

A PROBLEM ON INDEPENDENT r -TUPLES

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$G(n; l)$ denotes a graph of n vertices and l edges. A set of edges is called independent if no two of them have a vertex in common. GALLAI and I [1] proved that if

$$(1) \quad l > \max \left\{ \binom{2k-1}{2}, (k-1)(n-k+1) + \binom{k-1}{2} \right\}$$

then $G(n; l)$ contains k independent edges. It is easy to see that the above result is best possible since the complete graph of $2k-1$ vertices and the graph of vertices $x_1, \dots, x_{k-1}; y_1, \dots, y_{n-k+1}$ and edges (x_i, x_j) , $1 \leq i < j \leq k-1$; (x_i, y_j) , $1 \leq i \leq k-1$, $1 \leq y_j \leq n-k+1$ clearly does not contain k independent edges.

By an r -graph $G^{(r)}$ we shall mean a graph whose basic elements are its vertices and r -tuples; for $r = 2$ we obtain the ordinary graphs. $G^{(r)}(n; m)$ will denote an r -graph of n vertices and m r -tuples. For $r > 2$ these generalised graphs have not yet been investigated very much. A set of r -tuples is called independent if no two of them have a vertex in common.

$f(n; r, k)$ denotes the smallest integer so that every $G^{(r)}(n; f(n; r, k))$ contains k independent r -tuples. (1) implies that

$$(2) \quad f(n; 2, k) = 1 + \max \left\{ \binom{2k-1}{2}, (k-1)(n-k+1) + \binom{k-1}{2} \right\}.$$

It does not seem easy to determine $f(n; r, k)$ for $r > 2$ and every k . For $k = 2$ KO, RADO and I [2] proved that for $n \geq 2r$

$$(3) \quad f(n; r, 2) = \binom{n-1}{r-1} + 1.$$

The case $n < 2r$ is trivial since then no two r -tuples are independent.

Denote by $g(n; r, k-1)$ the number of those r -tuples formed from the elements x_1, \dots, x_n each of which contain at least one of the elements x_1, \dots, x_{k-1} . Clearly $f(n; r, k) > g(n; r, k-1)$ and a simple computation shows that

$$(4) \quad g(n; r, k-1) = \sum_i \binom{k-1}{i} \binom{n-k+1}{r-i} \geq (k-1) \binom{n-k+1}{r-1}$$

where the dash indicates that i runs from 1 to $\min(r, k-1)$.

Now we prove the following

THEOREM. For $n > c_r k$ (c_r is a constant which depends only on r)

$$f(n; r, k) = 1 + g(n; r, k-1).$$

The proof uses induction with respect to k . For $k = 2$ the result is known [2]. We assume that it holds for $k-1$ and prove it for k .

Let $n > c_r k$ and consider an arbitrary $G^{(r)}(n; 1 + g(n; r, k-1))$. Denote by $\nu(x_i)$ the number of r -tuples in our $G^{(r)}(n; 1 + g(n; r, k-1))$ which contain x_i . Without loss of generality we can assume that $\max_{1 \leq i \leq n} \nu(x_i) = \nu(x_1)$. We distinguish two cases. Assume first

$$(5) \quad \nu(x_1) < \frac{1 + g(n; r, k-1)}{(k-1)r}$$

and let R_1, \dots, R_l be a maximal system of independent r -tuples of our $G^{(r)}$. We show

$$(6) \quad l \geq k.$$

If (6) would be false our r -tuples R_1, \dots, R_l would contain at most $(k-1)r$ vertices and by (5) the number of r -tuples containing any of these vertices is less than

$$1 + g(n; r, k-1).$$

Thus our $G^{(r)}(n; 1 + g(n; r, k-1))$ contains an R_{l+1} which is independent of all the $R_i, 1 \leq i \leq l$, which contradicts the maximality of R_1, \dots, R_l , hence $l < k$ leads to a contradiction, which proves (6) and disposes of the first case.

Now we consider the second case, that is, we assume

$$(7) \quad \nu(x_1) \geq \frac{1 + g(n; r, k-1)}{(k-1)r}.$$

Consider now the r -graph $G^{(r)}$ whose vertices are x_2, \dots, x_n and whose r -tuples are those r -tuples of our $G^{(r)}(n; 1 + g(n; r, k-1))$ which do not contain x_1 . The number of r -tuples of $G_1^{(r)}$ is clearly at least

$$(8) \quad 1 + g(n; r, k-1) - \binom{n-1}{r-1} = 1 + g(n-1, r, k-1),$$

since there are at most $\binom{n-1}{r-1}$ r -tuples containing x_1 . Thus by our induction

hypothesis $G_1^{(r)}$ contains at least $k-1$ independent r -tuples R_1, \dots, R_{k-1} . The proof of our Theorem will be complete if we succeed to show that there is an r -tuple of our $G^{(r)}(n; 1 + g(n; r, k-1))$ containing x_1 which does not contain any of the $(k-1)r$ vertices of R_1, \dots, R_{k-1} . To see this observe that the number of r -tuples containing x_1 and x_i is at most $\binom{n-2}{r-2}$, and therefore the number of r -tuples containing x_1 and one of the vertices of R_1, \dots, R_{k-1} is at

most $(k-1)r \binom{n-2}{r-2}$. By (7) and (4) we obtain by a simple computation that for $n > c, k$

$$(k-1)r \binom{n-2}{r-2} < v(x_1);$$

hence there is an r -tuple of our $G^{(r)}(n; 1+g(n; r, k-1))$ containing x_1 which is disjoint from R_1, \dots, R_{k-1} , as stated. This completes the proof of our theorem.

It is not impossible that

$$(9) \quad f(n; r, k) = 1 + \max \left\{ \binom{rk-1}{r}, g(n; r, k-1) \right\}.$$

For $r = 2$ (9) is implied by (1) and for $k = 2$ (9) is proved in [2], but the general case seems elusive.

References

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