REMARKS ON A PROBLEM OF OBREANU

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Let $a_1 < a_2 < \ldots$ be any sequence of integers. Assume that the infinite sequence of numbers u_n satisfies the following condition: To every $\varepsilon > 0$ there is an $n_o = n_o(\varepsilon)$ such that for all $n > n_o$ and all k

$$(1) \qquad |u_{n+a_k} - u_n| < \varepsilon .$$

Obreanu asked (Problem P. 35 Can. Math. Bull.) under what conditions on the sequence $a_1 < a_2 < \ldots$ does (1) imply that the sequence u_n is convergent. N. G. de Bruijn and P. Erdős proved that a necessary and sufficient condition for (1) to imply the convergence of u_n is that the sequence $\{a_n\}$ be infinite and that the greatest common divisor of the a_n should be 1.

The condition (1) is very strong and is "nearly equivalent" to Cauchy's criterion for convergence. We discuss various conditions which are weaker than (1).

Assume first that the sequence u_n satisfies

$$(2) \qquad \lim_{n \to \infty} \overline{\lim_{r}} \, | u_{n+a_r} - u_n | = 0 .$$

Condition (2) means that to every $\varepsilon > 0$ there exists $n_o = n_o(\varepsilon)$ such that for $n > n_o$ we have $|u_{n+a_r} - u_n| < \varepsilon$ except for

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finitely many r (the number of exceptional r may of course depend on n). Denoting the sequence $a_1 < a_2 < \dots$ by A we shall prove

THEOREM 1 (2) implies the convergence of $\{u_n\}$ if and only if A satisfies the following two conditions:

(I) to every integer $d > 1$ there are infinitely many k with $a_k \not\equiv 0 \pmod{d}$,

(II) $a_{k+1} - a_k$ does not tend to infinity as $k \to \infty$.

First we prove that (I) and (II) are necessary. This is clear for (I) since if (I) is not satisfied for a certain $d > 1$ then the sequence u_n with

$$u_n = 0 \text{ if } n \equiv 0 \pmod{d} \text{ and } u_n = 1 \text{ otherwise,}$$

clearly satisfies (2) and does not converge.

Next we show that (II) is necessary. Suppose A does not satisfy (II), i.e. $a_{k+1} - a_k \to \infty$ as $k \to \infty$. Put

$$n = a_{i_1} + a_{i_2} + \dots + a_{i_r} + \ell$$

where a_{i_1} is the greatest a not exceeding n, a_{i_2} the greatest a not exceeding $n - a_{i_1}$, or a_{i_r} is the greatest a not exceeding $n - (a_{i_1} + \dots + a_{i_{r-1}})$, and $0 \leq \ell < a_1$ (thus if $a_1 = 1$, ℓ is always 0). Put

(3) $u_n = 0$ if $i_r = 1$ and $u_n = 1$ if $i_r \neq 1$,

e.g. if $(i > i_o)$ $n = a_i + a_1$ then $u_n = 0$, while if $n = a_i + a_2$ then $u_n = 1$. Thus u_n is infinitely often 0 and infinitely often 1 and hence does not converge. On the other hand it is easy to see that the sequence (3) satisfies (2) since from $a_{k+1} - a_k \to \infty$ we obtain that $a_{k+1} - a_k > n$ for $k > k_o(n)$ and hence for these k we have from (3) $u_{n+a_k} - u_n = 0$, so that (2) is satisfied. This shows that our conditions are necessary.

Next we show that our conditions are sufficient, in other words we shall show that if A satisfies (I) and (II) and the infinite sequence $\{u_n\}$ satisfies (2), then $\{u_n\}$ converges.

Since (II) is satisfied, there is a T for which

$$(4) \qquad a_{k+1} - a_k = T$$

has infinitely many solutions. First we show that for every i

$$(5) \qquad \lim_{\ell \to \infty} (u_{i+(\ell+1)T} - u_{i+\ell T}) = 0 .$$

Let $\varepsilon > 0$ be given; to prove (5) we shall show that for all $\ell > \ell_o(\varepsilon)$

$$(6) \qquad |u_{i+(\ell+1)T} - u_{i+\ell T}| < \varepsilon .$$

From (2) it follows that for sufficiently large fixed ℓ ($\ell = \ell(\varepsilon)$) and every $r > r_o(\varepsilon, \ell)$

$$(7) \qquad |u_{i+\ell T+a_r} - u_{i+\ell T}| < \varepsilon/2 \text{ and}$$

$$|u_{i+(\ell+1)T+a_r} - u_{i+(\ell+1)T}| < \varepsilon/2 .$$

Since (4) has infinitely many solutions there is a k (in fact infinitely many such k) for which $a_{k+1}-a_k = T$, $k > r_o(\varepsilon, \ell)$. Thus from (7)

(8) $\quad |u_{i+\ell T+a_{k+1}} - u_{i+\ell T}| < \varepsilon/2$ and

$$|u_{i+(\ell+1)T+a_k} - u_{i+(\ell+1)T}| < \varepsilon/2 .$$

(6) follows from (8) by subtraction (since $i+(\ell+1)T+a_k = i+\ell T+a_{k+1}$). (6) implies that for every s and i

(9) $$\lim_{\ell \to \infty} (u_{i+(\ell+s)T} - u_{i+\ell T}) = 0 .$$

From (9) we shall now deduce that for every fixed i

(10) $$\lim_{\ell \to \infty} u_{i+\ell T}$$

exists. If (10) did not exist there would exist an infinite sequence of integers ξ_j, λ_j satisfying

(11) $$\xi_j \equiv \lambda_j \equiv i \pmod{T}, \quad \xi_1 < \xi_2 < \dots, \ \xi_j < \lambda_j$$

and

(12) $$|u_{\xi_j} - u_{\lambda_j}| > c$$

for a certain positive absolute constant c. From (2) we obtain that for sufficiently large j and r

(13) $$|u_{\xi_j+a_r} - u_{\xi_j}| < c/4 \text{ and } |u_{\lambda_j+a_r} - u_{\lambda_j}| < c/4 .$$

From the first part of (11) we have $\xi_j - \lambda_j = sT$, and so from (9) we have for sufficiently large r

(14) $$|u_{\xi_j+a_r} - u_{\lambda_j+a_r}| < c/4 \quad (\xi_j + a_r = i+\ell T \text{ of } (9));$$

(13) and (14) imply $|u_{\xi_j} - u_{\lambda_j}| < 3c/4$ which contradicts (12).

and hence (10) is proved.

If the limit in (10) does not depend on $\underline{i}$ then $\{u_n\}$ converges and our theorem is proved. Assume thus that for two values $i_1 \not\equiv i_2 \pmod T$

$$(15)\quad \lim_{\ell \to \infty} u_{i_1+\ell T} = \alpha_1 , \quad \lim_{\ell \to \infty} u_{i_2+\ell T} = \alpha_2 , \quad \alpha_1 < \alpha_2 .$$

Choose $\varepsilon < (\alpha_2 - \alpha_1)/2T^2$ and let ℓ be so large that for all $n > \ell T$ and all r except possibly for finitely many exceptions

$$(16)\quad |u_{n+a_r} - u_n| < \varepsilon ,$$

and choose ℓ_o so large that for every $\ell > \ell_o$, $\ell_1 > \ell_o$

$$(17)\quad |u_{i_1+\ell T} - u_{i_2+\ell_1 T}| > (\alpha_2 - \alpha_1)/2 .$$

Denote by $j_1, \dots, j_r$ those residue classes (mod T) for which the congruence $a_n \equiv j_s \pmod T$ has infinitely many solutions. By (I), $(j_1, j_2, \dots, j_r, T) = 1$ and therefore the congruence

$$(18)\quad \sum_{s=1}^{r} X_s j_s \equiv i_2 - i_1 \pmod T, \quad 0 \le X_s < T$$

is solvable (in fact every residue class (mod T) can be represented in the form (18). We can find arbitrarily large a's satisfying ($a_n \equiv j_s \pmod T$ has infinitely many solutions)

$$a_{m_s} \equiv j_s \pmod T \qquad 1 \le s \le r .$$

Put

$$(19)\quad v = i+\ell T + \sum_{s=1}^{r} X_s a_{m_s} = i+\ell T + \sum_{j=1}^{y} b_j , \quad y = \sum_{s=1}^{r} X_s < T^2 \text{ (by (18))}$$

where X_s of the b's are equal to a_{m_s}. From (19) and (18) we have

$$(20) \quad v = i_2 + \ell_1 T, \quad \ell_1 \geq \ell .$$

We evidently have by (19), (as in the proof of Problem 35)

$$(21) \quad |u_v - u_{i_1+\ell T}| \leq |u_{i_1+\ell T+b_1} - u_{i_1+\ell T}|$$

$$+ |u_{i_1+\ell T+b_1+b_2} - u_{i_1+\ell T+b_1}| + \ldots$$

$$+ |u_{i_1+\ell T+\sum_{j=1}^{r} b_j} - u_{i_1+\ell T+\sum_{j=1}^{r-1} b_j}|$$

Now since each b is an a, we have from (16) and (17) that for sufficiently large ℓ and sufficiently large b's each summand at the right side of (21) is less than ε. Thus from (20), (21) and the definition of ε we obtain by the last inequality of (19)

$$(22) \quad |u_{i_2+\ell_1 T} - u_{i_1+\ell T}| < y\varepsilon = \varepsilon \sum_{j=1}^{r} X_j < \varepsilon T^2 < (\alpha_2 - \alpha_1)/2 .$$

(22) contradicts (17) and this contradiction proves the convergence of $\{u_n\}$ and hence the proof of our theorem is complete.

We also considered the following modification of (2):

$$(23) \quad \underline{\lim}_{n} \overline{\lim}_{r} |u_{n+a_r} - u_n| = 0 .$$

We proved

THEOREM 2 (23) implies the convergence of $\{u_n\}$ if and only if for every infinite sequence of integers $b_1 < b_2 < \ldots$ there is a t such that the sequence

(24) $\{a_r + b_i\}$ $1 \leq r < \infty$, $1 \leq i \leq t$

contains all but a finite number of the integers $1, 2, \ldots$.

We suppress the proof of Theorem 2. It is easy to see that (24) is equivalent to the following condition which is perhaps more manageable: Let $b_1 < b_2 < \ldots$ be any infinite sequence of integers; then all but a finite number of the natural numbers are of the form (a_i+b_j) where i and j are natural numbers.

Assume that we modify (2) as follows: To every $\varepsilon > 0$ there exists an n_o such that for $n > n_o$ we have $|u_{n+a_k} - u_n| < \varepsilon$ except for at most t_ε values of k where t_ε depends only on ε and not on n. We do not know what is the necessary and sufficient condition on the sequence $\{a_k\}$ that this should imply that $\{u_n\}$ converges.

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