$. The proof is easy and will be omitted. It follows from Theorems 28 and 29 that, for $a \geqq \aleph_0$ and $r \geqq 3$,

$$a^+ \rightarrow \left(\begin{bmatrix} r+1 \\ 2 \end{bmatrix}, a^+\right)^r, \tag{3}$$

$$a^+ \nrightarrow \left(\begin{bmatrix} r+1 \\ 3 \end{bmatrix}, a^+\right)^r. \tag{4}$$

Lemma 5B shows that whenever $r \geqq 3$ and $\aleph_{\alpha+1} \nrightarrow (r+1, \aleph_{\alpha+1})^r$ then, for every $s$,

$$\aleph_{\alpha+1+s} \nrightarrow (r+1+s, \aleph_{\alpha+1})^{r+s}.$$

The question arises whether such a stepping-up method works for our generalized relations. By an application of the Ramification Lemma we can step up the relation (3) and obtain

(∗) THEOREM 29A. *If* $\alpha \geqq 0$ *and* $r \geqq 3$, *then*

$$\aleph_{\alpha+1+s} \rightarrow \left(\begin{bmatrix} r+1+s \\ 2+s \end{bmatrix}, \aleph_{\alpha+1}\right)^{r+s} \qquad \textit{for all } s.$$

We omit the proof. The problem whether this theorem is best possible remains open. We cannot similarly step up the formula (4), i. e. we cannot prove that

$$\aleph_{\alpha+1+s} \nrightarrow \left(\begin{bmatrix} r+1+s \\ 3+s \end{bmatrix}, \aleph_{\alpha+1}\right)^{r+s},$$

not even in the simplest case $\alpha=0$; $r=3$; $s=1$.

In what follows we restrict ourselves for the sake of brevity to the generalized I-relation.

Consider the formula

$$a \nrightarrow \left(\begin{bmatrix} r+1 \\ j \end{bmatrix}, a\right)^{r}, \tag{5}$$

where $r \geqq 3$ and $a > a'$. We mention without proof that if $a'$ is accessible then, by means of the usual methods, one can prove that the least $j$ for which (5) holds is again $j=3$ the case $a > a' = \aleph_0$. The following result covers.

($*$) THEOREM 30. *Let $a > a'$ and $r \geqq 3$. Then*

$$a \rightarrow \left(\begin{bmatrix} r+1 \\ 3 \end{bmatrix}, a\right)^{r}, \quad \textit{provided that } a' \rightarrow (a', a')^{r}; \tag{6}$$

$$a \nrightarrow \left(\begin{bmatrix} r+1 \\ 4 \end{bmatrix}, a\right)^{r}. \tag{7}$$

PROOF OF (6). Let $|S|=a$; $[S]^r = I(0) +' I(1)$ (partition $\Delta$). Let $\omega(a')=n$. Then, by Lemma 3B, there is a set $S' = \Sigma'(\nu<n) S_\nu \subset S$ such that $\aleph_0 \leqq |S_\nu| < |S'| = a$ for $\nu<n$, and $\Delta$ is supercanonical in $(S_0,, \hat{S}_n)$. This means that there is $f(r_0,, \hat{r}_t) < 2$ such that $A \in I(f(r_0,, \hat{r}_t))$ whenever $A \in [S']^r$; $\{\nu : AS_\nu \neq \emptyset\} = \{\nu_0,, \hat{\nu}_t\}_<$; $|AS_{\nu_\lambda}| = r_\lambda$ for $\lambda < t$.

*Case* 1. $f(r_0,, \hat{r}_t)=1$ whenever $r_0 + + \hat{r}_t = r$. Then $[S']^r \subset I(1)$; $|S'|=a$.

*Case* 2. There is $(r_0,, \hat{r}_t)$ with $f(r_0,, \hat{r}_t)=0$.

*Case* 2a. $t \in \{1, r\}$. Then, clearly, there is $X \in [S']^{\aleph_0}$ with $[X]^r \subset I(0)$.

*Case* 2b. $1<t<r$. Then there is $\sigma<t$ with $r_\sigma \geqq 2$. Choose $X_0 \in [S']^{r+1}$ such that $|X_0 S_\tau| = r_\tau$ for $\tau \in [0, t) - \{\sigma\}$, and $|X_0 S_\sigma| = r_\sigma + 1$. Then there are exactly $r_\sigma+1$ sets $A \in [X_0]^r$ with $|AS_\sigma| = r_\sigma$, and all these sets $A$ belong to $I(0)$. Hence $|[X_0]^r I_0| \geqq r_\sigma + 1 \geqq 3$, and (6) follows.

PROOF OF (7). Let $S = \Sigma'(\nu<n) S_\nu$; $n=\omega(a')$; $|S_\nu| < |S| = a$ for $\nu<n$. Let $A \subset S$; $\{\nu : AS_\nu \neq \emptyset\} = \{\nu_0,, \hat{\nu}_t\}_<$; $|AS_{\nu_\tau}| = r_\tau$ for $\tau<t$. Put $g(A) = (r_0,, \hat{r}_t)$. Then there is a partition $[S]^r = I_0 +' I_1$ such that

$$I_0 = [S]^r\{A : g(A) \in \{(2,, 2), (1, 2,, 2)\}\}.$$

Then, if $X \subset S$ and $[X]^r \subset I_1$, it follows that $|X|<a$. Now let $Y \subset S$; $|Y| = r+1$;

$$g(Y) = (r_0,, \hat{r}_t); \quad \lambda = |[Y]^r I_0| \geqq 1.$$

We want to deduce that $\lambda < 4$.

*Case* 1. $r=2s$. Then $s \leqq t \leqq s+1$.

*Case* 1a. $t=s$. Then there is $\sigma<t$ with $r_\sigma=3$ and $r_\tau=2$ for $\tau\neq\sigma$. Then $\lambda=3$.

*Case* 1b. $t=s+1$. Then there is $\sigma<t$ with $r_\sigma=1$ and $r_\tau=2$ for $\tau\neq\sigma$. Then $\lambda=1$.

*Case* 2. $r=2s+1$. Then $s+1\leqq t\leqq s+2$.

*Case* 2a. $t=s+1$. Then $\lambda\leqq 3$.

*Case* 2b. $t=s+2$. Then $\lambda\leqq 2$. This proves (7) and completes the Proof of Theorem 30.

REMARK. It is worth noting that in the case of the relation (6) the stepping-up method does not seem to work, and so the following simple problem remains unsolved.

($*$) PROBLEM 5.

$$?\ \aleph_{\omega+1}\rightarrow\left(\begin{bmatrix}5\\4\end{bmatrix},\aleph_\omega\right)^4.$$

By Theorem 30,

$$\aleph_\omega\rightarrow\left(\begin{bmatrix}4\\3\end{bmatrix},\aleph_\omega\right)^3.$$

To conclude this section we shall apply some of the results so far obtained to prove Theorem 26 which was stated in 18. 10.

PROOF OF THEOREM 26. We are given that $a>a'$ and $a'\rightarrow(a',a')^3$, and we have to deduce that

$$a^+\nrightarrow[a'^{++},a,a^+]^3. \tag{8}$$

Let $|S|=a^+$. We have, by Theorems 28 and I respectively, $a^+\nrightarrow\left(\begin{bmatrix}4\\3\end{bmatrix},a^+\right)^3$ and $a^+\nrightarrow(a,a'^+)^3$. Hence there are partitions

$$[S]^3=I_0+'I_1=J_0+'J_1$$

such that

$$\text{if } X\in[S]^4, \text{ then } \quad |[X]^3I_0|<3, \tag{9}$$

$$a^+\notin[I_1]_3, \tag{10}$$

$$a\notin[J_0]_3, \tag{11}$$

$$a'^+\notin[J_1]_3. \tag{12}$$

We now form the new partition

$$[S]^3=K_0+K_1+K_2 \qquad \text{(partition } \Delta),$$

where

$$K_0=J_0-I_0;\quad K_1=J_1-I_0;\quad K_2=I_0.$$

We now show that $\Delta$ has the properties required by (8).

1. Let $[X]^3\subset K_1+K_2$. Then $|X|<a'^{++}$. For suppose that $|X|\geqq a'^{++}$. Then, by Theorem I, $|X|\rightarrow(a'^+,a')^3$ and hence $|X|\rightarrow\left(a'^+,\begin{bmatrix}4\\3\end{bmatrix}\right)^3$. There is $X'\subset X$ such that either (i) $|X'|=a'^+$ and $[X']^3\subset K_1\subset J_1$ which contradicts (12), or (ii) $|X'|=4$ and $3\leqq|[X']^3K_2|=|[X']^3I_0|$ which contradicts (9).

2. Let $[X]^3 \subset K_2 + K_0$. Then $|X| < a$. For suppose $|X| \geqq a$. Then, by Theorem 30, $|X| \to \left(\begin{bmatrix}4\\3\end{bmatrix}, a\right)^3$, and hence there is $X' \subset X$ such that either (i) $|X'| = 4$ and $3 \leqq \leqq |[X']^3 K_2| = |[X']^3 I_0|$ which contradicts (9), or (ii) $|X'| = a$ and $[X']^3 \subset K_0 \subset J_0$ which contradicts (11).

3. Let $[X]^3 \subset K_0 + K_1$. Then

$$[X]^3 \subset (J_0 - I_0) + (J_1 - I_0) = [S]^3 - I_0 = I_1$$

and hence, by (10), $|X| < a^+$. Our three results 1, 2 and 3 prove (8).

## 20. FURTHER PROBLEMS RELATED TO THE ORDINARY PARTITION RELATION I

In this section we shall formulate some general problems concerning partitions. Here we have no essentially new results. As applications of our results about I-relations we shall obtain the answers to a number of simple questions and we shall point out some interesting unsolved problems of a new type.

**20. 1.** Let $m \geqq 1$, and let $a, a_\nu, b_\nu, c_\nu$ be cardinals, for $\nu < m$. Let $|S| = a$. If $a \nrightarrow (b_0,, \hat{b}_m)^r$ then there is a partition $[S]^r = \Sigma(\nu < m) I_\nu$ (partition $\Delta$) such that $b_\nu \notin [I_\nu]_r$ for $\nu < m$. One can ask the question whether there is a partition $\Delta$ satisfying $b_\nu \notin [I_\nu]_r$ for $\nu < m$ and, in addition, having the property that whenever $\nu < m$ and $X \in [S]^{c_\nu}$ then $a_\nu \in [[X]^r I_\nu]_r$. The fact that the answer to this question is negative will be expressed by the relation

$$(1) \qquad (a, a_0,, \hat{a}_m) \to \begin{pmatrix} b_0,, \hat{b}_m \\ c_0,, \hat{c}_m \end{pmatrix}^r.$$

Explicitly, (1) has the following meaning: if $|S| = a$ and $[S]^r = I_0 + + \hat{I}_m$, then either (i) there is $\nu < m$ with $b_\nu \in [I_\nu]_r$, or (ii) there are $\nu < m$ and $X \in [S]^{c_\nu}$ with $a_\nu \notin [[X]^r I_\nu]_r$. It follows from this definition that the relation (1) is increasing in $a$ and in each $a_\nu$, and decreasing in each $b_\nu$ and in each $c_\nu$. Also, (1) is always true if there is $\nu$ with $a_\nu \geqq b_\nu$. We shall investigate the special case when (1) is of the form

$$(2) \qquad (a, a_0, 0) \to \begin{pmatrix} b_0, a^+ \\ c_0, a^+ \end{pmatrix}^r.$$

We shall write (2) more simply as

$$(a, a_0) \to (b_0, c_0)^r.$$

Thus the relation

$$(3) \qquad (a, b) \to (c, d)^r$$

means: if $|S| = a$ and $[S]^r = I_0 + I_1$, then either (i) $c \in [I_0]_r$ or (ii) there is $X \in [S]^d$ with $b \notin [[X]^r I_0]_r$. The relation (3) is increasing in $a$ and $b$, and decreasing in $c$ and $d$. The negation of (3) is

$$(a, b) \nrightarrow (c, d)^r$$

and means: if $|S| = a$, then there is a partition $[S]^r = I_0 + I_1$ such that (i) $c \notin [I_0]_r$

and (ii) whenever $X \in [S]^d$ then $b \in [[X]^r I_0]_r$. The following remarks establish connections between our new relation and the ordinary I-relation.

**20. 2.** (i) If $a \to (c, d)^r$ and $b \geqq r$, then $(a, b) \to (c, d)^r$.

(ii) If $a \nrightarrow (c, d_0)^r$ and $d \to (b, d_0)^r$, then $(a, b) \nrightarrow (c, d)^r$.

Proof of (i). Let $|S| = a$ and $[S]^r = I_0 + I_1$. Then there are two cases:

*Case* 1. $c \in [I_0]_r$.

*Case* 2. $d \in [I_1 - I_0]_r$. Then there is $X \in [S]^d$ with $[X]^r \subset I_1 - I_0$. Then $[X]^r I_0 = \emptyset$ and hence $b \notin [[X]^r I_0]_r$.

Proof of (ii). Let $|S| = a$. Then there is a partition $[S]^r = I_0 + I_1$ such that $c \notin [I_0]_r$ and $d_0 \notin [I_1]_r$. Let $X \in [S]^d$. Then there are two cases:

*Case* 1. $b \in [[X]^r I_0]_r$.

*Case* 2. $d_0 \in [[X]^r I_1]_r$. Then $d_0 \in [[X]^r I_1]_r \subset [I_1]_r$ which is a contradiction. This proves 20. 2.

As corollary of 20. 2 and Theorem I we have for $r = 2$:

($*$) **20. 3.**

(i) $(a, 2) \to (c, d)^2$ if $a \geqq \aleph_0$ and $c, d < a$.

(ii) $(a, b) \nrightarrow (a, a)^2$ if $b < a$ and $a$ is infinite and not inaccessible.

(iii) $(\aleph_\alpha, 2) \to (\aleph_{\mathrm{cr}(\alpha)}, \aleph_\alpha)^2$ except possibly when $\mathrm{cr}(\alpha) > 0$ and $\aleph_{\mathrm{cr}(\alpha)}$ is inaccessible.

(iv) $(\aleph_\alpha, \aleph_{\mathrm{cr}(\alpha)}) \nrightarrow (\aleph^+_{\mathrm{cr}(\alpha)}, \aleph_\alpha)^2$ except possibly when $\mathrm{cr}(\alpha) > 0$ and $\aleph_{\mathrm{cr}(\alpha)}$ is inaccessible.

Proof of (i). By Theorem I, $a \to (c, d)^2$. Hence the conclusion follows from 20. 2 (i).

Proof of (ii). Put $a = \aleph_\alpha$. By Theorem I we have $a \nrightarrow (a, \aleph^+_{\mathrm{cr}(\alpha)})^2$ and $a \to (b, \aleph^+_{\mathrm{cr}(\alpha)})^2$, and the conclusion follows from 20. 2 (ii).

Proof of (iii). By Theorem I, $\aleph_\alpha \to (\aleph_{\mathrm{cr}(\alpha)}, \aleph_\alpha)^2$, and the conclusion follows from 20. 2 (i).

Proof of (iv). By Theorem I we have $\aleph_\alpha \nrightarrow (\aleph^+_{\mathrm{cr}(\alpha)}, \aleph_\alpha)^2$ and $\aleph_\alpha \to (\aleph_{\mathrm{cr}(\alpha)}, \aleph_\alpha)^2$, and the conclusion follows from 20. 2 (ii).

The results just proved show that here we get new problems concerning the relation (3) only if $a = d = \aleph_\alpha$ and $\mathrm{cr}(\alpha) < \alpha \dot{-} 1$. Thus we have to investigate the cases

(i) $a = \aleph_{\beta+1}$, where $\beta > \mathrm{cf}(\beta)$,

and

(ii) $a > a'$.

In case (i) we have no further results and the following are the simplest of the open problems:

($*$) Problem 6.

$$? \ (\aleph_{\omega+1}, \aleph_1) \to (\aleph_\omega, \aleph_{\omega+1})^2.$$

$$? \ (\aleph_{\omega+1}, \aleph_1) \to (\aleph_n, \aleph_{\omega+1})^2 \quad \text{for} \quad 2 \leqq n < \omega.$$

In fact we cannot even decide whether $(\aleph_{\omega+1}, \aleph_n) \to (\aleph_\omega, \aleph_\omega)^2$ holds for any $n < \omega$. This problem seems to be interesting and difficult. There are many classes of problems where the first difficulty arises for the cardinal $\aleph_{\omega+1}$. We shall formulate some of these in connection with polarized partition relations. Here we are going to formulate another unsolved problem belonging into the general field under dicsussion.

PROBLEM 7. Let $|S|=\aleph_{\omega+1}$ and $[S]^2 = I_0+I_1$. Suppose that whenever $X\in[S]^{\aleph_1}$, then $\aleph_1\in[[X]^2 I_1]_2$. Does this imply that $\aleph_{\omega+1}\in[I_1]_2$?

It follows from the definition of the relation (1) that if

$$(\aleph_{\omega+1}, 0, \aleph_1) \nrightarrow \begin{pmatrix} \aleph_1 & \aleph_{\omega+1} \\ \aleph_1 & \aleph_1 \end{pmatrix}^2,$$

then the answer to the question in Problem 7 is in the affirmative.

Let us return to the relation (3). We want to consider the case $a>a'$. Here we have the following results which often allow us to reduce the cardinals which enter a relation under discussion.

**20. 4.** Let $r\geqq 1$ and $a>a'$.

(i) If $(a, b)\to(c, a)^r$, then $(a', b)\to(c, a')^r$.

$(*)$ (ii) If $c<a$ and $(a', b)\to(c, a')^r$, then $(a, b_0)\to(c, a)^r$, where $b_0=b$ if either $r=2$ or $b\geqq\aleph_0$, and $b_0 = (b-1)(r-1)+1$ otherwise.

PROOF. Let $\omega(a')=n$; $|S|=a$; $N=[0, n)$.

PROOF OF (i). Let $[N]^r=I_0^*+'I_1^*$; $S = \Sigma'(\nu<n)S_\nu$; $|S_\mu|<|S_\nu|<a$ for $\mu<\nu<n$. Then $[S]^r = I_0+'I_1$, where

$$I_0 = \Sigma(\{\nu_0,, \nu_{r-1}\}_<\in I_0^*)[S_{\nu_0},, S_{\nu_{r-1}}]^{1,,1}.$$

Then we have the following cases:

*Case* 1. $c\in[I_0]_r$. Then there is $X\in[S]^c$ with $[X]^r\subset I_0$. Put $N' = \{\nu: XS_\nu\neq\varnothing\}$. Then $|N'|=c$; $[N']^r\subset I_0^*$; $c\in[I_0^*]_r$.

*Case* 2. There is $X\in[S]^a$ with $b\notin[[X]^r I_0]_r$. Put $N'=\{\nu: XS_\nu\neq\varnothing\}$. Then $|N'|=a'$.

*Case* 2a. $b\in[[N']^r I_0^*]_r$. Then there is $N''\in[N']^b$ with $[N'']^r\subset I_0^*$. Then there are $x_\nu\in XS_\nu$ for $\nu\in N''$. Put $X'=\{x_\nu: \nu\in N''\}$. Then $X'\in[X]^b$; $[X']^r\subset I_0$; $b\in[[X]^r I_0]_r$ which is a contradiction.

*Case* 2b. $b\notin[[N']^r I_0^*]_r$. This proves (i). We did not require $(*)$ for this part.

PROOF OF (ii). Let $[S]^r = I_0+'I_1$ (partition $\Delta$). Then, by Lemma 3, there is a set $S' = \Sigma'(\nu<n)S_\nu\subset S$ with $c\leqq|S_\mu|<|S_\nu|<|S'|=a$ for $\mu<\nu<n$, such that $\Delta$ is canonical in $(S_0,, \hat{S}_n)$. Choose $x_\nu\in S_\nu$ for $\nu<n$. Then $[N]^r = I_0^*+'I_1^*$, where $I_0^* = \{\{\nu_0,, \nu_{r-1}\}_<: \{x_{\nu_0},, x_{\nu_{r-1}}\}\in I_0\}$. We have the cases:

*Case* 1. $c\in[I_0^*]_r$. Then there is $N'\in[N]^c$ with $[N']^r\subset I_0^*$. Put $X=\{x_\nu: \nu\in N'\}$. Then $|X|=c$; $[X]^r\subset I_0$; $c\in[I_0]_r$.

*Case* 2. There is $N'\in[N]^{a'}$ with $b\notin[[N']^r I_0^*]_r$. Put $X=\Sigma(\nu\in N')S_\nu$. Then $|X|=a$. Let $X'\subset X$; $[X']^r\subset I_0$. Put $N''=\{\nu: X'S_\nu\neq\varnothing\}$. Then $[N'']^r\subset I_0^*$; $N''\subset N'$, and hence $|N''|<b$.

*Case* 2a. $|X'S_\nu|<r$ for $\nu\in N''$.

*Case* 2a1. $b<\aleph_0$. Then $|X'|\leqq|N''|(r-1)\leqq(b-1)(r-1)<b_0$; $b_0\notin[[X]^r I_0]_r$.

*Case* 2a2. $b\geqq\aleph_0$. Then $|X'|\leqq|N''|(r-1)<b = b_0$; $b_0\notin[[X]^r I_0]_r$.

*Case* 2b. $|X'S_\nu| \geqq r$ for some $\nu \in N''$. Then, since $\Delta$ is canonical, we have $[S_\nu]^r \subset I_0$; $|S_\nu| \geqq c$; $c \in [I_0]_r$. This proves (ii).
The following is a corollary of 20. 4.

**20. 5.** Let $r \geqq 1$ and $a > a'$. Then

(i) $$(a, a'^+) \to (c, a)^r \qquad \text{for } c < a;$$

(ii) $$(a, a') \nrightarrow (c, a)^r \qquad \text{for } c > a'.$$

To prove (i) we notice that, trivially, $(a', a'^+) \to (c, a')^r$, and 20. 4 (ii) gives the result.

Proof of (ii). If $(a, a') \to (c, a)^r$ then, by 20. 4 (i), $(a', a') \to (c, a')^r$. But this is obviously false as is shown by the partition in which $I_1 = \emptyset$.

It follows from 20. 5 that if $a' < c < a$, then the relation $(a, b) \to (c, a)^r$ holds if and only if $b > a'$.

The first unsolved problems arise for $a = \aleph_{\omega_{\omega+1}}$, when $a' = \aleph_{\omega+1}$. We have, by 20. 5 (ii), $(a, a') \nrightarrow (c, a)^2$ for $c > a'$. The question is to decide if this is best possible. By 20. 3 (ii) we have $(a', a'^-) \nrightarrow (a', a')^2$ and hence, by 20. 4 (i), $(a, a'^-) \nrightarrow (a', a)^2$ which does not yet answer our question. The following problems remain open:

$$? \ (\aleph_{\omega_{\omega+1}}, \aleph_1) \to (\aleph_\omega, \aleph_{\omega_{\omega+1}})^2;$$

$$? \ (\aleph_{\omega_{\omega+1}}, \aleph_1) \to (\aleph_n, \aleph_{\omega_{\omega+1}})^2$$

for $2 \leqq n < \omega$. By 20. 4 this reduces to Problem 6.

Consider now the case $r = 3$. Here we have the following nontrivial results.

$(*)$ **20. 6.** If $3 \leqq b < c < \omega$, then

(i) $$(a^+, b) \to (c, a^+)^3 \qquad \text{if } a \geqq \aleph_0;$$

(ii) $$(a, b) \to (c, a)^3 \qquad \text{if } a > a' > a'^-.$$

The proof of (i) can be conducted by induction on $c$, and (ii) follows from (i) by means of 20. 4. We omit the details in order to save space.

By comparing these relations with trivial applications of 20. 2 we see that the following are among the simplest problems that remain unsolved.

$(*)$ Problem 8.

$$? \ (\aleph_3, \aleph_0) \to (\aleph_1, \aleph_3)^3,$$

$$? \ (\aleph_2, 4) \to (\aleph_1, \aleph_1)^3,$$

$$? \ (\aleph_2, \aleph_0) \to (\aleph_1, \aleph_1)^3.$$

Finally we formulate a typical instance of another class of unsolved problems for $r = 3$.

$(*)$ Problem 9. Let $|S| = \aleph_2$. Does there exist a partition $[S]^3 = I_0 + I_1$ such that:

(i) $\aleph_1 \notin [I_\nu]_3$ for $\nu < 2$;

(ii) whenever $X \in [S]^{\aleph_1}$, then there are sets $X_0, X_1 \in [X]^{\aleph_0}$ such that $[X_\nu]^3 \subset I_\nu$ for $\nu < 2$?

REMARK. We know that there is a 3-partition such that (i) holds since, by Theorem I, $\aleph_2 \nrightarrow (\aleph_1, \aleph_1)^3$. Also, the formula $\aleph_0 \rightarrow (\aleph_0, \aleph_0)^3$ implies that whenever $X \in [S]^{\aleph_1}$ then either $X_0$ or $X_1$ exists satisfying the requirement stated in (ii). But it does not necessarily follow that both, $X_0$ and $X_1$, always exist simultaneously.

# PART II

# POLARIZED PARTITIONS

We are now going to discuss relations of the form

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a_0, a_1 \\ b_0, b_1 \end{pmatrix}^{1,1}$$

which will briefly be denoted by

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a_0, a_1 \\ b_0, b_1 \end{pmatrix}.$$

The commas will be omitted whenever possible. When any of $a, b, a_\nu, b_\nu$ are infinite, we shall always put

$$a = \aleph_\alpha; \quad b = \aleph_\beta; \quad a_\nu = \aleph_{\alpha_\nu}; \quad b_\nu = \aleph_{\beta_\nu}.$$

It will be shown that if $a, b \geqq \aleph_0$ then the discussion of the general case can be reduced to that of the two special cases $\alpha = \beta$ and $\alpha + 1 = \beta$. We shall also discuss the "relation with alternatives"

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a_0 \vee a_1, a_2 \vee a_3 \\ b_0 \vee b_1, b_2 \vee b_3 \end{pmatrix}$$

which was defined in 3. 3.

## 21. PRELIMINARIES

**21. 1.** DEFINITIONS. Let $ST = \varnothing$. We shall write instead of $[S, T]^{1,1}$, which was defined in Section 2, the symbol $[S, T]$. Let

(1) $$[S, T] = I_0 + I_1.$$

Relative to a partition (1) we put, for $x_0 \in S$; $y_0 \in T$; $\nu < 2$,

$$P_\nu(x_0) = T\{y : \{x_0, y\} \in I_\nu\},$$
$$Q_\nu(y_0) = S\{x : \{x, y_0\} \in I_\nu\}.$$

Also, $[I_\nu]$ denotes the set of all pairs $(c, d)$ such that there are sets $X \in [S]^c$ and $Y \in [T]^d$ with* $[X, Y] \subset I_\nu$.

First we prove a negative result.

**21. 2.** $\begin{pmatrix} a \\ a \end{pmatrix} \nrightarrow \begin{pmatrix} a+1 & 1 \\ 1 & a \end{pmatrix}$ for $a \geqq 0$.

* It will always be clear from the context to which partition (1) these notions refer.

PROOF. Let $\omega(a)=n$; $S=\{x_0,, \hat{x}_n\}_{\neq}$; $T=\{y_0,, \hat{y}_n\}_{\neq}$; $ST=\varnothing$;

$$[S, T] = I_0 +' I_1,$$

where $I_0=\{\{x_\mu, y_\nu\}: \mu \leqq \nu < n\}$. Let $\mu, \nu < n$. Then

$$|Q_0(y_\nu)| = |\nu+1| \leqq a; \quad |P_1(x_\mu)| = |\mu| < a.$$

This proves the result. The case $\alpha=\beta$ is closely connected with the theory of set mappings.

**21. 3.** DEFINITION. Let $|S|=a$. A set mapping on $S$ is a function $f: S \to \mathbf{P}(S)$ such that $x \notin f(x)$ for $x \in S$. The set mapping is of order $p$ if $|f(x)|<p$ for $x \in S$. If $S' \subset S$ then we put $f(S')=\Sigma(x \in S')f(x)$. The set $S'$ is called $f$-free if $S'f(S')=\varnothing$.

**21. 4.** A set family $(A_\nu: \nu \in N)$ is said to possess the property $D(c, d)$ if $N' \in [N]^c$ implies $|\Pi(\nu \in N')A_\nu| < d$. We express this by writing $(A_\nu: \nu \in N) \in D(c, d)$. The set mapping $f(x)$ on $S$ has the property $D(c, d)$ whenever the set family $(f(x): x \in S) \in D(c, d)$. The property $D(c, d)$, with a different notation, was introduced in [29], p. 871, definition (1. 3).

For the sake of brevity we introduce the following definition

**21. 5.** DEFINITION. The relation

$$a \to [[p, c, d, q]] \tag{2}$$

expresses the following condition. Whenever $|S|=a$, and $f$ is a set mapping on $S$ of order $p$ having the property $D(c, d)$, then there exists a $f$-free set of cardinal $q$. It follows that the relation (2) is increasing in $a$ and decreasing in each of $p, c, d, q$. The relation (2), in a different notation, was introduced in [14], p. 281. We need

($*$) LEMMA 7. *Let $a=a'$ and $|S|=a^+$. Then there is a set family $\mathbf{F}=(A_\nu: \nu \in N)$ such that:*

(i) $|N|=a^+$; (ii) $A_\nu \in [S]^a$ *for* $\nu \in N$; (iii) $\mathbf{F} \in D(2, a)$; (iv) *whenever* $N' \in [N]^{a^+}$, *then* $|S-\Sigma(\nu \in N')A_\nu| < a^+$.

COROLLARY. If $a=a'$ then $a^+ \to [[a,^+ 2, a, a^+]]$. Lemma 7 is a theorem of A. HAJNAL [14]. We shall apply it to obtain partition relations, and we shall also deduce further results on set mappings. Some of the open problems stated in [14] will be settled. In section 27 we shall return to the theory of set mappings and shall formulate the simplest problems which still remain unsolved.

As a corollary of Lemma 7 we have

($*$) THEOREM 31. *If $a=a'$ then* $\begin{pmatrix} a^+ \\ a^+ \end{pmatrix} \nrightarrow \begin{pmatrix} a^+, a \vee a^+ \\ a^+, 2 \vee 1 \end{pmatrix}$.

PROOF. Let $n=\omega(a^+)$;

$$S=\{x_0,, \hat{x}_n\}_{\neq}; \quad T=\{y_0,, \hat{y}_n\}_{\neq}; \quad ST=\varnothing.$$

Let the family $\mathbf{F}=(A_\nu: \nu \in N)$ have the properties stated in Lemma 7. By (i) and (iii) we can write $\{A_\nu: \nu \in N\}=\{F_0,, \hat{F}_n\}_{\neq}$. Then we have

$$[S, T] = I_0 +' I_1 \qquad \text{(partition } \Delta\text{)},$$

where $I_1 = \{\{x_\mu, y_\nu\}: x_\mu \in F_\nu\}$. Then $\Delta$ has the required properties. For let $T' \in [T]^{a^+}$. Put $\mathbf{F}' = \{F_\nu : y_\nu \in T'\}$. Then $|\mathbf{F}'| = a^+$ and hence, by (iv), $|S - \Sigma(y_\nu \in T') Q_1(y_\nu)| = |S - \Sigma(F_\nu \in \mathbf{F}') F_\nu| < a^+$. Hence, if $S' \subset S$ and $[S', T'] \subset I_0$, then $|S'| < a^+$. On the other hand, by (iii), $|Q_1(y_\mu) Q_1(y_\nu)| = |F_\mu F_\nu| < a$ for $\mu < \nu < n$ and, by (ii), $|Q_1(y_\nu)| < a^+$ for $\nu < n$. This proves Theorem 31.

REMARK. If $a > a'$ then the conclusion of Theorem 31 is in general false. We shall return to this point later.

(*) THEOREM 32. *If $a \geqq \aleph_0$ then, putting $b = a^+$, we have*

$$\begin{pmatrix} b \\ b \end{pmatrix} \nrightarrow \begin{pmatrix} b \vee a, & b \vee a \\ a \vee b, & a \vee b \end{pmatrix}. \tag{3}$$

PROOF. Let $S$ and $T$ be as in the preceding proof. By Theorem 17A there is a partition $[S]^2 = I_0^* +' I_1^*$ such that, whenever $S' \in [S]^a$; $S'' \in [S - S']^b$; $\lambda < 2$, then $[S', S''] I_\lambda^* \neq \varnothing$. Then the partition $[S, T] = I_0 +' I_1$, defined by

$$I_0 = \{\{x_\mu, y_\nu\}: \{x_{2\mu}, x_{2\nu+1}\} \in I_0^*\},$$

has the properties required by (3), and Theorem 32 follows.

(*) COROLLARY 16. *If $a \geqq \aleph_0$ then*

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \nrightarrow \begin{pmatrix} a & a \\ a^+ & a^+ \end{pmatrix}.$$

We shall need the following theorem of A. TARSKI*.

(*) LEMMA 8. *Let $a, b \geqq \aleph_0$; $|S| = a$; $\mathbf{F} = (A_\nu : \nu \in N)$; $|N| = a^+$; $A_\nu \in [S]^{\geqq b}$ for $\nu \in N$. Then*

(i) *if $a' \neq b'$ then $\mathbf{F} \notin D(a^+, b)$;*

(ii) *if $c < b$ then $\mathbf{F} \notin D(a^+, c)$.*

## 22. POSITIVE RESULTS FOR THE CASES $\alpha = \beta$ AND $\alpha + 1 = \beta$

(*) THEOREM 33. *If $a \geqq \aleph_0$ and $a_1 < a$, then* $\begin{pmatrix} a \\ a^+ \end{pmatrix} \rightarrow \begin{pmatrix} a & a_1 \\ a^+ & a^+ \end{pmatrix}$.

PROOF. *Case* 1. $a = \aleph_0$. The result is trivial for $a_1 = 0$. The conclusion would follow for all $a_1$ if we can prove the following more general proposition:

$$\left\{ \begin{array}{ll} \text{If} \quad a \geqq \aleph_0;\ c < a < b';\ \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & b \end{pmatrix}, & \\ \text{then} & \begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a & c+1 \\ b & b \end{pmatrix}. \end{array} \right. \tag{4}$$

PROOF OF (4). Let $ST = \varnothing$; $|S| = a$; $|T| = b$;

$$[S, T] = I_0 + I_1; \quad (a, b) \notin [I_0].$$

* [15] p. 211 Theorem 51, and p. 213 Corollary 6 for (i) and (ii) respectively.

Then there are $X_0 \in [S]^c$ and $Y_0 \in [T]^b$ with $[X_0, Y_0] \subset I_1$. Put $Y_1 = Y_0 - \Sigma(x \in S - X_0) P_1(x)$. Then $[S - X_0, Y_1] \subset I_0$; $|S - X_0| = a$; $|Y_1| < b$; $|\Sigma(x \in S - X_0) Y_0 P_1(x)| = b$; $|S - X_0| = a < b'$, and hence there is $x_0 \in S - X_0$ with $|Y_0 P_1(x_0)| = b$. Then $[X_0 + \{x_0\}, Y_0 P_1(x_0)] \subset I_1$; $|X_0 + \{x_0\}| = c + 1$, and (4) follows.

*Case* 2. $a > \aleph_0$. We define $a_2$ as follows. If $a = a'$ then $a_2 = a_1$, and if $a > a'$ then $a_2 = a_1^+ + a'^+$. Then in any case, $a_1 \leqq a_2 < a$ and $a_2' \neq a'$. Put $a^+ = b$. Let $ST = \emptyset$;

$$|S| = a; \quad |T| = b; \quad [S, T] = I_0 +' I_1.$$

Put $T_0 = T\{y : |Q_1(y)| \geqq a_2\}$; $\mathbf{F} = (Q_1(y) : y \in T_0)$.

*Case* 2a. $\mathbf{F} \notin D(b, a_2)$. Then there is $T_1 \in [T_0]^b$ such that $|\Pi(y \in T_1) Q_1(y)| \geqq a_2$. Then

$$[\Pi(y \in T_1) Q_1(y), T_1] \subset I_1; \quad (a_1, b) \in [I_1].$$

*Case* 2b. $\mathbf{F} \in D(b, a_2)$. Then, by Lemma 8 (i), $|T_0| < b$, and hence $|T - T_0| = b$. Since $a_2^+ \leqq a$, there is a partition $S = \Sigma'(\mu < m) S_\mu$ with $|S_\mu| = a$ for $\mu < m$, where $m = \omega(a_2^+)$. But $|Q_1(y)| < a_2$ for $y \in T - T_0$. Hence, given $y \in T - T_0$ there is $\mu(y) < m$ with $S_{\mu(y)} Q_1(y) = \emptyset$. Then there are a number $\mu_0 < m$ and a set $Y_0 \in [T - T_0]^b$ such that $\mu(y) = \mu_0$ for $y \in Y_0$. Put $X_0 = S_{\mu_0}$. Then $|X_0| = a$; $X_0 Q_1(y) = \emptyset$ for $y \in Y_0$, and $[X_0, Y_0] \subset I_0$; $(a, b) \in [I_0]$, and Theorem 33 follows.

Our next theorem is now almost trivial.

THEOREM 34. *Let* $c < \aleph_0 \leqq a$. *Then* $\begin{pmatrix} a \\ a^+ \end{pmatrix} \rightarrow \begin{pmatrix} a & a \\ a^+ & c \end{pmatrix}$.

PROOF. Let this be true for $c = c_0 < \aleph_0$. It suffices to deduce that it holds for $c = c_0 + 1$. Let $|S| = a$; $|T| = a^+$; $ST = \emptyset$; $[S, T] = I_0 +' I_1$; $(a, a^+) \notin [I_0]$. Then there are sets $S' \in [S]^a$; $T' \in [T]^{c_0}$ with $[S', T'] \subset I_1$.

*Case* 1. $|S' Q_1(y)| < a$ for $y \in T - T'$. Put

$$T(a_0) = (T - T')\{y : |S' Q_1(y)| = a_0\} \text{ for } a_0 < a.$$

Then there is $a_0 < a$ with $|T(a_0)| = a^+$. We can write $S' = \Sigma'(\mu < m) S_\mu$, where $m = \omega(a_0^+)$ and $|S_\mu| = a$ for $\mu < m$. If $y \in T(a_0)$ then $|S' Q_1(y)| = a_0 < |m|$, and hence there is $\mu(y) < m$ with $S_{\mu(y)} Q_1(y) = \emptyset$. Then there are $Y_0 \in [T(a_0)]^{a^+}$ and $\mu_0 < m$ with $\mu(y) = \mu_0$ for $y \in Y_0$. Put $X_0 = S_{\mu_0}$. Then $X_0 Q_1(y) = \emptyset$ for $y \in Y_0$, and $[X_0, Y_0] \subset I_0$; $(a, a^+) \in [I_0]$ which is a contradiction.

*Case* 2. There is $y_0 \in T - T'$ with $|S' Q_1(y_0)| = a$. Then

$$[S' Q_1(y_0), T' + \{y_0\}] \subset I_1; \quad (a, c_0 + 1) \in [I_1],$$

and Theorem 34 follows.

LEMMA 9. *Let* $f(x)$ *be a set mapping of order* $p$ *on* $S$, *where* $|S| = a \geqq \aleph_0$ *and* $p < a$. *Then there exists a f-free set of cardinal* $a$.

This lemma was first proved by P. ERDŐS using $(*)$. It can be proved without assuming $(*)$. See [16] and [17].

A corollary of Lemma 9 is

THEOREM 35. *If* $a \geqq \aleph_0$ *and* $b_1 < a$, *then* $\begin{pmatrix} a \\ a \end{pmatrix} \rightarrow \begin{pmatrix} a & 1 \\ a & b_1 \end{pmatrix}$.

PROOF. Let $n=\omega(a)$; $S=\{x_0,, \hat{x}_n\}_{\neq}$; $T=\{y_0,, \hat{y}_n\}_{\neq}$; $ST=\varnothing$; $[S, T] = I_0+'I_1$. Assume that $(1, b_1)\notin[I_1]$. Define a set mapping $f$ on $[0, n)$ by putting

$$f(\mu) = ([0, n)-\{\mu\})\{\nu: y_\nu\in P_1(x_\mu)\} \quad \text{for} \quad \mu<n.$$

Then $|f(\mu)| \leqq |P_1(x_\mu)|<b_1$ for $\mu<n$, and by Lemma 9 there is a $f$-free set $N\in[[0, n)]^a$. Then $y_\nu\notin P_1(x_\mu)$ for $\mu, \nu\in N'$ and hence $[\{x_\mu: \mu\in N\}, \{y_2: \nu\in N\}]\subset I_0$; $(a, a)\in[I_0]$, and Theorem 35 follows.

($*$) THEOREM 36. *Let $a=a'$ and $d^+<a$. Then $a\to[[a, a, d, a]]$.*

PROOF. Let this be false. Then there is a set mapping $f$ of order $a$ on a set $S$, where $|S|=a$, such that $f\in D(a, d)$ and, at the same time, there is no $f$-free set of cardinal $a$. Put $n=\omega(d^+)$. We define sets $S_0,, \hat{S}_n$. Let $\nu<n$, and let $S_0,, \hat{S}_\nu\in[S]^{<a}$. We now define $S_\nu$. Put $S_\nu^* = S_0++\hat{S}_\nu$; $S_\nu^{**}=f(S_\nu^*)$. Since $a=a'$ we have $|S_\nu^*|<a$ and $|S_\nu^{**}|<a$. Hence $S_\nu' = S-(S_\nu^*+S_\nu^{**})\neq\varnothing$. Let $S_\nu$ be a maximal $f$-free subset of $S_\nu'$. Then $1\leqq|S_\nu|<a$ and $S_\mu S_\nu=\varnothing$ for $\mu<\nu$. This defines $S_0,, \hat{S}_n$. Put $S^* = S_0++\hat{S}_n$ and $S^{**}=f(S^*)$. Then $|S^*|\geqq|n| = d^+$. Also, since $|n|<a$, we have $|S^*+S^{**}|<a$. Put $T = S-(S^*+S^{**})$. Then $|T|=a$. Let $x\in T$. Then, by the maximality of $S_\nu$, we have $f(x)S_\nu\neq\varnothing$ for $\nu<n$. Hence, for $x\in T$,

$$|f(x)S^*| = \Sigma(\nu<n)|f(x)S_\nu|\geqq d^+.$$

Put $|S^*|=c$.

*Case* 1. $c^+<a$. Then $|\{f(x)S^*: x\in T\}|\leqq 2^{|S^*|}=c^+<a$, and there are a set $T'\in[T]^a$ and a set $A\in[S]^{\geqq d^+}$ such that $f(x)S^*=A$ for $x\in T'$. Then

$$|\Pi(x\in T')f(x)S^*| = |A|\geqq d^+$$

which contradicts the hypothesis $f\in D(a, d)$.

*Case* 2. $c^+\geqq a$. Then $c^+=a$. Consider the family $\mathbf{F}=(f(x)S^*: x\in T)$. Since $f\in D(a, d)=D(c^+, d)$, we have $\mathbf{F}\in D(c^+, d)$. On the other hand an application of Lemma 8 (ii), with $a, b, c, S, N$ in the Lemma replaced by $c, d^+, d, S^*, T$ respectively, yields $\mathbf{F}\notin D(c^+, d)$. This contradiction proves Theorem 36.

($*$) THEOREM 37. *Let $a>a'$. Then*

(i) $\qquad a\to[[a, a, d, a']]$ *for* $d<a$;

(ii) $\qquad a\nrightarrow[[a, a, 1, a'^+]]$.

Part (i) can be deduced from Theorem 36 in the usual way by means of Lemma 3, and part (ii) is established by means of a trivial "canonical" counter example. Since Theorem 37 will not be used in our discussion we omit the proof.

($*$) THEOREM 38. *If $a\geqq\aleph_0$ and $c_0^+, c_1^+<a$, then*

$$\begin{pmatrix} a \\ a \end{pmatrix}\to\begin{pmatrix} a, & a\vee c_0 \\ a, & c_1\vee a \end{pmatrix}.$$

PROOF. Let $n=\omega(a)$; $N=[0, n)$; $S=\{x_0,, \hat{x}_n\}_{\neq}$; $T=\{y_0,, \hat{y}_n\}_{\neq}$; $ST=\varnothing$; $[S, T] = I_0+'I_1$.

*Case* 1. $a=a'$. We define set mappings $f_0$ and $f_1$ on $N$ as follows.

$$f_0(\nu)=[0,\nu)\,\{\mu: x_\mu \in Q_1(y_\nu)\} \qquad \text{for } \nu<n;$$

$$f_1(\mu)=[0,\mu)\,\{\nu: y_\nu \in P_1(x_\mu)\} \qquad \text{for } \mu<n.$$

Then $f_0$ and $f_1$ are of order $a$.

*Case* 1a. $f_0 \notin D(a, c_0)$. Then there is $N' \in [N]^a$ with $|\Pi(\nu \in N')f_0(\nu)| \geqq c_0$. Put $X'=\Pi(\nu \in N')\,Q_1(y_\nu)$ and $Y'=\{y_\nu: \nu \in N'\}$. Then $[X', Y'] \subset I_1$; $|X'| \geqq c_0$; $|Y'|=a$; $(c_0, a) \in [I_1]$.

*Case* 1b. $f_1 \notin D(a, c_1)$. Then, by symmetry, $(a, c_1) \in [I_1]$.

*Case* 1c. $f_\varkappa \in D(a, c_\varkappa)$ for $\varkappa<2$. Then, by Theorem 36, there is a set $N_0 \in [N]^a$ which is $f_0$-free. By applying Theorem 36 to the set mapping $N_0 f_1(\mu)$ we obtain a set $N_1 \in [N_0]^a$ which is both $f_0$-free and $f_1$-free. Let $N_1=\{\lambda_0,, \hat{\lambda}_n\}_<$; $X= =\{x_{\lambda_{2\mu}}: \mu<n\}$; $Y=\{y_{\lambda_{2\nu+1}}: \nu<n\}$. Let $x_\sigma \in X$ and $y_\tau \in Y$. If $\sigma<\tau$ then $\sigma \notin f_0(\tau)$; $x_\sigma \notin Q_1(y_\tau)$, and if $\sigma \geqq \tau$ then $\tau<\sigma$; $\tau \notin f_1(\sigma)$; $y_\tau \notin P_1(x_\sigma)$. In either case, $\{x_\sigma, y_\tau\} \in I_0$. Hence $[X, Y] \subset I_0$; $(a, a) \in [I_0]$.

*Case* 2. $a>a'$. Let $\varrho=\omega(a')$ and

$$c_0, c_1 \leqq a_0 < \cdots < \hat{a}_\varrho < a = \sup(\sigma<\varrho)a_\sigma.$$

By Lemma 3A there are sets $S_\sigma$, $T_\sigma$ and numbers $h(\sigma, \tau)<2$ such that $\Sigma'(\sigma<\varrho)\,S_\sigma \subset S$; $\Sigma'(\sigma<\varrho)\,T_\sigma \subset T$;

$$|S_\sigma|=|T_\sigma|=a_\sigma \qquad \text{for } \sigma<\varrho,$$

and

$$[S_\sigma, T_\tau] \subset I_{h(\sigma,\tau)} \qquad \text{for } \sigma, \tau<\varrho.$$

Let

$$U=\{u_0,, \hat{u}_\varrho\}_{\neq}; \quad W=\{w_0,, \hat{w}_\varrho\}_{\neq}; \quad UW=\varnothing.$$

Then

$$[U, W] = I_0^* +' I_1^*$$

where

$$I_0^*=\{\{u_\sigma, w_\tau\}: h(\sigma, \tau)=0\}.$$

Since $a'$ is regular we have, by case 1,

$$\begin{pmatrix} a' \\ a' \end{pmatrix} \to \begin{pmatrix} a', a' \vee 1 \\ a', 1 \vee a' \end{pmatrix}.$$

Then there are the following cases:

*Case* 2a. $(a', a') \in [I_0^*]$. Then there are sets $U_0 \in [U]^{a'}$ and $W_0 \in [W]^{a'}$ with $[U_0, W_0] \subset I_0^*$. Put $X_0=\Sigma(u_\sigma \in U_0)\,S_\sigma$ and $Y_0=\Sigma(w_\tau \in W_0)\,T_\tau$. Then

$$[X_0, Y_0] \subset I_0; \quad |X_0|=|Y_0|=a; \quad (a, a) \in [I_0].$$

*Case* 2b. $(a', 1) \in [I_1^*]$. Then there are a set $U_0 \in [U]^{a'}$ and an element $w_{\tau_0} \in W$ with $[U_0, \{w_{\tau_0}\}] \subset I_1^*$. Put $X_0=\Sigma(u_\sigma \in U_0)\,S_\sigma$ and $Y_0=T_{\tau_0}$. Then $[X_0, Y_0] \subset I_1$; $|X_0|=a$; $|Y_0|=a_{\tau_0} \geqq c_1$; $(a, c_1) \in [I_1]$.

*Case* 2c. $(1, a') \in [I_1^*]$. Then, by symmetry, $(c_0, a) \in [I_1]$. This proves Theorem 38.

($*$) THEOREM 39. *If* $a \geqq \aleph_0$ *then*

$$\binom{a^+}{a^+} \to \binom{a,\ a^+ \vee a}{a,\ a \vee a^+}.$$

PROOF. We shall deal with the cases $a = a'$ and $a > a'$ simultaneously. Put $\varrho = \omega(a')$; $n = \omega(a)$; $m = \omega(a^+)$. Let $X_0 Y_0 = \emptyset$; $|X_0| = |Y_0| = a^+$; $[X_0, Y_0] = I_0 +' I_1$. We assume that

$$(5) \qquad (a^+, a), (a, a^+) \notin [I_1]$$

and we shall deduce that $(a, a) \in [I_0]$. If $a > a'$ then we choose $a_0,, \hat{a}_\varrho$ with $a_\sigma = a'_\sigma$ for $\sigma < \varrho$ and

$$(6) \qquad a' < a_0 < < \hat{a}_\varrho < a = \sup(\sigma < \varrho) a_\sigma.$$

If $A \in [X_0]^a$ and $B \in [Y_0]^{a^+}$, then

$$[A, B - \Sigma(x \in A) P_0(x)] \subset I_1.$$

Hence, by (5),

$$(7) \qquad |B - \Sigma(x \in A) P_0(x)| \leqq a.$$

By symmetry, if $A \in [X_0]^{a^+}$ and $B \in [Y_0]^a$, then

$$(8) \qquad |A - \Sigma(y \in B) Q_0(y)| \leqq a.$$

Now let

$$A \in [X_0]^a; \quad B \in [Y_0]^{a^+}; \quad a' < p = p' < a.$$

We shall show that there are a set $\mathbf{A}^* \subset \mathbf{P}(A)$ and a function $U(X)$ from $\mathbf{A}^*$ into $\mathbf{P}(B)$ such that

$$(9) \qquad |\mathbf{A}^*| \leqq a;$$

$$(10) \qquad \mathbf{A}^* \subset [A]^p;$$

$$(11) \qquad [X, U(X)] \subset I_0 \quad \text{for} \quad X \in \mathbf{A}^*;$$

$$(12) \qquad |B - \Sigma(X \in \mathbf{A}^*) U(X)| \leqq a.$$

This is a generalization of Theorem 33 for the case $a > a'$. In the proof that follows we cannot apply Lemma 8 directly but our proof is based on the same ideas as Tarski's proof in the corresponding case.

Put $Y_1 = B\{y: |AQ_0(y)| < p\}$ and $Y_2 = B - Y_1$. If we assume that $|Y_1| = a^+$ then, since by Theorem 34 $\binom{a}{a^+} \to \binom{a\ a}{1\ a^+}$, it follows from $[A, Y_1] \subset I_0 + I_1$ that either $(a, 1) \in [[A, Y_1] I_0]$ which contradicts the definition of $Y_1$, or $(a, a^+) \in [[A, Y_1] I_1]$ which contradicts (5). Hence $|Y_1| \leqq a$. There is a partition $A = \Sigma'(\sigma < \varrho) A_\sigma$ with $|A_\sigma| = a_\sigma$ for $\sigma < \varrho$. Put $\mathbf{A}^* = \Sigma(\sigma < \varrho)[A_\sigma]^p$. Then $|\mathbf{A}^*| \leqq \Sigma(\sigma < \varrho)\, a_\sigma^p = a$, and (9) and (10) hold. We put $U(X) = Y_2\{y: X \subset AQ_0(y)\}$ for $X \in \mathbf{A}^*$. If, now, $y \in Y_2$ then there is $X \in \mathbf{A}^*$ with $y \in U(X)$. For otherwise $|A_\sigma Q_0(y)| < p$ for $\sigma < \varrho$ and, using the definition of $Y_2$ and the fact that $|\varrho| = a' < p = p'$, we obtain the contradiction $p \leqq |AQ_0(y)| = \Sigma(\sigma < \varrho)|A_\sigma Q_0(y)| < p$. This shows that $Y_2 = \Sigma(X \in \mathbf{A}^*) U(X)$, and (11) and (12) follow. By symmetry we have: If $A \in [X_0]^{a^+}$;

$B \in [Y_0]^a$; $a' < p = p' < a$, then there are a set $\mathbf{B}^* \subset \mathbf{P}(B)$ and a function $V(Y)$ from $\mathbf{B}^*$ into $\mathbf{P}(A)$ such that

(13) $$|\mathbf{B}^*| \leqq a;$$

(14) $$\mathbf{B}^* \subset [B]^p;$$

(15) $$[V(Y), Y] \subset I_0 \quad \text{for} \quad Y \in \mathbf{B}^*;$$

(16) $$|A - \Sigma(Y \in \mathbf{B}^*) V(Y)| \leqq a.$$

Let $X_0 = \{x_0,, \hat{x}_m\}_{\neq}$; $Y_0 = \{y_0,, \hat{y}_m\}_{\neq}$; $S = [0, m)$. For $X \subset X_0$; $Y \subset Y_0$; $S' \subset S$ put $M(X) = \{\mu : x_\mu \in X\}$; $M(Y) = \{\nu : y_\nu \in Y\}$;

$$X(S') = \{x_\mu : \mu \in S'\}; \quad Y(S') = \{y_\nu : \nu \in S'\}.$$

We now define a ramification system $\mathbf{R}$ on $S$ of length $\varrho$ and order $n$. We use the notation of Lemma 1. Let $\sigma < \varrho$. We assume that for $\tau < \sigma$ and $\nu_0,, \nu_\tau < n$ the sets $F(\nu_0,, \hat{\nu}_\tau)$ and $S(\nu_0,, \nu_\tau)$ have already been defined. Now let $\nu_0,, \hat{\nu}_\sigma < n$. We have to define $F(\nu_0,, \hat{\nu}_\sigma)$ and $S(\nu_0,, \nu_\sigma)$ for $\nu_\sigma < n$. Throughout the rest of the whole proof we abbreviate, whenever possible, the sequence $\nu_0,, \hat{\nu}_\sigma$ to the single letter $\nu$. If $|S'(\nu)| \leqq a$ then we put $F(\nu) = S'(\nu)$ and $S(\nu, \nu_\sigma) = \emptyset$ for $\nu_\sigma < n$. Now let $|S'(\nu)| = a^+$. We recall that ordinals of the form $2\lambda$ are called even and those of the form $2\lambda + 1$ odd.

*Case* 1. $a = a'$.

*Case* 1a. $\sigma$ even. Then we choose

$$R_0(\nu) = \{x(\nu, \nu_\sigma) : \nu_\sigma < n\}_{\neq} \subset [X(S'(\nu))]^a.$$

Put

$$E_1(\nu) = Y\big(S'(\nu)\big) - \Sigma(\nu_\sigma < n) P_0(x(\nu, \nu_\sigma));$$

$$R(\nu) = M\big(R_0(\nu)\big); \quad E(\nu) = M\big(E_1(\nu)\big); \quad F(\nu) = R(\nu) + E(\nu).$$

Then $S'(\nu) - F(\nu) = \Sigma(\nu_\sigma < n) S(\nu, \nu_\sigma)$, where

$$S(\nu, \nu_\sigma) = S'(\nu) M\big(P_0(x(\nu, \nu_\sigma))\big) - F(\nu) \qquad \text{for } \nu_\sigma < n.$$

This follows by a straightforward application of the definitions given above.

*Case* 1b. $\sigma$ odd. Then choose

$$R_1(\nu) = \{y(\nu, \nu_\sigma) : \nu_\sigma < n\}_{\neq} \subset [Y(S'(\nu))]^a.$$

Put

$$E_0(\nu) = X\big(S'(\nu) - \Sigma(\nu_\sigma < n) Q_0(y(\nu, \nu_\sigma)\big);$$

$$R(\nu) = M\big(R_1(\nu)\big); \quad E(\nu) = M\big(E_0(\nu)\big); \quad F(\nu) = R(\nu) + E(\nu).$$

Then $S'(\nu) - F(\nu) = \Sigma(\nu_\sigma < n) S(\nu, \nu_\sigma)$, where

$$S(\nu, \nu_\sigma) = S'(\nu) M\big(Q_0(y(\nu, \nu_\sigma))\big) - F(\nu) \qquad \text{for } \nu_\sigma < n.$$

This follows by symmetry from case 1a.

*Case* 2. $a > a'$.

*Case* 2a. $\sigma$ even. Then choose $R_0(v) \in [X(S'(v))]$. Let*

$$\mathbf{A}^*(v) = \{X(v, v_\sigma) : v_\sigma < n\}$$

be a subset of $\mathbf{P}(R_0(v))$ and $U(v, X)$ a function on $\mathbf{A}^*(v)$ such that (9)—(12) hold for $\mathbf{A}^* = \mathbf{A}^*(v)$; $U(X) = U(v, X)$;

$$A = R_0(v); \quad B = Y(S'(v)); \quad p = a_\sigma.$$

Put

$$E_1(v) = Y(S'(v)) - \Sigma(v_\sigma < n)\, U(v, X(v, v_\sigma));$$

$$R(v) = M(R_0(v)); \quad E(v) = M(E_1(v)); \quad F(v) = R(v) + E(v).$$

Then $S'(v) - F(v) = \Sigma(v_\sigma < n)\, S(v, v_\sigma)$, where

$$S(v, v_\sigma) = M(U(v, X(v, v_\sigma))) - F(v) \qquad \text{for } v_\sigma < n.$$

This follows from our definitions.

*Case* 2b. $\sigma$ odd. Then choose $R_1(v) \in [Y(S'(v))]^a$. Let

$$\mathbf{B}^*(v) = \{Y(v, v_\sigma) : v_\sigma < n\}$$

be a subset of $\mathbf{P}(R_1(v))$ and $V(v, Y)$ a function on $\mathbf{B}^*(v)$ such that (13)—(16) hold for $\mathbf{B}^* = \mathbf{B}^*(v)$; $V(Y) = V(v, Y)$;

$$A = X(S'(v)); \quad B = R_1(v); \quad p = a_\sigma.$$

Put

$$E_0(v) = X(S'(v)) - \Sigma(v_\sigma < n)\, V(v, Y(v, v_\sigma));$$

$$R(v) = M(R_1(v)); \quad E(v) = M(E_0(v)); \quad F(v) = R(v) + E(v).$$

Then $S'(v) - F(v) = \Sigma(v_\sigma < n)\, S(v, v_\sigma)$, where

$$S(v, v_\sigma) = M(V(v, Y(v, v_\sigma))) - F(v) \qquad \text{for } v_\sigma < n.$$

This follows by symmetry from case 2a. We have completed the definition of the ramification system $\mathbf{R}$ on $S$ and, irrespective whether $a = a'$ or $a > a'$,

(17) $\mathbf{R}$ is of length $\varrho$ and of order $n$.

We now define for $v_\sigma < n$ a set $f(v, v_\sigma) \subset R(v)$. If $|S'(v)| \leqq a$ then we put $f(v, v_\sigma) = \emptyset$. Now let $|S'(v)| = a^+$.

*Case* I. $a = a'$.

*Case* Ia. $\sigma$ even. Then put $f(v, v_\sigma) = M(\{x(v, v_\sigma)\})$. Then

$$f(v, v_\sigma) \subset M(R_0(v)) = R(v).$$

*Case* Ib. $\sigma$ odd. Then put $f(v, v_\sigma) = M(\{y(v, v_\sigma)\})$. Then

$$f(v, v_\sigma) \subset M(R_1(v)) = R(v).$$

*Case* II. $a > a'$.

* There is no risk of confusing the function $X(S')$ defined for subsets $S'$ of $S$, and the function $X(v, v_\sigma)$ defined for sequences $v_0,, v_\sigma$. Similarly later with $Y(S')$ and $Y(v, v_\sigma)$.

*Case* IIa. $\sigma$ even. Then put $f(v, v_\sigma) = M(X(v, v_\sigma))$. Then

$$f(v, v_\sigma) \subset M(R_0(v)) = R(v).$$

*Case* IIb. $\sigma$ odd. Then put $f(v, v_\sigma) = M(Y(v, v_\sigma))$. Then

$$f(v, v_\sigma) \subset M(R_1(v)) = R(v).$$

This completes the definition of $f(v, v_\sigma)$ for $\sigma < \varrho$ and $v_0,, v_\sigma < n$. We have in any of our cases

$$F(v) = R(v) + E(v); \quad f(v, v_\sigma) \subset R(v)$$

and, by definition,

(18) $$|R(v)| \leqq a.$$

We also have

(19) $$|E(v)| \leqq a.$$

For in case 1 this follows from (7) and (8), and in case 2 from (12) and (16). Now we have, using (17), (18) and (19),

$$|S| = a^+; \quad |\varrho| < a^+; \quad |n| < a^+; \quad |F(v)| < a^+.$$

Hence Lemma 1 (v) applies to **R** and yields numbers $v_0,, \hat{v}_\varrho < n$ such that

$$\Pi(\sigma < \varrho)\, S(v, v_\sigma) \neq \varnothing.$$

From now on $v_0,, \hat{v}_\varrho$ are fixed. Put $Z_\sigma = f(v, v_\sigma)$ for $\sigma < \varrho$;

$$S_0 = \{2\tau : \tau < \varrho\}; \quad S_1 = \{2\tau + 1 : \tau < \varrho\};$$

$$X^* = \Sigma(\sigma \in S_0)\, X(Z_\sigma); \quad Y^* = \Sigma(\sigma \in S_1)\, Y(Z_\sigma).$$

To complete the proof it suffices to show that

(20) $$X^* \in [X_0]^a; \quad Y^* \in [Y_0]^a;$$

(21) $$[X^*, Y^*] \subset I_0.$$

Proof of (20). $X^* \subset X_0$; $Y^* \subset Y_0$. By Lemma 1 (i),

$$R(v_0,, \hat{v}_\sigma)\, R(v_0,, \hat{v}_\tau) = \varnothing \qquad \text{for } \sigma < \tau < \varrho.$$

*Case* A. $a = a'$. Then $|Z_\sigma| = |f(v, v_\sigma)| = 1$ and hence

$$|X^*| = |S_0| = a; \quad |Y^*| = |S_1| = a.$$

*Case* B. $a > a'$. Let $\sigma < \varrho$. If $\sigma$ is even then, by definition of $f(v, v_\sigma)$ and (10), $|Z_\sigma| = |M(X(v, v_\sigma))| = a_\sigma$, and if $\sigma$ is odd then $|Z_\sigma| = |M(Y(v, v_\sigma))| = a_\sigma$. Hence $|X^*| = \Sigma(\sigma \in S_0)\, a_\sigma = a$; $|Y^*| = \Sigma(\sigma \in S_1)\, a_\sigma = a$.

Proof of (21). Let $\sigma \in S_0$; $\tau \in S_1$. Then $\sigma \neq \tau$. It suffices to show that

(22) $$[X(Z_\sigma), Y(Z_\tau)] \subset I_0.$$

*Case* $\alpha$. $\sigma < \tau$. Then

$$Z_\tau = f(v_0,, v_\tau) \subset R(v_0,, \hat{v}_\tau) \subset S'(v_0,, \hat{v}_\tau) \subset S(v_0,, v_\sigma).$$

*Case* $\alpha 1$. $a = a'$. Then

$$[X(Z_\sigma), Y(Z_\tau)] \subset [\{x(v_{0},, v_\sigma), Y(S(v_{0},, v_\sigma))] \subset [\{x(v, v_\sigma)\} \, Y(M(P_0(x(v, v_\sigma))))] = \\ = [\{x(v, v_\sigma)\}, P_0(x(v, v_\sigma))] \subset I_0 .$$

*Case* $\alpha 2$. $a > a'$. Then $Z_\sigma = M\big(X(v, v_\sigma)\big)$ and

$$S(v, v_\sigma) \subset M\big(U(v, X(v, v_\sigma))\big).$$

Hence

$$Y(Z_\tau) \subset Y\big(S(v_{0},, v_\sigma)\big) \subset U\big(v, X(v, v_\sigma)\big).$$

But $X(Z_\sigma) = X(v, v_\sigma)$. By (11), when applied to $\mathbf{A}^*(v)$,

$$[X(Z_\sigma), Y(Z_\tau)] \subset [X(v, v_\sigma), U(v, X(v, v_\sigma))] \subset I_0 .$$

This completes the proof of (22) in case $\alpha$.

*Case* $\beta$. $\tau < \sigma$. Then (22) follows exactly as in case $\alpha$, for reasons of symmetry. This proves Theorem 39.

As an immediate consequence of Theorem 39 we have

($*$) COROLLARY 17.

$$\begin{pmatrix} a^+ \\ a^+ \end{pmatrix} \to \begin{pmatrix} a & a \\ a & a \end{pmatrix} \quad \text{for} \quad a \geqq \aleph_0 .$$

In [1] it is proved that $\begin{pmatrix} \aleph_0 \\ \aleph_1 \end{pmatrix} \to \begin{pmatrix} \aleph_0 & \aleph_0 \\ \aleph_1 & \aleph_0 \end{pmatrix}$. Using ($*$) we shall now prove the following generalization of this result.

($*$) THEOREM 40. *If* $a' = \aleph_0$ *then*

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \to \begin{pmatrix} a & a \\ a^+ & \aleph_0 \end{pmatrix}.$$

PROOF. Let $|A| = a$; $|B| = a^+$; $AB = \emptyset$; $[A, B] = I_0 + I_1$;

$$(a, a^+) \notin [I_0].$$

In the proof that follows we shall always suppose that $A_\nu \in [A]^a$ and $B_\nu \in [B]^{a^+}$. We can write $a = a_0 + + \hat{a}_\omega$, where $a_{0},, \hat{a}_\omega < a$. We define inductively $D_\nu, B_\nu, A_\nu, y_\nu$ for $\nu < \omega$ as follows. By Theorem 33 there are $D_0 \in [A]^{a_0}$ and $B_0 \subset B$ with $[D_0, B_0] \subset I_1$. By Theorem 34 there are $A_0 \subset A$ and $y_0 \in B_0$ with $[A_0, \{y_0\}] \subset I_1$. Generally, for $1 \leqq \nu < \omega$: By Theorem 33 there are $D_\nu \in [A_{\nu-1}]^{a_\nu}$ and $B_\nu \subset B_{\nu-1} - \{y_{\nu-1}\}$ with $[D_\nu, B_\nu] \subset I_1$. By Theorem 34 there are $A_\nu \subset A_{\nu-1}$ and $y_\nu \in B_\nu$ with $[A_\nu, \{y_\nu\}] \subset I_1$. Put $X = D_0 + + \hat{D}_\omega$; $Y = \{y_{0},, \hat{y}_\omega\}$. Then $X \in [A]^a$ and $Y \in [B]^{\aleph_0}$. If $r \leqq s$ then $[D_r, \{y_s\}] \subset [D_r, B_s] \subset [D_r, B_r] \subset I_1$. If $r > s$ then $[D_r, \{y_s\}] \subset [A_s, \{y_s\}] \subset I_1$. Hence $[X, Y] \subset I_1$, and Theorem 40 follows.

($*$) THEOREM 41. *If* $a > a'$ *and* $b < a$, *then*

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \to \begin{pmatrix} a & a \\ a & b \end{pmatrix}.$$

PROOF. We may assume that $a' < b = b' < a$. Put $\varrho = \omega(a')$. Let $b < a_0 < \cdots < \hat{a}_\varrho < a = \sup(\sigma < \varrho) a_\sigma$; $a_\sigma = a'_\sigma$ for $\sigma < \varrho$.

$$X_0 Y_0 = \emptyset; \quad |X_0| = a; \quad |Y_0| = a^+; \quad [X_0, Y_0] = I_0 +' I_1.$$

We may suppose that

$$(a, b) \notin [I_1]. \tag{23}$$

Now let $a' < p = p' < a$. Let $A \in [X_0]^a$ and $B \in [Y_0]^{a^+}$. Then there are a set $\mathbf{A}^* \subset \mathbf{P}(A)$ and a function $U(X)$ from $\mathbf{A}^*$ into $\mathbf{P}(B)$ such that

$$|\mathbf{A}^*| \leqq a; \tag{24}$$

$$\mathbf{A}^* \subset [A]^p; \tag{25}$$

$$[X, U(X)] \subset I_0 \quad \text{for} \quad X \in \mathbf{A}^*; \tag{26}$$

$$|B - \Sigma(X \in \mathbf{A}^*) U(X)| \leqq a. \tag{27}$$

The proof is identical, including the notation, with that of (9)—(12) in the proof of Theorem 39. We now define a ramification system $\mathbf{R}$ on $S = Y_0$ of length $\varrho$ and order $n = \omega(a)$. Let $\sigma < \varrho$ and $v_0,, \hat{v}_\sigma < n$. We write $v$ in place of $v_0,, \hat{v}_\sigma$.

If $|S'(v)| \leqq a$ then put $R(v) = \emptyset$ and $E(v) = F(v) = S'(v)$ and $S(v, v_\sigma) = \emptyset$ for $v_\sigma < n$.

Now let $|S'(v)| = a^+$. Then we choose any set $R(v) \in [S'(v)]^a$. Then there are a set $\mathbf{A}^*(v) \subset \mathbf{P}(X_0)$ and a function $U(v, X)$ on $\mathbf{A}^*(v)$ such that (24)—(27) hold for $A = X_0$; $B = S'(v) - R(v)$; $\mathbf{A}^* = \mathbf{A}^*(v)$; $U(X) = U(v, X)$ and $p = a_\sigma$. We can write $\mathbf{A}^*(v) = \{X(v, v_\sigma) : v_\sigma < n\}$. Put $S(v, v_\sigma) = U(v, X(v, v_\sigma))$ for $v_\sigma < n$;

$$E(v) = S'(v) - \Sigma(v_\sigma < n) S(v, v_\sigma); \quad F(v) = R(v) + E(v).$$

Then

$$S'(v) - F(v) = \Sigma(v_\sigma < n) S(v, v_\sigma).$$

This defines $\mathbf{R}$. Lemma 1 (v) applies. For we have $|S| = a^+$; $|\varrho| = a' < a^+$; $|n| = a < a^+$;

$$|F| = |R + (E - R)| = |R + ((S'(v) - R) - \Sigma(v_\sigma < n) U(v, X(v, v_\sigma)))| \leqq$$
$$\leqq |R| + |(S'(v) - R) - \Sigma(X \in \mathbf{A}^*(v)) U(v, X)| \leqq a.$$

By Lemma 1 (v) there are $v_0,, \hat{v}_\varrho < n$ with $S'(v_0,, \hat{v}_\varrho) \neq \emptyset$. From now on $v_0,, \hat{v}_\varrho$ are fixed. Let $\sigma < \varrho$. Then $|S'(v)| = a^+$. There is a one-one map

$$X \to H(v, X)$$

of $[X_0]^{a_\sigma}$ onto $[R(v)]^{a_\sigma}$. Put

$$f(\sigma) = H(v, X(v, v_\sigma)).$$

Then $f(\sigma) \in [R(v)]^{a_\sigma}$. Put $A_\sigma = X(v, v_\sigma)$; $B_\sigma = f(\sigma)$;

$$A = A_0 + + \hat{A}_\varrho; \quad B = B_0 + + \hat{B}_\varrho.$$

Then

$$A \in [X_0]^a; \quad B \in [Y_0]^a. \tag{28}$$

If $\sigma<\tau<\varrho$, then $B_\tau=f(\tau)\subset R(v_{0,},\hat{v}_\tau)\subset S'(v_{0,},\hat{v}_\tau)\subset S(v,v_\sigma)$. By (26), $[X(v,v_\sigma), U(v,X(v,v_\sigma))]\subset I_0$. Hence

(29) $$[A_\sigma, B_\tau]\subset I_0 \quad \text{for} \quad \sigma<\tau<\varrho.$$

By Lemma 3A there are sets $A'_\sigma\in[A]^{a_\sigma}$ and $B'_\sigma\in[B]^{a_\sigma}$ and numbers $h(\sigma,\tau)<2$ such that $A'_\sigma A'_\tau=B'_\sigma B'_\tau=\varnothing$ for $\sigma<\tau<\varrho$, and

(30) $$[A'_\sigma, B'_\tau]\subset I_{h(\sigma,\tau)} \quad \text{for} \quad \sigma,\tau<\varrho.$$

Let $M_0=\{u_{0,},\hat{u}_\varrho\}_{\neq}$; $M_1=\{v_{0,},\hat{v}_\varrho\}_{\neq}$; $M_0M_1=\varnothing$;

(31) $$[M_0, M_1] = I_1^*+'I_1^*,$$

where
$$I_0^*=\{\{u_\sigma, v_\tau\}: h(\sigma,\tau)=0\}.$$

By Theorem 38,
$$\binom{a'}{a'}\to\binom{a',\, a'\vee 1}{a',\, 1\vee a'}.$$

By applying this formula to the partition (31) we see that we have only to consider the following three cases:

*Case* 1. There are $M\in[[0,\varrho)]^{a'}$ and $\tau_0<\varrho$ such that
$$h(\sigma,\tau_0)=1 \quad \text{for} \quad \sigma\in M.$$

Then, by (30), $[A'', B'_{\tau_0}]\subset I_1$, where $A''=\Sigma(\sigma\in M)A'_\sigma$. Then $|A''|=a$; $(a, a_{\tau_0})\in[I_1]$ which contradicts (23).

*Case* 2. There are $\sigma_0<\varrho$ and $M\in[[0,\varrho)]^{a'}$ such that $h(\sigma_0,\tau)=1$ for $\tau\in M$. Then $[A'_{\sigma_0}, B'']\subset I_1$, where $B''=\Sigma(\tau\in M)B'_\tau$. Then $|B''|=a$. Choose $z_0\in A'_{\sigma_0}$. Then $[\{z_0\}, B'']\subset I_1$. Also, there is $\sigma_1<\varrho$ such that $z_0\in A_{\sigma_1}$. Hence, by (29), $B''\subset B_0++B_{\sigma_1}$ and we obtain the contradiction
$$a=|B''|\leqq|B_0++B_{\sigma_1}|=a_0++a_{\sigma_1}<a.$$

*Case* 3. There are sets $M', M''\in[[0,\varrho)]^{a'}$ such that
$$h(\sigma,\tau)=0 \quad \text{for} \quad \sigma\in M' \text{ and } \tau\in M''.$$

Put $A''=\Sigma(\sigma\in M')A'_\sigma$ and $B''=\Sigma(\tau\in M'')B'_\tau$. Then $A''\in[X_0]^a$; $B''\in[Y_0]^a$; $[A'', B'']\subset\subset I_0$. This proves Theorem 41.

(*) THEOREM 42. *If* $a'=\aleph_0$ *then* $\binom{a}{a^+}\to\binom{a\ a}{a\ a}$.

PROOF. If $a=\aleph_0$ the assertion follows from Theorem 40. Now let $a>\aleph_0$ so that $a>a'=\aleph_0$. Let $|S|=a$; $|T|=a^+$; $ST=\varnothing$; $[S,T]=I_0+I_1$. The letter $A$ will always denote subsets of $S$ and the letter $B$ subsets of $T$. Assume

(32) $$(a,a)\notin[I_0].$$

Let
$$a_0<<\hat{a}_\omega<a=a_0++\hat{a}_\omega;\quad |A|=a;\quad |B|=a^+;\quad n<\omega.$$

Then, by (32) and Theorem 33, there are $A', B'$ such that

(33) $$A'\in[A]^{a_n};\quad B'\in[B]^{a^+};\quad [A',B']\subset I_1.$$

By (32) and Theorem 41, there are $A''$, $B''$ such that

(34) $$A'' \in [A]^{a}; \quad B'' \in [B]^{a_n}; \quad [A'', B''] \subset I_1.$$

We now define inductively for $n < \omega$ the sets

$$A_n, A'_n, A''_n, A^*_n, B_n, B'_n, B''_n, B^*_n.$$

Put $A^*_0 = S$; $B^*_0 = T$. Let $n < \omega$, and suppose that $A^*_\nu$ and $B^*_\nu$ have been defined for $\nu \leqq n$ and that $A_\nu, A'_\nu, A''_\nu, B_\nu, B'_\nu, B''_\nu$ have been defined for $\nu < n$. Suppose also that $|A^*_n| = a$ and $|B^*_n| = a^+$. Then, applying (33) to $A = A^*_n$ and $B = B^*_n$, we find $A'_n$ and $B'_n$ such that

(35) $$A'_n \in [A^*_n]^{a_n}; \quad B'_n \in [B^*_n]^{a^+}; \quad [A'_n, B'_n] \subset I_1.$$

Applying (34) to $A = A^*_n$ and $B = B'_n$, we find $A''_n$ and $B''_n$ such that

(36) $$A''_n \in [A^*_n]^{a}; \quad B''_n \in [B'_n]^{a_n}; \quad [A''_n, B''_n] \subset I_1.$$

Put

(37) $$A_n = A'_n; \quad B_n = B''_n; \quad A^*_{n+1} = A''_n; \quad B^*_{n+1} = B'_n.$$

This completes the inductive definition. We have, for $n < \omega$,

(38) $$A^*_n \supset A^*_{n+1}; \quad B^*_n \supset B^*_{n+1}; \quad |A^*_n| = a; \quad |B^*_n| = a^+;$$

(39) $$|A_n| = |B_n| = a_n.$$

Put $A = A_0 + + \hat{A}_\omega$ and $B = B_0 + + \hat{B}_\omega$. Then, by (39), $|A| = |B| = a$. Let $m, n < \omega$. It suffices to prove that

(40) $$[A_m, B_n] \subset I_1.$$

If $m \leqq n$ then, by (37), (36), (38), (35),

$$A_m = A'_m; \quad B_n = B''_n \subset B'_n = B^*_{n+1} \subset B^*_{m+1} = B'_m,$$

$$[A_m, B_n] \subset [A'_m, B'_m] \subset I_1.$$

If $m > n$ then, by (37), (35), (38), (36),

$$B_n = B''_n; \quad A_m = A'_m \subset A^*_m \subset A^*_{n+1} = A''_n,$$

$$[A_m, B_m] \subset [A''_n, B''_n] \subset I_1.$$

This proves (40) and completes the proof of Theorem 42.

## 23. COUNTER EXAMPLES FOR THE CASES $\alpha = \beta$ AND $\alpha + 1 = \beta$

(∗) THEOREM 43.

(i) *If* $a' > \aleph_0$, *then* $\begin{pmatrix} a^+ \\ a^+ \end{pmatrix} \nrightarrow \begin{pmatrix} a, & a \vee a^+ \\ a^+, & \aleph_0 \vee 1 \end{pmatrix}$.

(ii) *If* $a' = \aleph_0$, *then* $\begin{pmatrix} a^+ \\ a^+ \end{pmatrix} \nrightarrow \begin{pmatrix} a, & a \vee a^+ \\ a^+, & \aleph_2 \vee 1 \end{pmatrix}$.

($*$) COROLLARY 18. *If* $a' > \aleph_0$, *then*

$$\binom{a^+}{a^+} \nrightarrow \begin{pmatrix} a & a \\ a^+ & \aleph_0 \end{pmatrix} \quad \text{and} \quad \binom{a}{a^+} \nrightarrow \begin{pmatrix} a & a \\ a^+ & \aleph_0 \end{pmatrix}.$$

($*$) COROLLARY 19. *If* $a' = \aleph_0$, *then*

$$\binom{a^+}{a^+} \nrightarrow \begin{pmatrix} a & a \\ a^+ & \aleph_2 \end{pmatrix} \quad \text{and} \quad \binom{a}{a^+} \nrightarrow \begin{pmatrix} a & a \\ a^+ & \aleph_2 \end{pmatrix}.$$

PROOF OF THEOREM 43. Put $m = \omega(a)$ and $n = \omega(a^+)$. Let

$$ST = \varnothing; \quad |S| = |T| = a^+; \quad T = \{y_0,, \hat{y}_n\}_{\neq}.$$

By ($*$) we can write $[S]^a = \{X_0,, \hat{X}_n\}_{\neq}$. Put $\mathbf{F}_\nu = \{X_0,, \hat{X}_\nu\}$ for $\nu < n$. We shall define a partition $[S, T] = I_0 +' I_1$ by defining the sets $Q_1(y_\nu)$ for $\nu < n$. First we shall define elements $x(\nu, \sigma)$ of $S$ for $\nu < n$ and $\sigma < m$. Let $\nu < n$ and suppose that $x(\mu, \sigma)$ has been defined for $\mu < \nu$ and $\sigma < m$. We have to define $x(\nu, \sigma)$ for $\sigma < m$.

If $\nu < m$ then we choose any set

$$(1) \qquad \{x(\nu, 0),, \hat{x}(\nu, m)\}_{\neq} \subset S - \Sigma(\mu < \nu)\{x(\mu, 0),, \hat{x}(\mu, m)\}.$$

Now let $\nu \geqq m$. Then we can write $[0, \nu) = \{\mu(\nu, 0),, \hat{\mu}(\nu, m)\}_{\neq}$. Put $X_{\mu(\nu,\varrho)} = X(\nu, \varrho)$ and $y_{\mu(\nu,\varrho)} = y(\nu, \varrho)$ for $\varrho < m$. Then $\mathbf{F}_\nu = \{X(\nu, \sigma) : \sigma < m\}_{\neq}$. We now define $x(\nu, 0),,$ $\hat{x}(\nu, m)$ inductively. Let $\sigma < m$ and, let $x(\nu, 0),, \hat{x}(\nu, \sigma)$ be defined already. Put

$$(2) \qquad T(\nu, \sigma) = \{x(\nu, \tau) : \tau < \sigma\} + \Sigma(\varrho < \sigma)\{x(\mu(\nu, \varrho), \tau) : \tau \leqq \sigma\}.$$

Then

$$(3) \qquad |T(\nu, \sigma)| \leqq |\sigma| + |\sigma + 1| \cdot |\sigma| < a.$$

Hence, since $|X(\nu, \sigma)| = a$, we can choose

$$(4) \qquad x(\nu, \sigma) \in X(\nu, \sigma) - T(\nu, \sigma).$$

This completes the definition of the $x(\nu, \sigma)$. We now put

$$Q(y_\nu) = \{x(\nu, \sigma) : \sigma < m\} \quad \text{for} \quad \nu < n.$$

By (1), (2) and (4) we have

$$(5) \qquad x(\nu, \tau) \neq x(\nu, \sigma) \quad \text{for} \quad \nu < n \quad \text{and} \quad \tau < \sigma < m.$$

Also,

$$(6) \qquad Q_1(y_\nu) \subset S; \quad |Q_1(y_\nu)| = a \quad \text{for} \quad \nu < n.$$

We now prove that

$$(7) \qquad \text{if } A_0 \in [S]^a \text{ and } B_0 \in [T]^{a^+}, \text{ then } [A_0, B_0] \not\subset I_0.$$

There is $\mu_0 < n$ with $A_0 = X_{\mu_0}$. Then, since $|B_0| = a^+$, there is $\nu_0$ such that

$$m, \mu_0 < \nu_0 < n, \quad \text{and} \quad y_{\nu_0} \in B_0.$$

Then $X_{\mu_0} \in \mathbf{F}_{\nu_0}$ and hence $X_{\mu_0} = X(\nu_0, \sigma_0)$ for some $\sigma_0 < m$. Then, by (4), $x(\nu_0, \sigma_0) \in$ $\in X(\nu_0, \sigma_0)$. Also, $x(\nu_0, \sigma_0) \in Q_1(y_{\nu_0})$. Hence $\{x(\nu_0, \sigma_0), y_{\nu_0}\} \in [A_0, B_0] I_1$, and (7) follows. Next, we prove that

$$(8)\qquad \begin{cases} \text{if } \mu<\nu<n, \text{ then there is } \varrho(\mu,\nu)<m \text{ such that} \\ \text{whenever } \sigma>\varrho(\mu,\nu) \text{ and } x(\mu,\tau)=x(\nu,\sigma), \text{ then } \tau>\sigma. \end{cases}$$

In fact, if $\nu<m$ then we may put $\varrho(\mu,\nu)=0$. Now let $\nu\geqq m$. Then $\mu=\mu(\nu,\varrho_0)$ for some $\varrho_0<m$. We put $\varrho(\mu,\nu)=\varrho_0$. Then, if $\tau\leqq\sigma$ we have, by (2), $x(\mu,\tau)= = x(\mu(\nu,\varrho_0),\tau)\in T(\nu,\sigma)$ and, by (4), $x(\mu,\tau)\neq x(\nu,\sigma)$. This proves (8). In view of (6) and (7) the parts (i) and (ii) of Theorem 43 follow from (9) and (10) respectively, where:

(9) If $a'>\aleph_0$ and $B\in[T]^{\aleph_0}$, then $|\Pi(y\in B)\,Q_1(y)|<a$.

(10) If $a'=\aleph_0$ and $B'\in[T]^{\aleph_2}$, then $|\Pi(y\in B')\,Q_1(y)|<a$.

PROOF OF (9). $B=\{y_{\nu_0},, \hat{y}_{\nu_l}\}$, where $\nu_0<<\hat{\nu}_l<n$ and $l\geqq\omega$. Since $a'>\aleph_0$, there is $\varrho_0<m$ such that

$$(11)\qquad \varrho(\nu_p,\nu_q)<\varrho_0 \quad \text{for} \quad p<q<\omega.$$

Let $x\in\Pi(p<\omega)\,Q_1(y_{\nu_p})$. Then there are $\sigma_p<m$ with $x=x(\nu_p,\sigma_p)$, for $p<\omega$. If $\sigma_0,,\hat{\sigma}_\omega>\varrho_0$ then, by (8), $\sigma_0>>\hat{\sigma}_\omega$ which is impossible. Hence there is $p_0<\omega$ with $\sigma_{p_0}\leqq\varrho_0$. Then $x\in\{x(\nu_{p_0},\sigma):\sigma\leqq\varrho_0\}$ and hence

$$|\Pi(p<\omega)\,Q_1(y_{\nu_p})| \leqq |\Sigma(p<\omega)\,\{x(\nu_p,\sigma):\sigma\leqq\varrho_0\}| \leqq |\varrho_0+1|\,\aleph_0<a.$$

This proves (9).

PROOF OF (10). Let $a_0<<\hat{a}_\omega<a=\sup(p<\omega)a_p$. Put

$$f(\varrho)=\min(\omega(a_\nu)>\varrho)\nu \quad \text{for} \quad \varrho<m.$$

Then $f(\varrho)<\omega$ and $\varrho<\omega(a_{f(\varrho)})$ for $\varrho<m$. We have

$$[B']^2 = \Sigma(\lambda<\omega)\,I^*_\lambda \qquad (\text{partition } \Delta^*),$$

where, for $\lambda<\omega$,

$$I^*_\lambda = \{\{y_\mu,y_\nu\}:\mu<\nu<n\wedge y_\mu,y_\nu\in B'\wedge f(\varrho(\mu,\nu)) = \lambda\}.$$

By Theorem I, $\aleph_2\to(\aleph_0)^2_{\aleph_0}$. Hence there are a set $B\in[B']^{\aleph_0}$ and a number $\lambda<\omega$ such that $[B]^2\subset I^*_\lambda$. We have $B=\{y_{\nu_0},,\hat{y}_{\nu_l}\}$, where $l\geqq\omega$ and $\nu_0<<\hat{\nu}_l<n$. It suffices to prove that $|\Pi(p<\omega)\,Q_1(y_{\nu_p})|<a$. By definition of $I^*_\lambda$ we have

$$f(\varrho(\nu_p,\nu_q))=\lambda \quad \text{for} \quad p<q<\omega.$$

Put $\omega(a_\lambda)=\varrho_0$. Then

$$\varrho(\nu_p,\nu_q)<\omega(a_{f(\varrho(\nu_p,\nu_q))})=\omega(a_\lambda)=\varrho_0$$

for $p<q<\omega$. From here on the proof of (10) is identical with the proof of (9) from (11) onwards. This proves Theorem 43.

REMARK. In the proof of (ii), more precisely, in the proof of (10), we used the relation $\aleph_2\to(\aleph_0)^2_{\aleph_0}$ although by Theorem I the stronger relation $\aleph_2\to(\aleph_1)^2_{\aleph_0}$ holds. But in spite of this our proof does not in fact establish (ii) with $\aleph_2$ replaced by $\aleph_1$ since $\aleph_1\nrightarrow(\aleph_0)^2_{\aleph_0}$, by Theorem I, and we cannot prove any special property of the partition $\Delta^*$ which would lead to the sharper result.

## (*) 24. SUMMARY OF THE RESULTS ABOUT THE CASES $\alpha=\beta$ AND $\alpha+1=\beta$. PROBLEMS

Throughout this section (*) is supposed. We use the notation introduced at the beginning of Part II. We shall always assume that

$$\aleph_0 \leqq a \leqq b; \quad 0 < a_0, a_1 \leqq a; \quad 0 < b_0, b_1 \leqq b.$$

*Case* I. $a^+ = b$. We have to discuss the relation

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \tag{1}$$

where $a \geqq \aleph_0$.

*Case* IA. $\min(a_0, a_1) < a$. Then (1) holds by Theorem 33.

*Case* IB. $a_0 = a_1 = a$.

*Case* IB (i). $\max(b_0, b_1) = a^+$. Then it suffices to discuss the relation

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \to \begin{pmatrix} a & a \\ a^+ & b_1 \end{pmatrix}. \tag{2}$$

Here we have the following results:

(2) is true if $a' > \aleph_0$ and $b_1 < \aleph_0$ (Theorem 34)
(2) is true if $a' = \aleph_0$ and $b_1 = \aleph_0$ (Theorem 40)
(2) is false if $a' > \aleph_0$ and $b_1 = \aleph_0$ (Corollary 18)
(2) is false if $a \geqq \aleph_0$ and $b_1 = a^+$ (Corollary 16)
(2) is false if $a' = \aleph_0$ and $b_1 = \aleph_2$ (Corollary 19).

We do not know if (2) holds when $a > a' = \aleph_0$ and $b_1 = \aleph_1$. Here the simplest unsolved problem is

PROBLEM 10.

$$? \begin{pmatrix} \aleph_\omega \\ \aleph_{\omega+1} \end{pmatrix} \to \begin{pmatrix} \aleph_\omega & \aleph_\omega \\ \aleph_{\omega+1} & \aleph_1 \end{pmatrix}.$$

*Case* IB (ii). $b_0, b_1 < a^+$. Here one might conjecture the following best possible result:

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \to \begin{pmatrix} a & a \\ a & a \end{pmatrix} \quad \text{for} \quad a \geqq \aleph_0. \tag{3}$$

We distinguish three cases:

*Case* IB (ii) a. $a' = \aleph_0$. Then (3) holds by Theorem 42.

*Case* IB (ii) b. $a > a' > \aleph_0$. Then Theorem 41 shows that the following result, which is weaker than (3), holds:

$$\begin{pmatrix} a \\ a^+ \end{pmatrix} \to \begin{pmatrix} a & a \\ a & b_1 \end{pmatrix} \quad \text{for} \quad b_1 < a.$$

We do not know whether (3) holds for any $a$ with $a > a' > \aleph_0$. Here the simplest open problem is:

PROBLEM 11.

$$? \begin{pmatrix}\aleph_{\omega_1}\\ \aleph_{\omega_1+1}\end{pmatrix} \to \begin{pmatrix}\aleph_{\omega_1} & \aleph_{\omega_1}\\ \aleph_{\omega_1} & \aleph_{\omega_1}\end{pmatrix}.$$

*Case* IB (ii) c. $a = a' > \aleph_0$. In this case we only have the following trivial remark. If, for some $b \leqq a$ and some $b_0, b_1 \leqq b$, we have $\begin{pmatrix}a\\ b\end{pmatrix} \to \begin{pmatrix}a & a\\ b_0 & b_1\end{pmatrix}$, then $\begin{pmatrix}a\\ a^+\end{pmatrix} \to \begin{pmatrix}a & a\\ b_0 & b_1\end{pmatrix}$. Here the simplest unsolved problems are:

PROBLEM 12.

$$1) \qquad ? \begin{pmatrix}\aleph_1\\ \aleph_2\end{pmatrix} \to \begin{pmatrix}\aleph_1 & \aleph_1\\ \aleph_0 & \aleph_0\end{pmatrix}; \quad ? \begin{pmatrix}\aleph_1\\ \aleph_2\end{pmatrix} \to \begin{pmatrix}\aleph_1 & \aleph_1\\ \aleph_1 & \aleph_1\end{pmatrix}.$$

$$2) \qquad ? \begin{pmatrix}\aleph_2\\ \aleph_3\end{pmatrix} \to \begin{pmatrix}\aleph_2 & \aleph_2\\ \aleph_1 & \aleph_1\end{pmatrix}; \quad ? \begin{pmatrix}\aleph_2\\ \aleph_3\end{pmatrix} \to \begin{pmatrix}\aleph_2 & \aleph_2\\ \aleph_2 & \aleph_0\end{pmatrix},$$

$$? \begin{pmatrix}\aleph_2\\ \aleph_3\end{pmatrix} \to \begin{pmatrix}\aleph_2 & \aleph_2\\ \aleph_2 & \aleph_2\end{pmatrix}.$$

REMARK. We know that $\begin{pmatrix}\aleph_2\\ \aleph_3\end{pmatrix} \to \begin{pmatrix}\aleph_2 & \aleph_2\\ \aleph_1 & \aleph_0\end{pmatrix}$ since in fact, by Theorem 33,

$$\begin{pmatrix}\aleph_2\\ \aleph_1\end{pmatrix} \to \begin{pmatrix}\aleph_2 & \aleph_2\\ \aleph_1 & \aleph_0\end{pmatrix}.$$

*Case* II. $a = b$. Here the problems are more ramified.

*Case* IIA. $a = a^-$. First of all, let us consider relations without alternatives. We assert that

$$(4) \qquad \begin{pmatrix}a\\ a\end{pmatrix} \to \begin{pmatrix}a_0 & a_1\\ b_0 & b_1\end{pmatrix}$$

holds if and only if

$$(5) \qquad \min(a_0, b_1) < a \quad \text{and} \quad \min(a_1, b_0) < a.$$

In fact, if (4) holds then, by 21. 2, the condition (5) follows. Now, vice versa, assume that (5) is true. To deduce (4) we have to establish the following propositions:

$$(6) \qquad \text{If } b_0, b_1 < a, \text{ then } \begin{pmatrix}a\\ a\end{pmatrix} \to \begin{pmatrix}a & a\\ b_0 & b_1\end{pmatrix}.$$

$$(7) \qquad \text{If } a_1, b_1 < a, \text{ then } \begin{pmatrix}a\\ a\end{pmatrix} \to \begin{pmatrix}a & a_1\\ a & b_1\end{pmatrix}.$$

In fact, (7) follows from Theorem 38. We shall deduce (6) from Theorem 44 which will be proved in section 26.

From Theorem 38 it is clear that in the case under discussion the only genuine problem for relations with alternatives is that of deciding the truth of the statement:

$$\begin{pmatrix}a\\ b\end{pmatrix} \to \begin{pmatrix}a, a \vee a_1\\ a, b_1 \vee a\end{pmatrix} \quad \text{for} \quad a_1, b_1 < a,$$

and again Theorem 38 shows that this statement is true. Thus case II A is settled.

*Case* II B. $a = c^+$. Again we begin with relations without alternatives. We have to discuss the relation

$$\binom{c^+}{c^+} \to \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix}. \tag{8}$$

If either $a_0 = b_1 = c^+$ or $a_1 = b_0 = c^+$, then (8) is false by 21. 2. Now suppose that $\min(a_0, b_1) \leqq c$ and $\min(a_1, b_0) \leqq c$. For reasons of symmetry it suffices to consider the following cases:

*Case* II Ba. $a_0, a_1, b_0, b_1 \leqq c$. Then (8) holds by Corollary 17.

*Case* II Bb. $a_0 = a_1 = c^+$; $b_0, b_1 \leqq c$. Then (8) holds if and only if $\min(b_0, b_1) < c$. For if $\min(b_0, b_1) < c$ then (8) follows from Theorem 33, and if $b_0 = b_1 = c$ then (8) is false by Corollary 16.

*Case* II Bc. $a_0 = b_0 = c^+$; $a_1, b_1 \leqq c$.

*Case* II Bc (i). $a_1, b_1 < c$. Then (8) holds by Theorem 38.

*Case* II Bc (ii). $a_1 \leqq c$; $b_1 = c$. Here we have the following results:

(9) If $a_1 = 1$ then (8) holds by Theorem 35;

(10) if $a_1 > 1$ and $c = c'$, then (8) is false by Theorem 31;

(11) $\begin{cases} \text{if either } a_1 \geqq \aleph_2 \text{ and } c' = \aleph_0, \\ \text{or } a_1 \geqq \aleph_0 \text{ and } c' > \aleph_0, \text{ then (8) is false by Theorem 43.} \end{cases}$

Thus here the simplest unsolved problems are:

PROBLEM 13.

$$? \ \binom{\aleph_{\omega+1}}{\aleph_{\omega+1}} \to \begin{pmatrix} \aleph_{\omega+1} & a_1 \\ \aleph_{\omega+1} & \aleph_\omega \end{pmatrix} \quad \text{for} \quad 2 \leqq a_1 \leqq \aleph_1.$$

$$? \ \binom{\aleph_{\omega_1+1}}{\aleph_{\omega_1+1}} \to \begin{pmatrix} \aleph_{\omega_1+1} & a_1 \\ \aleph_{\omega_1+1} & \aleph_{\omega_1} \end{pmatrix} \quad \text{for} \quad 2 \leqq a_1 < \aleph_0.$$

*Case* II Bd. $a_0 = c^+$; $a_1, b_0, b_1 \leqq c$. Here we have: If $\min(b_0, b_1) < c$ then (8) holds by Theorem 33. There remains to consider the relation

$$\binom{c^+}{c^+} \to \begin{pmatrix} c^+ & a_1 \\ c & c \end{pmatrix}. \tag{12}$$

If $a_1 < \aleph_0$ then (12) is true by Theorem 34. If either $c' > \aleph_0$ and $a_1 \geqq \aleph_0$, or $c' = \aleph_0$ and $a_1 \geqq \aleph_2$, then (12) is false by Theorem 43. If $c = \aleph_0$ and $a_1 = \aleph_1$, then (12) is false by Theorem 32. If $c' = \aleph_0$ and $a_1 = \aleph_0$, then (12) is true by Theorem 40. Here the simplest unsolved problem is

PROBLEM 14.

$$? \ \binom{\aleph_{\omega+1}}{\aleph_{\omega+1}} \to \begin{pmatrix} \aleph_{\omega+1} & \aleph_1 \\ \aleph_\omega & \aleph_\omega \end{pmatrix}.$$

Thus many cases under II B are settled if we restrict ourselves to relations without alternatives. Let us now consider in case II B relations with alternatives. We shall show, without going into details, that the theorems proved so far, together with

Problems 13 and 14, essentially cover all cases. We want to consider relations of the form

$$(13) \qquad \binom{c^+}{c^+} \to \binom{a_0 \vee a_1,\ a_2 \vee a_3}{b_0 \vee b_1,\ b_2 \vee b_3},$$

where $c \geqq \aleph_0$ and $1 \leqq a_\nu, b_\nu \leqq c^+$ for $\nu < 4$. We make a fresh start with our subdivision into cases.

*Case* A*. There is $\nu < 4$ with $a_\nu, b_\nu \leqq c$. Then we may assume $a_0, b_0 \leqq c$. By Theorem 39 we have $\binom{c^+}{c^+} \to \binom{a_0,\ c^+ \vee c}{b_0,\ c \vee c^+}$. Hence (13) is true if there are $r, s \in \{2, 3\}$ with $a_r, b_s \leqq c$. Hence we may assume that $a_2 = a_3 = c,^+$ and it suffices to discuss relations of the form

$$(14) \qquad \binom{c^+}{c^+} \to \binom{a_0 \vee a_1,\ c^+}{b_0 \vee b_1,\ d},$$

where $a_0, b_0 \leqq c$ and $1 \leqq d \leqq c^+$. We shall in fact not make any use of these last inequalities. If $d_0, d_1 < c$ then, by Theorem 38, $\binom{c^+}{c^+} \to \binom{c^+ \vee d_1,\ c^+}{d_0 \vee c^+,\ c^+}$. Therefore (14) holds whenever there are $r, s \in \{0, 1\}$ with $a_r, b_s < c$. Hence we may assume that either $a_0, a_1 \geqq c$ or $b_0, b_1 \geqq c$.

*Case* A* 1. $d = c^+$. Then we may suppose that $a_0, a_1 \geqq c$. If $a_0 = a_1 = c$ and $b_0 = b_1 = 1$, then (14) holds by Theorem 35. On the other hand we have the following negative results:

If $c = c'$ then $\binom{c^+}{c^+} \nrightarrow \binom{c \vee c^+,\ c^+}{2 \vee 1,\ \ c^+}$ (Theorem 31).

If $c > c' = \aleph_0$ then $\binom{c^+}{c^+} \nrightarrow \binom{c \vee c^+,\ \ c^+}{\aleph_2 \vee 1,\ \ c^+}$ (Theorem 43).

If $c > c' > \aleph_0$ then $\binom{c^+}{c^+} \nrightarrow \binom{c \vee c^+,\ c^+}{\aleph_0 \vee 1,\ c^+}$ (Theorem 43).

Thus the following refinements of Problem 11 remain open:

$$? \ \binom{\aleph_{\omega+1}}{\aleph_{\omega+1}} \to \binom{\aleph_\omega \vee \aleph_{\omega+1},\ \aleph_{\omega+1}}{b_0 \ \vee 1,\qquad \aleph_{\omega+1}} \quad \text{for} \quad 2 \leqq b_0 \leqq \aleph_1;$$

$$? \ \binom{\aleph_{\omega_1+1}}{\aleph_{\omega_1+1}} \to \binom{\aleph_{\omega_1} \vee \aleph_{\omega_1+1},\ \aleph_{\omega_1+1}}{b_0 \ \vee 1,\qquad \aleph_{\omega_1+1}} \quad \text{for} \quad 2 \leqq b_0 < \aleph_0.$$

*Case* A* 2. $d = c$. If $b_0, b_1 < c$ then (14) holds by Theorem 33, and if $b_0 = {} = b_1 = c^+$ then (14) is false by 21. 2. These remarks lead us to consider relations of the form

$$(15) \qquad \binom{c^+}{c^+} \to \binom{a_0 \vee a_1,\ c^+}{c \ \vee \ c^+,\ c}.$$

We note that the case

$$\binom{c^+}{c^+} \to \binom{a_0 \vee a_1,\ c^+}{c \vee b_1,\quad c}, \quad \text{where } b_1 \leqq c,$$

will not be covered but we shall omit its discussion. If $a_0 < \aleph_0$ then (15) holds by Theorem 34, and if $c' > \aleph_0$ and $a_0 = \aleph_0$, then (15) is false by Theorem 43.

*Case* A* 3. $d < c$. If $\min(b_0, b_1) \leqq c$ then (14) holds by Theorem 33, and in the remaining case $b_0 = b_1 = c^+$ the relation (14) is false by 21. 2.

*Case* B*. $\max(a_\nu, b_\nu) = c^+$ for $\nu < 4$. If there is $\nu \in \{0, 2\}$ such that either $a_\nu = a_{\nu+1} = c^+$ or $b_\nu = b_{\nu+1} = c^+$, then (13) reduces to a relation of the form (14) which has already been discussed. Hence it only remains to discuss relations of the form

$$\begin{pmatrix} c^+ \\ c^+ \end{pmatrix} \to \begin{pmatrix} c^+ \vee a_1, c^+ \vee a_3 \\ b_0 \vee c^+, b_2 \vee c^+ \end{pmatrix}, \tag{16}$$

where $a_1, a_3, b_0, b_2 \leqq c$. If $\min(a_1, a_3, b_0, b_2) < c$ then (16) is true by Theorem 33, and in the remaining case $a_1 = a_3 = b_0 = b_2 = c$ (16) is false by Theorem 32.

## 25. LEMMAS FOR THE CASE $\beta > \alpha + 1$

LEMMA 10. *Let* $\aleph_0 < 2^a < b$; $c < b$; $b > b'$. *Then*

(i) *If* $\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b & b \end{pmatrix}$, *then* $\begin{pmatrix} a \\ b' \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b' & b' \end{pmatrix}$.

(ii) *If* $\begin{pmatrix} a \\ b' \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b' & b' \end{pmatrix}$, *then* $\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b & b \end{pmatrix}$.

(iii) *If* $\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b & 1 \end{pmatrix}$, *then* $\begin{pmatrix} a \\ b' \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b' & 1 \end{pmatrix}$.

(iv) *If* $\begin{pmatrix} a \\ b' \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b' & 1 \end{pmatrix}$, *then* $\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b & c \end{pmatrix}$.

PROOF. Let $m = \omega(b')$; $|A| = a$; $|B| = b$; $AB = AS = \emptyset$, where $S = [0, m)$. There are numbers $b_0,., \hat{b}_m$ and sets $B_0,., \hat{B}_m$ such that $2^a, c < b_0 < \cdots < \hat{b}_m < b$;

$$B = \Sigma'(\nu < m) B_\nu; \quad |B_\nu| = b_\nu = b'_\nu \quad \text{for} \quad \nu < m.$$

PROOF OF (i). Let $[A, S] = I_0^* +' I_1^*$. Then $[A, B] = I_0 +' I_1$, where

$$I_0 = \Sigma(\{x, \nu\} \in I_0^*)[\{x\}, B_\nu].$$

By hypothesis there are a number $\varkappa < 2$ and sets $X \in [A]^{a_\varkappa}$; $Y \in [B]^b$ with $[X, Y] \subset I_\varkappa$. Put $Y' = \{\nu : YB_\nu \neq \emptyset\}$. Then $Y' \in [S]^{b'}$; $[X, Y'] \subset I_\varkappa^*$, and (i) follows.

PROOF OF (ii). Let $[A, B] = I_0 +' I_1$ and $\nu < m$. Then

$$|\{Q_0(y) : y \in B_\nu\}| \leqq 2^a < b_\nu = |B_\nu|,$$

and there are sets $A'_\nu \subset A$ and $B'_\nu \in [B_\nu]^{b_\nu}$ such that $Q_0(y) = A'_\nu$ for $y \in B'_\nu$. Then $[A, S] = I_0^* +' I_1^*$, where $I_0^* = \{\{x, \nu\} : x \in A'_\nu \wedge \nu < m\}$. By hypothesis there are a number $\varkappa < 2$ and sets $X \in [A]^{a_\varkappa}$; $Y' \in [S]^{b'}$ such that $[X, Y'] \subset I_\varkappa^*$. Put $Y = \Sigma(\nu \in Y') B'_\nu$. Then $Y \in [B]^b$; $[X, Y] \subset I_\varkappa$, and (ii) is proved.

PROOF OF (iii). Let $I_\varkappa^*$, $I_\varkappa$ be as in the proof of (i). Put $c_0 = b$; $c_1 = 1$. By hypothesis there are a number $\varkappa < 2$ and sets $X \in [A]^{a_\varkappa}$; $Y \in [B]^{c_\varkappa}$ such that $[X, Y] \subset I_\varkappa$. Put $Y' = \{v : YB_v \neq \emptyset\}$. Then $Y' \subset S$; $[X, Y'] \subset I^*_\varkappa$. If $\varkappa = 0$ then $|Y| = b$; $|Y'| = b'$, and if $\varkappa = 1$ then $|Y'| = 1$. This proves (iii).

Proof of (iv). Let $I_\varkappa$, $B'_v$, $A'_v$, $I^*_\varkappa$ be as in the proof of (ii). Put $c_0 = b'$; $c_1 = 1$. Then there are a number $\varkappa < 2$ and sets $X \in [A]^{a_\varkappa}$; $Y' \in [S]^{c_\varkappa}$ such that $[X, Y'] \subset I^*_\varkappa$. Put $Y = \Sigma(v \in Y')B'_v$. Then $Y \subset B$; $[X, Y] \subset I_\varkappa$. If $\varkappa = 0$ then $|Y'| = b'$; $|Y| = b$, and if $\varkappa = 1$ then $Y' = \{v_0\}$ for some $v_0 < m$, and we have $|Y| = |B'_{v_0}| = b_{v_0} > c$. This proves (iv) and completes the proof of Lemma 10.

Lemma 11. *Let* $\aleph_0 < 2^a < b'$. *Then* $\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a\ a \\ b\ b \end{pmatrix}$.

Remark. Lemma 11 is obviously not best possible. It will be used in the proof of Theorem 44 which gives a necessary and sufficient condition for $\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a\ a \\ b\ b \end{pmatrix}$.

Proof. Let $|A| = a$; $|B| = b$; $AB = \emptyset$; $[A, B] = I_0 +' I_1$. Then

$$b = \Sigma(A' \subset A)|B\{y : Q_0(y) = A'\}|.$$

Since $|\mathbf{P}(A)| = 2^a < b'$, there are sets $A' \subset A$ and $B' \in [B]^b$ such that $Q_0(y) = A'$ for $y \in B'$.

*Case* 1. $|A'| = a$. Then $[A', B'] \subset I_0$.

*Case* 2. $|A'| < a$. Then $|A - A'| = a$; $[A - A', B'] \subset I_1$. This proves Lemma 11.

## 26. POLARIZED PARTITION RELATIONS IN THE GENERAL CASE. DISCUSSION AND PROBLEMS

In the general case we do not investigate relations with alternatives. It is obvious that such relations only lead to interesting question provided $a' = b'$ but we omit this. We shall discuss the relation

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a_0\ a_1 \\ b_0\ b_1 \end{pmatrix}.$$

We begin by considering the special case when $a_0 = a_1 = a$ and $b_0 = b_1 = b$. We need some definitions.

**26. 1.** Definition. The *distance* $d(\alpha, \beta)$ of two ordinals $\alpha, \beta$ is defined by the equations

$$d(\alpha, \alpha + \delta) = d(\alpha + \delta, \alpha) = \delta \quad \text{for all } \alpha, \delta.$$

We put $d(\aleph_\alpha, \aleph_\beta) = d(\alpha, \beta)$.

**26. 2.** Definition. Cardinals $a, b$ are called *disjoint* if $a, b \geqq \aleph_0$ and

$$d(x, y) > 1 \quad \text{for } x \in \{a, a'\} \text{ and } y \in \{b, b'\}.$$

($*$) Theorem 44. *Let* $a, b \geqq \aleph_0$. *Then* $\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a\ a \\ b\ b \end{pmatrix}$ *if and only if* $a$ *and* $b$ *are disjoint*.

Proof. Put

$$P = \left\{(a, b) : a, b \geqq \aleph_0 \wedge \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a\ a \\ b\ b \end{pmatrix}\right\}.$$

Throughout the proof we suppose $a, b \geqq \aleph_0$. We shall frequently use, without reference, the following propositions:

1. If $a^+ < b$, then $(a, b) \in P$ if and only if $(a, b') \in P$ (Lemma 10).
2. If $a^+ < b'$ then $(a, b) \in P$ (Lemma 11).
3. If $d(a, b) \leqq 1$ then $(a, b) \notin P$ (21. 2 and Corollary 16).

In proving the theorem we may assume $a \leqq b$.

*Case* 1. $a, b$ disjoint. Then $a^+ < b$.

*Case* 1a. $a^+ < b'$. Then $a^+ < b' = b''$; $(a, b') \in P$; $(a, b) \in P$.

*Case* 1b. $a^+ \geqq b'$. Then $b'^+ < a$. Since $d(a', b') > 1$ we have $(a', b') \in P$. Then $(a, b') \in P$; $(a, b) \in P$.

*Case* 2. $a, b$ not disjoint.

*Case* 2a. $b \leqq a^+$. Then $d(a, b) \leqq 1$; $(a, b) \notin P$.

*Case* 2b. $b > a^+$. Then $d(a', b) \geqq d(a, b) > 1$, and since $a, b$ are not disjoint, at least one of the following four cases arises:

*Case* 2b1. $d(a, b') \leqq 1$. Then $(a, b') \notin P$; $(a, b) \notin P$.

*Case* 2b2. $a' = b'$. Then $(a', b') \notin P$; $(a', b) \notin P$. Let us assume that $(a, b) \in P$. Then $(a, b') \in P$. Then, since $(a, b') \in P$ but $(a', b') \notin P$, we have $a \leqq b'^+$. Then $a \leqq b'^+ = a'^+$; $a' \leqq a \leqq a'^+$; $a \in \{a', a'^+\}$. Hence $a$ is regular; $(a, a) = (a, b') \in P$ which is a contradiction. Therefore $(a, b) \notin P$.

*Case* 2b3. $a'^+ = b'$. Then $(a', b') \notin P$. Let us assume that $(a, b) \in P$. Then $(a, b') \in P$; $d(a, a'^+) = d(a, b') > 1$; $a > a'$; $b' = a'^+ < a$; $b'^+ < a$; $(a', b') \in P$ which is a contradiction. Hence $(a, b) \notin P$.

*Case* 2b4. $a' = b'^+$. Then $(a', b') \notin P$. Let us assume that $(a, b) \in P$. Then $(a, b') \in P$. Then, since $(a, b') \in P$ but $(a', b') \notin P$, we have $a \leqq b'^+ = a'$; $a = a'$; $(a', b') = (a, b') \in P$ which is a contradiction. Hence $(a, b) \notin P$, and Theorem 44 follows.

**26. 3.** We now turn to the general case

$$\begin{pmatrix} a \\ b \end{pmatrix} \rightarrow \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix}. \tag{1}$$

Throughout we suppose that $a, b \geqq \aleph_0$ and

$$1 \leqq a_\nu \leqq a; \ 1 \leqq b_\nu \leqq b \quad \text{for} \quad \nu < 2.$$

Also, $(*)$ will be assumed. The case $d(a, b) \leqq 1$ was discussed in section 24. We may therefore assume $a^+ < b$. By Theorem 44 the relation (1) holds whenever $a, b$ are disjoint. We may now assume that $a, b$ are not disjoint. Then at least one of the following six cases arises:

1. $a^+ = b'$; 2. $a = b'$; 3. $a = b'^+$;

4A. $a'^+ = b'$; 5A. $a' = b'$; 6A. $a' = b'^+$.

We now show that if $a \leqq b'^+$ and if one of the cases 4A, 5A, 6A holds, say the case $(3+k)A$, then already case $k$ holds $(k = 1, 2, 3)$. In fact, under these circumstances we have $a' \leqq a \leqq b'^{+} \leqq a'^{++}$, so that $a \in \{a', a'^{+}, a'^{++}\}$, and $a$ is regular. Then case $k$ applies. Thus we have the following six cases to deal with:

1. $a^{+} = b'$; 2. $a = b'$; 3. $a = b'^{+}$;
4. $a > b'^{+}$ and $a'^{+} = b'$; 5. $a > b'^{+}$ and $a' = b'$; 6. $a > b'^{+}$ and $a' = b'^{+}$

Clearly, the cases 1, 2, 3 are exclusive, and the cases 4, 5, 6 are exclusive. Also, in each of the cases 1, 2, 3, we have $a \leqq b'^{+}$ so that all six cases 1—6 are exclusive.

Our assumptions are therefore: $\aleph_0 < a^{+} < b$, and exactly one of the cases 1—6 applies. We note that by Theorem 44 in any of these cases,

$$\binom{a}{b} \nrightarrow \binom{a\ a}{b\ b}. \tag{2}$$

The discussion that follows will settle the truth of (1) in all cases except for the sub-case of case 3 which is given by the conditions

$$a = b'^{+}; \quad a^{+} < b;$$

$$a_0, a_1 < a; \quad b_0 = b_1 = b.$$

*Case* 1. $a^{+} = b'$ and $a^{+} < b$. Here we have the following two results which settle this case:

$$\binom{a}{b} \rightarrow \binom{a\ a_1}{b\ b} \quad \text{for} \quad a_1 < a. \tag{3}$$

PROOF. By Lemma 10 (ii) it suffices to prove

$$\binom{a}{b'} \rightarrow \binom{a\ a_1}{b'\ b'},$$

and this last relation follows from Theorem 33.

$$\binom{a}{b} \rightarrow \binom{a\ a}{b\ b_1} \quad \text{for} \quad b_1 < b. \tag{4}$$

PROOF. By Lemma 10 (iv) it suffices to prove $\binom{a}{b'} \rightarrow \binom{a\ a}{b'\ 1}$, and this last relation follows from Theorem 34.

We see from (2), (3), (4) that in case 1 the relation (1) holds if and only if either $\min(a_0, a_1) < a$ or $\min(b_0, b_1) < b$.

*Case* 2. $a = b'$ and $a^{+} < b$. We shall settle this case. By 21. 2 we have $\binom{a}{b'} \nrightarrow \binom{a\ 1}{1\ b'}$ and hence, by Lemma 10 (iii),

$$\binom{a}{b} \nrightarrow \binom{a\ 1}{1\ b}. \tag{5}$$

*Case* 2a. $b_0, b_1 < b$. Then

$$\binom{a}{b} \rightarrow \binom{a\ \ a}{b_0\ b_1}. \tag{6}$$

PROOF. Since $b' = a < a^{+} < b$ we may suppose that $a^{+} < b_0 = b_1 = b_1' < b$, and then (6) follows from Lemma 11.

*Case* 2b. $b_1 < b_0 = b$. Then

$$\binom{a}{b} \to \begin{pmatrix} a & a_1 \\ b_0 & b_1 \end{pmatrix} \tag{7}$$

holds if and only if $a_1 < a$.

Proof. If (7) holds then, by (5), $a_1 < a$. Vice versa, if $a_1 < a$ then, by Theorem 35, $\binom{a}{b'} \to \begin{pmatrix} a & a_1 \\ b' & 1 \end{pmatrix}$ which implies (7), by Lemma 10 (iv). This establishes the assertion relating to (7).

*Case* 2c. $b_0 < b_1 = b$. Then, by symmetry,

$$\binom{a}{b} \to \begin{pmatrix} a_0 & a \\ b_0 & b_1 \end{pmatrix}$$

holds if and only if $a_0 < a$.

*Case* 2d. $b_0 = b_1 = b$. Then (1) holds if and only if

$$a_0, a_1 < a \quad \text{and} \quad \min(a_0, a_1) < a^-. \tag{8}$$

Proof. We have $b' = a < a^+ < b$. First of all, suppose that (1) holds. If we then assume that $a_0 = a$ then

$$\binom{b'}{b} \to \begin{pmatrix} b' & 1 \\ b & b \end{pmatrix}$$

and so, by Lemma 10 (i), $\binom{b'}{b'} \to \begin{pmatrix} b' & 1 \\ b' & b' \end{pmatrix}$ which contradicts (5). Hence $a_0, a_1 < a$. Next, if we assume that $\min(a_0, a_1) \geqq a^-$ then $b'^- = a^- \leqq a_0 < a = b'$; $b' = c^+$ for some $c$, and

$$\binom{c^+}{b} \to \begin{pmatrix} c & c \\ b & b \end{pmatrix}.$$

Therefore, by Lemma 10 (i), $\binom{c^+}{c^+} \to \begin{pmatrix} c & c \\ c^+ & c^+ \end{pmatrix}$ which contradicts Theorem 32. Hence $\min(a_0, a_1) < a^-$, and (8) follows.

Now suppose, vice versa, that (8) is satisfied. By Lemma 10 (ii), the relation (1) is a consequence of

$$\binom{a}{b'} \to \begin{pmatrix} a_0 & a_1 \\ b' & b' \end{pmatrix}.$$

Hence it suffices to prove the following two propositions:

(9) If $a = a^-$ and $a_0, a_1 < a$, then $\binom{a}{b'} \to \begin{pmatrix} a_0 & a_1 \\ b' & b' \end{pmatrix}$.

(10) If $a = c^+$ and $a_1 < c$, then $\binom{a}{b'} \to \begin{pmatrix} c & a_1 \\ b' & b' \end{pmatrix}$.

In fact, (9) follows from Lemma 11, and (10) follows from Theorem 33.

By analysing the results in cases 2a-2d it will be seen that we have proved:

In case 2 the relation (1) holds if and only if at least one of the following three conditions $(\alpha)$, $(\beta)$, $(\gamma)$ is satisfied:

($\alpha$) $b_0, b_1 < b$;

($\beta$) there is $\nu < 2$ with $a_\nu < a$ and $b_\nu < b$;

($\gamma$) $\min(a_0, a_1) < a^-$ and $a_0, a_1 < a$.

*Case* 3. $a = b'^+$ and $a^+ < b$. Here there remain some open questions. First we prove:

(11)
$$\binom{a}{b} \to \begin{pmatrix} a & a \\ b & b_1 \end{pmatrix} \quad \text{for} \quad b_1 < b.$$

PROOF. By Theorem 33, $\binom{a}{b'} \to \begin{pmatrix} a & a \\ b' & 1 \end{pmatrix}$, and (11) follows from Lemma 10 (iv).

By comparing (2) and (11) we see that to settle case 3 we have to investigate the relation

(12)
$$\binom{a}{b} \to \begin{pmatrix} a_0 & a_1 \\ b & b \end{pmatrix}$$

when $\min(a_0, a_1) < a$. By Lemma 10 (i) and (ii) the relation (12) is equivalent to $\binom{a}{b'} \to \begin{pmatrix} a_0 & a_1 \\ b' & b' \end{pmatrix}$, and this last relation was discussed in section 24.

Corresponding to the cases IB (i) and IB (ii) in section 24 we distinguish now the cases $\max(a_0, a_1) = a$ and $a_0, a_1 < a$.

*Case* 3a. $\max(a_0, a_1) = a$. We have to study the relation

(13)
$$\binom{a}{b} \to \begin{pmatrix} a & a_1 \\ b & b \end{pmatrix}.$$

This relation can be discussed completely. As pointed out above, (13) is equivalent to

(14)
$$\binom{a}{b'} \to \begin{pmatrix} a & a_1 \\ b' & b' \end{pmatrix}.$$

If we put $b' = c$ then (14) becomes

$$\binom{c^+}{c} \to \begin{pmatrix} c^+ & a_1 \\ c & c \end{pmatrix}.$$

Here $c = c' = b'$. By section 24, case IB (i) we have:

(15)
$$\begin{cases} \text{If } b' > \aleph_0 \text{ then (13) is true for } a_1 < \aleph_0 \text{ and false for } a_1 \geqq \aleph_0. \\ \text{If } b' = \aleph_0 \text{ then (13) is true for } a_1 \leqq \aleph_0 \text{ and false for } a_1 > \aleph_0. \end{cases}$$

*Case* 3b. $a_0, a_1 < a$. This case reduces to section 24, case IB (ii) c where we had only trivial results. Although the problems in our present case are equivalent to those stated in the earlier section we state here explicitly those equivalent to problem 12. 1:

PROBLEM 15.

$$? \quad \binom{\aleph_{\omega_1}}{\aleph_2} \to \begin{pmatrix} \aleph_{\omega_1} & \aleph_{\omega_1} \\ \aleph_0 & \aleph_0 \end{pmatrix};$$

$$? \quad \binom{\aleph_{\omega_1}}{\aleph_2} \to \begin{pmatrix} \aleph_{\omega_1} & \aleph_{\omega_1} \\ \aleph_1 & \aleph_1 \end{pmatrix}.$$

*Case* 4. $a'^+ = b'$; $a > b'^+$; $a^+ < b$. Here we shall obtain a complete discussion. We prove:

(16) $$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a\, a_1 \\ b\, b \end{pmatrix} \quad \text{for} \quad a_1 < a.$$

PROOF. By (11) we have, for $a_1 < a$,

$$\begin{pmatrix} a \\ b' \end{pmatrix} \to \begin{pmatrix} a & a_1 \\ b' & b' \end{pmatrix}.$$

Hence, by Lemma 10 (ii), (16) follows.

(17) $$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a\, a \\ a\, b_1 \end{pmatrix} \quad \text{for} \quad b_1 < b.$$

PROOF. By (15),

$$\begin{pmatrix} a \\ b' \end{pmatrix} \to \begin{pmatrix} a & a \\ b' & 1 \end{pmatrix},$$

and (17) follows by Lemma 10 (iv).

It follows from (2), (16), (17) that in case 4 the relation (1) holds if and only if either $\min(a_0, a_1) < a$ or $\min(b_0, b_1) < b$.

*Case* 5. $a' = b'$; $a > b'^+$; $a^+ < b$. Here we obtain a complete discussion.

(18) $$\begin{pmatrix} a \\ a \end{pmatrix} \nrightarrow \begin{pmatrix} 1\, a \\ b\, 1 \end{pmatrix}.$$

PROOF. By (5), $\begin{pmatrix} a \\ b' \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & a \\ b' & 1 \end{pmatrix}$, and (18) follows by Lemma 10 (iii).

(19) $$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a & a \\ b_0 & b_1 \end{pmatrix} \quad \text{if} \quad b_0, b_1 < b.$$

PROOF. We may assume $a^+ < b_0 = b_1 = b_1' < b$. Then (19) follows from Lemma 11.

(20) $$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a\, a_1 \\ a\, b_1 \end{pmatrix} \quad \text{if } a_1 < a \text{ and } b_1 < b.$$

PROOF. By (7), $\begin{pmatrix} a \\ b' \end{pmatrix} \to \begin{pmatrix} a & a_1 \\ b' & 1 \end{pmatrix}$, and (20) follows by Lemma 10 (iv).

(21) $$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b & b \end{pmatrix} \quad \text{if} \quad a_0, a_1 < a.$$

PROOF. We may assume $b'^+ < a_0 = a_1 = a_1' < a$. Then, by Theorem 44, $\begin{pmatrix} a \\ b' \end{pmatrix} \to \begin{pmatrix} a_0 & a_1 \\ b' & b' \end{pmatrix}$, and (21) follows by Lemma 10 (ii).

It now follows from (18)–(21) that in case 5 the relation (1) holds if and only if either $a_0, a_1 < a$, or $b_0, b_1 < b$, or there is $\nu < 2$ with $a_\nu < a$ and $b_\nu < b$.

*Case* 6. $a' = b'^+$; $a > b'^+$; $a^+ < b$. Here we have a complete discussion.

(22) $$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a\, a_1 \\ b\, b \end{pmatrix} \quad \text{if } a_1 < a.$$

PROOF. By (4), $\binom{a}{b'} \to \binom{a\ a_1}{b'\ b'}$, and (22) follows from Lemma 10 (ii).

(23) $$\binom{a}{a} \to \binom{a\ a}{a\ b_1} \quad \text{if} \quad b_1 < b.$$

PROOF. By (3), $\binom{a}{b'} \to \binom{a\ a}{b'\ 1}$, and (23) follows from Lemma 10 (iv).

It follows from (2), (22), (23) that in case 6 the relation (1) holds if and only if either $\min(a_0, a_1) < a$ or $\min(b_0, b_1) < b$.

## 27. THE THEORY OF SET MAPPINGS

In this section we shall state some problems and results concerning the set mapping relation

(1) $$a \to [[p, c, d, q]]$$

which was defined in 21. 5. It follows almost immediately from that definition that

(2) $$\text{if} \quad a \to (c+d, q)^2 \quad \text{then} \quad a \to [[a^+, c, d, q]].$$

Problems about this relation were first stated in [21] and discussed in [14]. Throughout this section let $a \geqq \aleph_0$. Our Theorem 36 states that

$$(*)\ a \to [[a, a, d, a]] \quad \text{if} \quad a = a' \quad \text{and} \quad d^+ < a,$$

and Theorem 37 states that

$$(*)\ a \to [[a, a, d, a']] \quad \text{if} \quad a > a' \quad \text{and} \quad d < a;$$

$$a \nrightarrow [[a, a, 1, a'^+]] \quad \text{if} \quad a > a'.$$

On the other hand, using the method of canonical partitions employed in Lemma 3, it is easy to prove

$(*)$ THEOREM 45. *If* $c, d < a$ *and* $a > a'$, *then* $a \to [[a, c, d, a]]$.

We omit the proof. Thus if $p \leqq a$ and $a > a'$ then the relation (1) is completely discussed. The case on inaccessible cardinals is settled by Theorem 36. We therefore restrict ourselves to a discussion of the relation

(3) $$a^+ \to [[a^+, c, d, q]].$$

When $d < a$ then Theorem 36 gives a best possible positive result. We are therefore led to consider the relation

$$a^+ \to [[a^+, c, a, q]].$$

Here Lemma 9, which is a theorem of [14], yields:

$$a^+ \nrightarrow [[a^+, 2, a, a^+]] \quad \text{if} \quad a = a'.$$

On the other hand, from $a^+ \to (a^+, a')^2$ we obtain, by (2),

($*$) THEOREM 46. $a^+ \to [[a^+, a^+, a, a']]$.

Thus (3) is settled in the case when $a$ is regular. For singular $a$ the following relations are still left undecided by the theorems listed so far:

(4) $$?\ a^+ \to [[a^+, c, a, q]] \text{ for } 2 \leqq c \leqq a^+ \text{ and } a' < q \leqq a^+.$$

In [14], Problem 1, the simplest case of (4) is stated, which is to decide the truth of

$$\aleph_{\omega+1} \to [[\aleph_{\omega+1}, 2, \aleph_\omega, \aleph_1]].$$

We cannot solve the problem (4) but by a direct application of some of the results of the present paper we can obtain a partial solution.

By using the method of proof of $a^+ \nrightarrow (a^+, (3)_{a'})^2$ (Theorem 10) as well as the relation $a'^{++} \to (3)^2_{a'}$ one can obtain

($*$) THEOREM 47. $a^+ \nrightarrow [[a^+, a^+, a, a'^{++}]]$.

By putting $d = a$ it is seen that Theorems 46 and 47 leave undecided only the following case:

($*$) PROBLEM 16. ? $a^+ \to [[a^+, a^+, a, a'^+]]$ if $a > a'$. On the other hand, Theorem 43 implies

($*$) THEOREM 48. *Let $a > a'$, and suppose that either* (i) $c \geqq \aleph_2$ *and* $a' = \aleph_0$, *or* (ii) $c \geqq \aleph_0$ *and* $a' > \aleph_0$. *Then*

$$a^+ \nrightarrow [[a^+, c, a, a^+]].$$

Using the ramification method of Lemma 1 but also some new ideas one can prove:

($*$) THEOREM 49. *If $c < a$ and $a > a'$, then* $a^+ \to [[a^+, c, a, a]]$.

Detailed proofs of Theorems 47, 48, 49 are reserved for a later publication. The following problems remain unsolved:

(A) $$?\ a^+ \to [[a^+, c, a, a^+]]$$

if either (i) $2 \leqq c \leqq \aleph_1$ and $a' = \aleph_0$, or (ii) $2 \leqq c < \aleph_0$ and $a > a' > \aleph_0$.

(B) $$?\ a^+ \to [[a^+, a, a, q]] \text{ if } a' < q \leqq a.$$

Thus the simplest open questions in this connection are

($*$) PROBLEM 17.

(i) ? $\aleph_{\omega+1} \to [[\aleph_{\omega+1}, c, \aleph_\omega, \aleph_{\omega+1}]]$ for $2 \leqq c \leqq \aleph_1$;

? $\aleph_{\omega_1+1} \to [[\aleph_{\omega_1+1}, c, \aleph_{\omega_1}, \aleph_{\omega_1+1}]]$ for $2 \leqq c < \aleph_0$.

(ii) ? $\aleph_{\omega+1} \to [[\aleph_{\omega+1}, \aleph_\omega, \aleph_\omega, q]]$ for $\aleph_1 \leqq q \leqq \aleph_\omega$;

? $\aleph_{\omega_1+1} \to [[\aleph_{\omega_1+1}, \aleph_{\omega_1}, \aleph_{\omega_1}, q]]$ for $\aleph_2 \leqq q \leqq \aleph_{\omega_1}$.

Finally we mention that we have omitted the investigation of the case

$$a^{+} \rightarrow [[a^{++}, c, d, q]],$$

i. e. the case where the set mapping is not restricted at all. Most of the problem in this case are equivalent to problems about the ordinary partition relation I and their complete discussion would be very lengthy.

*Added in proof (20. IV. 1965):* Recently A. H. KRUSE (A note on the partition calculus of P. Erdős and R. Rado, *Journ. London Math.* Soc., **40** (1965), pp. 135—148) proved a number of negative partition relations mostly of the form $a \nrightarrow (b, r+1)^r$.

MATHEMATICAL INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
BUDAPEST

MATHEMATICAL INSTITUTE
EÖTVÖS LORÁND UNIVERSITY,
BUDAPEST

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY,
READING, ENGLAND

*(Received 23 April 1964)*

## References

[1] P. ERDŐS and R. RADO, A partition calculus in set theory, *Bull. Amer. Math. Soc.*, **62** (1956), pp. 427—489.
[2] A. TARSKI, Quelques théorèmes sur les alephs, *Fundam. Math.*, **7** (1925), pp. 1—14.
[3] P. ERDŐS and R. RADO, Combinatorial theorems on classifications of subsets of a given set, *Proc. London Math. Soc.*, (**3**) **2** (1952), pp. 417—439.
[4] P. ERDŐS and A. HAJNAL, On the structure of set mappings, *Acta Math. Acad. Sci. Hung.*, **9** (1958), pp. 111—131.
[5] A. TARSKI, Some problems and results relevant to the foundation of set theory, *Proc. Internat. Congress for Logic, Methodology and Philosophy of Sciences*, Stanford, Calif., USA, 1960.
[6] P. ERDŐS and G. FODOR, Some remarks on set theory. VI, *Acta Sci. Math. Szeged*, **18** (1957), pp. 243—260.
[7] P. ERDŐS and A. TARSKI, On families of mutually exclusive sets, *Annals of Math.*, **44** (1943), pp. 315—329.
[8] P. ERDŐS and G. SZEKERES, A combinatorial problem in geometry, *Comp. Math.*, **2** (1935), pp. 463—470.
[9] P. ERDŐS, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.*, **53** (1947), pp. 292—294.
[10] P. ERDŐS, Remarks on a theorem of Ramsey, *Bull. Research Council of Israel*, Section F., 1947.
[11] P. ERDŐS, Graph theory and probability, *Canad. Journ. of Math.*, **11** (1959), pp. 34—38.
[12] R. E. GREENWOOD and R. M. GLEASON, Combinatorial relations and chromatic graphs, *Canad. Journ. of Math.*, **7** (1955), pp. 1—7.
[13] P. ERDŐS, Graph theory and probability II, *Canad. Journ. of Math.*, **13** (1961), pp. 346—352.
[14] A. HAJNAL, Some results and problems on set theory, *Acta Math. Acad. Sci. Hung.*, **11** (1960), pp. 277—298.
[15] A. TARSKI, Sur la décomposition des ensembles en sousensembles presque disjoint, *Fundam. Math.* **14** (1929), pp. 205—215.
[16] P. ERDŐS, Some remarks on set theory, *Proc. Amer. Math. Soc.*, **1** (1950), pp. 127—141.
[17] A. HAJNAL, Proof of a conjecture of S. Ruziewicz, *Fundam. Math.*, **50** (1961), pp. 123—128.

[18] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.*, (**2**) **30** (1930), pp. 264–286.
[19] W. Sierpiński, Sur un problème de la théorie des relations, *Annali R. Scuola Normale Superiore de Pisa*, Ser. 2, **2** (1933), pp. 285–287.
[20] G. Kurepa, On the cardinal number of ordered sets and of symmetrical structures in dependence of the cardinal numbers of its chains and antichains, *Glanick Mat. Fiz. i Astr.*, **14** (1952), pp. 183–203.
[21] P. Erdős and G. Fodor, Some remarks on set theory. V, *Acta Sci. Math. Szeged*, **12** (1956), pp. 250–260.
[22] W. Sierpiński, Concernant l'hypothèse du continu, *Acad. Royale Serbe Bull. de l'acad. Sci. Math. et Naturelles*, Sér. A. Sc. Math. et Phys., **1** (1933), pp. 67–73.
[23] P. Erdős and S. Kakutani, On non-denumerable graphs, *Bull. Amer. Math. Soc.*, **49** (1963), pp. 457–461.
[24] P. Erdős and A. Tarski, On some problems involving inaccessible cardinals, *Essays on the foundations of mathematics*, Jerusalem (1961), pp. 50–82.
[25] P. Hanf, On a problem of Erdős and Tarski, *Notices of the AMS*, **9** (1962), p. 229.
[26] A. Hajnal, Remarks on a theorem of W. P. Hanf, *Fundam. Math.*, **54** (1964), pp. 109–113.
[27] Dana Scott, Measurable cardinals and constructible sets, *Bull. Acad. Polon. Sci., Ser. des Sci. Math., Astr. et Phys.*, **7** (1959), pp. 145–149.
[28] H. J. Kiesler, Some applications of the theory of models to set theory, *Logic, Methodology and Philosophy of Science, Proceedings of the 1960. International Congress*, Stanford (1962), pp. 80–86.
[29] P. Erdős and A. Hajnal, On a property of families of sets, *Acta Math. Acad. Sci. Hung.*, **12** (1961), pp. 87–123.
[30] H. J. Kiesler and A. Tarski, From accessible to inaccessible cardinals, *Fundam Math.*, **53** (1964), pp. 225–308.