SOME APPLICATIONS OF PROBABILITY TO GRAPH THEORY AND COMBINATORIAL PROBLEMS

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In the first chapter of my lecture I will discuss applications of probabilistic methods to RAMSAY's theorem, next I will speak on problems of chromatic graphs and finally I will briefly mention some other problems. $\mathfrak{G}^{(n)}$ will denote a graph of n vertices; a graph is called complete if all its vertices are adjacent. $\overline{\mathfrak{G}}^{(n)}$ will denote the complementary graph of $\mathfrak{G}^{(n)}$ (i.e. two vertices in $\overline{\mathfrak{G}}^{(n)}$ are adjacent if and only if they are not adjacent in $\mathfrak{G}^{(n)}$). $\langle k \rangle$ will denote a complete graph of k vertices.

1. Denote by $f(k, l)$ the smallest integer such that for $n = f(k, l)$ either $\mathfrak{G}^{(n)}$ contains a $\langle k \rangle$ or $\overline{\mathfrak{G}}^{(n)}$ an $\langle l \rangle$. It is known that [1]

$$(1) \qquad f(k, l) \leqq \binom{k + l - 2}{k - 1}.$$

By probabilistic methods [2], [3] I showed that

$$(2) \qquad f(k, k) > 2^{k/2}$$

and that

$$(3) \qquad f(k, 3) > ck^2/(\log k)^2 .$$

It is very likely that my method will show that for every $l > 3$ and $\varepsilon > 0$, if $k > k_0(l, \varepsilon)$ then

$$f(k, l) > k^{l-1-\varepsilon},$$

but I have not worked out the formidable details. It would be very desirable to decide whether $f(k, 3) > ck^2$ holds and to determine the limit of $f(k, k)^{1/k}$. I cannot even prove the existence of this limit. It would also be of interest to prove (2) and (3) by a constructive method. I only succeeded to show by such methods that $f(k, 3) > k^{1+c}$ for a certain $c > 0$, [4].

2. ZYKOV and TUTTE [5] were the first to construct, for every k, a graph which contains no triangle (i.e. no $\langle 3 \rangle$) and whose chromatic number is k. J. B. KELLY and L. M. KELLY [6] showed that such graphs exist which contain no circuits of length $\leqq 5$ and they asked if there exist, for every k and l, k-chromatic graphs which contain no polygons of length $\leqq l$. By probabilistic methods I proved that such graphs exist [7] (it would be very desirable to give an explicit construction). In fact I showed that there exists an $\varepsilon_{k,l} > 0$ so that for every n there is a $\mathfrak{G}^{(n)}$ which contains no circuit of length $\leqq l$ and for which $\overline{\mathfrak{G}}^{(n)}$ does not contain an $\langle [n^{\varepsilon_{k,l}}] \rangle$ (the chromatic number of our $\mathfrak{G}^{(n)}$ is then clearly $> n^{1-\varepsilon_{k,l}}$).

More generally let S_n be a set of n elements and let $A_i \subset S_n$ be subsets of S_n. We say that the system $\{A_i\}$ is k-chromatic if the set S_n can be split up into k disjoint subsets S_j, $1 \leqq j \leqq k$ so that no A_i is contained in an S_j and that such a splitting is impossible into fewer than k subsets. HAJNAL and I now showed that for every k, r and l there exists an $n_0 = n_0(k, r, l)$ so that for every $n > n_0(k, r, l)$ there exists a k-chromatic system $\{A_i\}$, $A_i \subset S_n$ where each A_i has r elements and for $u \leqq l$ the union of any u A's contains at least $1 + u(r - 1)$ elements (for $r = 2$ this condition means that our graph contains no circuit of length $\leqq l$). We use probabilistic methods; our proof has not yet been published.

A well known theorem of BROOKS [8] states that every graph which does not contain a $\langle k + 1 \rangle$ and every vertex of which has valency $\leqq k$ is at most k-chromatic. GRÜNBAUM constructed a 4-chromatic graph which does not contain a triangle and a quadrilateral and every vertex of which has valency 4 (Grünbaum's construction has not been published yet). Grünbaum now asked: Does there exist, for every k and l, a k-chromatic graph every vertex of which has valency k and which does not contain a circuit of length $\leqq l$? I am very far from being able to solve this difficult question and can only show by probabilistic methods that there exists an absolute constant c so that there exists a graph of chromatic number $> ck/\log k$ every vertex of which has valency k and the graph contains no circuit of length $\leqq l$. I very much doubt whether Grünbaum's problem can be settled by probabilistic methods.

Denote by $f(m, k, n)$ the maximum of the chromatic numbers of all graphs $\mathfrak{G}^{(n)}$ every subgraph of m vertices of which has chromatic number $\leqq k$. By probabilistic methods I proved [9] that

$$f(m, 3, n) > c \left(\frac{n}{m}\right)^{1/3} \Big/ \log \frac{n}{2m}. \tag{4}$$

(4) in particular implies that to every k there is an $\varepsilon > 0$ so that if $n > n_0(\varepsilon, k)$ then there exists a k-chromatic $\mathfrak{G}^{(n)}$ every subgraph of which having $[\varepsilon n]$ vertices is at most 3-chromatic. The situation is radically different if we consider $f(m, 2, n)$. GALLAI proved (his proof will appear in Publ. Math. Inst. Hung. Acad. Sci.) that $f([n^{1/2}], 2, n) = 4$, in other words there exists a four-chromatic $\mathfrak{G}^{(n)}$ every odd circuit of which has length $> n^{1/2}$. Gallai and I conjectured that the largest value of m for which $f(m, 2, n) = k$ is of the order of magnitude $n^{1/k+2}$, but we have not even been able to prove that for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$, $f([\varepsilon n], 2, n) = 3$.

3. A theorem of RÉDEI [10] states that the directed complete graph of n vertices always has an open Hamiltonian path. SZELE denotes by T_n the maximum number of open Hamiltonian paths for all possible orientations of $\langle n \rangle$. Szele [11] proves by elementary probabilistic methods (calculation of the first moment) that

$$T_n \geqq \frac{n!}{2^{n-1}}\,. \tag{5}$$

As far as I know this is the first use of probabilistic methods for a combinatorial problem. Szele's paper unfortunately has been almost completely unknown. The same method was rediscovered and used in several other papers [12]. It would be very desirable to construct a directed $\langle n \rangle$ which has at least $n!/2^{n-1}$ open Hamilton lines. Szele proved that $\lim_{n=\infty} (T_{n/n!})^{1/n}$ exists, he conjectures that this limit is $\frac{1}{2}$.

SCHÜTTE asked the following question: Determine the smallest integer $f(k)$ for which there exists a directed $\langle f(k) \rangle$ so that for every k vertices of our $\langle f(k) \rangle$ there is another vertex of it from which edges go out to each of the k vertices. Trivially $f(1) = 3$, Schütte observed $f(2) = 7$, I proved [13]

$$2^{k+1} - 1 \leqq f(k) < ck^2 2^k \tag{6}$$

where the upper bound is obtained by probabilistic arguments; perhaps $f(k) = 2^{k+1} - 1$ always holds.

In connection with some work on set theory HAJNAL and I raised the following question [14]: What is the smallest integer $m(p)$ for which there exists a family of finite sets $A_1, \ldots, A_{m(p)}$ each having p elements and such that every set S which has a non-empty intersection with each A_i, $1 \leqq i \leqq m(p)$ contains at least one of the A_i? We observed $m(p) \leqq \binom{2p-1}{p}$. By probabilistic methods I showed [15] $m(p) > 2^{p-1}$, and for $p > p_0(\varepsilon)$, $m(p) > 2^p (\log 2 - \varepsilon)$. I cannot even prove that

$$\lim_{p=\infty} m(p)^{1/p} \tag{7}$$

exists. It is easy to see that $m(2) = 3$, $m(3) = 7$, $m(4)$ is unknown.

Added in proof: Schütte's problem was also raised by RYSER and (6) was proved by MOSER independently. Moser and I will publish a paper on this subject in the Bull. Canad. Math. Soc.

Recently I learned that a result substantially equivalent to (6) was proved by GLEASON, but he published nothing about this subject.

The fact that the limit in (7) exists was proved by ABBOT and MOSER (will appear in Bull. Canad. Math. Soc.). Later I proved that the limit is 2. In fact

$$2^p \left(1 + O\left(\frac{1}{p}\right)\right) < m(p) < cp^2 2^p\,. \tag{8}$$

The lower bound in (8) is due to W. SCHMIDT, the upper bound to me. Our papers will appear in Acta Math. Hung. Acad.

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