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ON AN EXTREMAL PROBLEM IN GRAPH THEORY

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In the present paper $G(n; l)$ denotes a graph of n vertices and l edges, K_p — the complete graph of p vertices, i. e. $G\left(p; \binom{p}{2}\right)$, $K(p_1, \ldots, p_r)$ — the complete r-chromatic graph with p_i vertices of the i-th colour in which every two vertices of different colour are adjacent.

Vertices of our graphs will be denoted by $x, y, \ldots$, edges by (x, y). The *valence* $v(x)$ of x is the number of edges adjacent to x.

Denote by $m(n; p)$ the smallest integer so that every $G(n; m(n; p))$ contains a K_p. Turán [6] (comp. also [7]) determined $m(n; p)$ and also showed that the only $G(n; m(n; p)-1)$ which contains no K_p is $K(m_1, \ldots, m_{p-1})$, where

$$\sum_{i=1}^{p-1} m_i = n \quad \text{and} \quad m_i = \left[\frac{n}{p-1}\right] \quad \text{or} \quad \left[\frac{n}{p-1}\right]+1.$$

Dirac [1] and I (independently) proved that every $G(n; m(n; p))$ contains a K_{p+1} from which one edge is missing. In fact, the following stronger result also holds:

There is a constant c_p so that every $G(n; m(n; p))$ contains a K_{p-1} and $c_p n$ vertices each of which is joined to every vertex of our K_{p-1} ([2], Lemma 2 [1]).

Denote by $u(n; p)$ the smallest integer such that every $G(n; u(n; p))$ contains a $K(p, p)$. The value of $u(n; p)$ is not known and its determination seems to be a very difficult problem. As far as I know the first result in this direction is due to E. Klein and myself [3]; we proved

$$a_1 n^{3/2} < u(n; 2) < a_2 n^{3/2}. \tag{1}$$

[1] This lemma concerns only the case $p = 3$ but the same proof works in the general case.

Probably $\lim_{n\to\infty} u(n;2)/n^{3/2} = 1/2\sqrt{2}$, but it is not even known that this limit exists. The best result in this direction is due to Reiman [5] who among others proved that

$$\limsup_{n\to\infty} u(n;2)/n^{3/2} \leqslant \frac{1}{2}, \quad \liminf_{n\to\infty} u(n;2)/n^{3/2} \geqslant \frac{1}{2\sqrt{2}}.$$

Kövári, Sós and Turán [4] and independently I proved that for a suitable constant β_n

$$u(u;p) < \beta_p n^{2-1/p}. \tag{2}$$

Probably $u(n;p) > \beta'_p n^{2-1/p}$, but this is known only for $p=2$ (see [1]).

In this note we prove the following refinement of (2):

THEOREM 1. *There is a constant γ_p such that every $G(n;[\gamma_p n^{2-1/p}])$ contains a $K(p+1,p+1)$ from which one edge is missing.*

Remarks. Clearly the structure of a $K(p+1,p+1)$ from which one edge is missing is uniquely determined.

One could conjecture (by analogy to [1]) that every $G(n;u(n;p))$ contains a $K(p+1,p+1)$ from which one edge is missing. This would of course be a much stronger result than Theorem 1, but, if true, it will be hard to prove since we do not know the value of $u(n;p)$ and have no idea of the structure of the extremal graphs $G(n;u(n;p)-1)$ which do not contain a $K(p,p)$.

Instead of Theorem 1 we shall prove the following sharper

THEOREM 2. *Let $l > p$ be any integer. Then there is a constant $\gamma_{p,l}$ such that for $n > n_0(p,l)$ every $G(n;[\gamma_{p,l} n^{2-1/p}])$ contains a subgraph $H(p,l,l)$ of the following structure: the vertices of $H(p,l,l)$ are $x_1,\dots,x_l$; $y_1,\dots,y_l$ and its edges are all (x_i,y_j), where at least one of the indices i or j is $\leqslant p$.*

In other words, $H(p,l,l)$ is $K(l,l)$ from which the edges (x_i,y_j), $\min(i,j) > p$, are missing.

First we prove two Lemmas.

LEMMA 1. *Every $G(n,m)$ contains a subgraph G' each vertex of which has valence (in G') not less than $[m/n]$.*

If Lemma 1 would be false we could clearly order the vertices of $G(n;m)$ into a sequence $x_1, x_2, \dots, x_n$ where for every i, $1 \leqslant i \leqslant n$, x_i is joined to fewer than $[m/n]$ vertices x_j, $i < j \leqslant n$. But this would imply that the number of edges of $G(n;m)$ is less than m. This contradiction proves the Lemma.

Consider now our $G(n; [\gamma_{p,l} n^{2-1/p}])$. By Lemma 1 it has a subgraph $G(N; m)$ each vertex of which has valence $u = \{\gamma_{p,l} n^{1-1/p}]$. Now we prove

LEMMA 2. *Let $c_{p,l} > 0$ be any constant. Then if $\gamma_{p,l}$ is sufficiently large, our $G(N; m)$ contains a $K(p-1, s)$ with $s = [c_{p,l}\ n^{1/p}]$.*

For each vertex y of $G(N; m)$ consider all the $(p-1)$-tuples formed from the vertices which are joined to y. Since by assumption y is joined to at least u vertices, the number of these $(p-1)$-tuples counted for each y separately is at least $N\binom{u}{p-1}$. Now since $N \leqslant n$, we obtain by a simple calculation that for sufficiently large $\gamma_{p,l}$

$$N\binom{u}{p-1} > c_{p,l}\, n^{1/p} \binom{N}{p-1}. \tag{3}$$

Thus to some $(p-1)$-tuples correspond more than $s = [c_{p,l}\ n^{1/p}]$ vertices y, i. e. (3) implies that there are $p-1$ vertices $x_1, \ldots, x_{p-1}$ which are all joined to the same s vertices $y_1, \ldots, y_s$. In other words, our graph contains a $K(p-1, s)$ and Lemma 2 is proved.

Now we are ready to prove Theorem 2. Denote by $z_1, \ldots, z_{N-p-s+1}$ the remaining vertices of $G(N; m)$, i. e. those vertices which are not included in $K(p-1, s)$. By our assumption the valence (in $G(N; m)$) of each y is at least u and clearly for $\gamma_{p,l} > 2c_{p,l}$ and sufficiently large n, $s+p < u/2$, hence each y is joined to more than $u/2$ z's. Hence there are more than $us/2$ edges joining the y's with the z's. Denote now by $v'(z_j)$ the number of y's which are joined to z_j $(1 \leqslant j \leqslant N-p-s+1)$. Clearly

$$\sum_{j=1}^{N-p-s+1} v'(z_j) > \frac{us}{2} \tag{4}$$

and ($\sum'$ denotes that the summation is extended only over the z_j for which $v'(z_j) \geqslant p+l$)

$$\sum{}' v'(z_j) > \frac{us}{2} - (p+l)(N-p-s+1) > \frac{us}{2} - n(p+l) > \frac{1}{4}\,\gamma_{p,l} c_{p,l} n \tag{5}$$

for sufficiently large $c_{p,l}$ and $\gamma_{p,l}$.

Form now for every z_j satisfying $v'(z_j) \geqslant p+l$ all the p-tuples from the y's which are joined to z_j. The number of these p-tuples, counted for each z_j separately, clearly equals

$$\sum{}' \binom{v'(z_j)}{p}. \tag{6}$$

Using (5) we obtain from an elementary inequality that the sum (6) is minimal if all the $v'(z_j)$ are as nearly equal as possible and if their number is as large as possible (it is $\leqslant n$). Thus by a simple computation we get

$$(7) \qquad \sum{}' \binom{v'(z_j)}{p} > n \binom{([\frac{1}{4}c_{p,l}\gamma_{p,l}]}{p} > (l-p+1)\binom{s}{p}$$

for sufficiently large $\gamma_{p,l}$. Formula (7) implies that the number of these multiply counted p-tuples is larger than $l-p+1$ times the number of all the p-tuples formed from the s distinguished y's of $K(p-1,s)$. Hence there are $l-p+1$ z's, say $z_1, \ldots, z_{l-p+1}$, satisfying

$$(8) \qquad v'(z_i) \geqslant p+l, \quad 1 \leqslant i \leqslant l-p+1$$

(only $v'(z_1) \geqslant l$ will be needed) and which are all joined to the same p y's, say to $y_1, \ldots, y_p$. By (8) we can further assume that z_1 is joined to $y_{p+1}, \ldots, y_l$. Let $x_1, \ldots, x_{p-1}$ be the distinguished $p-1$ x's of $K(p-1, s)$. Now the even graph spanned by $x_1, \ldots, x_{p-1}, z_1, \ldots, z_{l-p+1}$; $y_1, \ldots, y_p, y_{p+l}, \ldots, y_1$ is clearly an $H(p, l, l)$, since, by Lemma 2, $x_1, \ldots, x_{p-1}$ are all joined to all the y's, $y_1, \ldots, y_p$ are joined to all the z_j $(1 \leqslant j \leqslant l-p+1)$ by the argument following (7) and z_1 is joined to z_j $(p+1 \leqslant j \leqslant l)$ by construction. Thus the proof of Theorem 2 is complete.

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