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On a Problem of A. Zygmund

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Dedicated to Professor G. Pólya on his 75th birthday

1. Introduction

We shall consider in the present paper the theory of lacunary power series (and also Fourier series), an area of study in which Professor Pólya has made a number of important contributions—see, for example, his beautiful and already classical paper [1]. Many results of this theory may be characterized (somewhat vaguely) as follows: The behavior of a function $f(z)$ having a "sufficiently" lacunary power series is essentially the same on every arc of its circle of convergence (if the series has a finite radius convergence), or in every angle $\alpha \leqq \arg z \leqq \beta$ as $|z| \to +\infty$ (if it is an entire function). Among results of this type we mention only one.

Wiener [2] proved that if a lacunary power series

$$f(z) = \sum_{k=1}^{\infty} C_k z^{\lambda_k} \qquad (\lambda_{k+1} - \lambda_k \to +\infty)\,,$$

satisfies $\lim_{r\to 1} f(re^{i\vartheta}) = f(e^{i\vartheta})$ for $\alpha < \vartheta < \beta$ and $f(e^{i\vartheta}) \in L^2$ on the interval (α, β), then $\lim_{r\to 1} f(re^{i\vartheta}) = f(e^{i\vartheta})$ exists almost everywhere, and $f(e^{i\vartheta}) \in L^2$ on the interval $[-\pi, \pi]$. This result can be formulated in the language of Fourier series as follows: consider a lacunary trigonometic series with gaps tending to $+\infty$, i.e., a series of the form

$$\sum_{k=1}^{\infty} a_k \cos \lambda_k x + b_k \sin \lambda_k x\,, \tag{1.1}$$

where λ_k is an increasing sequence of positive integers such that

$$\lim_{k\to+\infty} (\lambda_{k+1} - \lambda_k) = +\infty\,. \tag{1.2}$$

If such a series is Abel-summable almost everywhere in some interval (α, β) to a function $f(\vartheta)$ which belongs to the class $L^2(\alpha, \beta)$ for $-\pi \leqq \alpha < \beta \leqq \pi$, then (1.1) is the Fourier series of a function $f(\vartheta)$ in $L^2(-\pi, \pi)$; i.e, the series $\sum_{k=1}^{-\pi}(a_k^2 + b_k^2)$ is convergent. Wiener's inequality for trigonometric polynomials, from which he deduced this result, has been generalized by Ingham [3] and by Turán [4]. For an elementary proof of a somewhat

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weaker inequality see [5]. It follows particularly from Wiener's theorem that if (1.1) is the Fourier series of an L-integrable function $f(\vartheta)$ and $f(\vartheta)$ belongs to L^2 in (α, β), then it belongs to L^2 in the whole interval.

About 20 years ago Zygmund suggested to the first-named author that he consider the problem of whether a similar result holds for the space L^q with $q \neq 2$ $(q > 1)$, instead of the space L^2. In other words, he asked, if the Fourier series (1.1) of $f(\vartheta) \in L$ is lacunary in the sense of (2.2) and if $f(\vartheta)$ belongs in the subinterval (α, β) to the space L^q, then does it follow that $f(\vartheta)$ belongs to L^q in the whole interval $[-\pi, \pi]$? The problem has so far remained unsolved, and Zygmund mentions it in his book [6, vol. I, p. 380] as an open question.

In Sec. 3 of this paper we shall show that the answer to Zygmund's query is negative for $q > 2$; i.e., the theorem of Wiener cannot be generalized for the space L^q $(q > 2)$. We shall prove even more: There exist functions $f(\vartheta) \in L^2$ having a lacunary Fourier series which do not belong to any class L^q $(q > 2)$ in the full interval $[-\pi, \pi]$ but which, however, are bounded in every closed subinterval of the interval $[-\pi, \pi]$ not containing the point $\vartheta = 0$.

We shall prove this by the use of probability theory. We consider a class of random lacunary Fourier series and prove that almost all series of this class have the above-mentioned properties. Thus our proof is not constructive; only the existence of a function (as a matter of fact, of an infinity of functions) having the required properties will be proved. Such a method has often been used in similar situations. Usually in such proofs the coefficients a_k, b_k are taken as random variables and the exponents λ_k are explicitly given (not random) numbers. In our proof, however, the coefficients a_k, b_k will be given numbers and the exponents λ_k will be positive, integer-valued random variables.

Lacunary random power series in which the exponents are random variables have already been used for a similar purpose in a previous joint paper [7] of the authors. The method used in this paper is essentially the same as that developed in [7], only it has been modified to some extent. The proof is based on a lemma presented in Sec. 2, similar to the lemma of [7]. The modification of the method is as follows: In the lemma of [7] we considered random exponents, each of which is uniformly distributed on a set of consecutive integers; in the lemma of Sec. 3, the exponents are random integers having a binomial distribution.

In Sec. 4 we consider some additional questions. For another related problem where a probabilistic method was also applied with success, see [10].

2. A Lemma on Random Cosine Polynomials

In this section we prove the following

LEMMA. *Let $d > 2$ and let $s < m_1 < m_2 < \cdots < m_d$ be arbitrary positive integers. Let $\nu_1, \nu_2, \cdots, \nu_d$ be independent random variables, each of which takes on the values* $0, \pm 1, \pm 2, \cdots, \pm s$ *with the corresponding probabilities**

* Here and in what follows $P(\ )$ denotes the probability of the event in the brackets.

$$(2.1)\qquad P(\nu_j = l) = \binom{2s}{s+l}\frac{1}{2^{2s}} \qquad (l = 0, \pm 1, \cdots, \pm s;\ j = 1, \cdots, d)\,.$$

Put

$$(2.2)\qquad C = \sum_{j=1}^{d} \cos{(m_j + \nu_j)\varphi}\,,$$

where $0 < |\varphi| \leqq \pi$. *Let* ε *be an arbitrary positive number with* $0 < \varepsilon < \frac{1}{2}$. *Then we have*

$$(2.3)\qquad P(C \geqq d^{(1/2+\varepsilon)} \leqq 2\exp\{-d^{2\varepsilon}/16\}\,,$$

provided that

$$(2.4)\qquad s \geqq \frac{8\log d}{\varphi^2}$$

and

$$(2.5)\qquad d^{\varepsilon} \geqq 2\,.$$

PROOF. The proof can be carried out by the method of S. Bernstein [9, pp. 162–65] (see also [7] and [8], where this method has been similarly applied).

Let t be a real number, $|t| \leqq 1$. Then we have

$$(2.6)\qquad M(e^{tC}) = \prod_{j=1}^{d} M(\exp\{t\cos{(m_j + \nu_j)\varphi}\})\,,$$

where $M(*)$ denotes the mean value of the random variable enclosed in parentheses. Since $|e^x - 1 - x| \leqq |x|^2$ for $|x| \leqq 1$, we have

$$(2.7)\qquad M(\exp\{t\cos{(m_j + \nu_j)\varphi}\}) \leqq 1 + |t| \cdot |M(\cos{(m_j + \nu_j)\varphi})| + t^2\,.$$

Since by (2.1) clearly

$$(2.8)\qquad M(\cos{(m_j + \nu_j)\varphi}) = (\cos m_j\varphi) \cdot \left(\cos\frac{\varphi}{2}\right)^{2s}\,,$$

we obtain

$$(2.9)\quad M(e^{tC}) \leqq \left[1 + |t|\left(\cos\frac{\varphi}{2}\right)^{2s} + t^2\right]^d \leqq \exp\left\{d\left[|t|\left(\cos\frac{\varphi}{2}\right)^{2s} + t^2\right]\right\}.$$

Clearly, for $0 < t < 1$,

$$(2.10)\qquad P(C \geqq d^{(1/2)+\varepsilon}) = P(\exp\{Ct\} \geqq \exp\{td^{(1/2)+\varepsilon}\}) \leqq \exp\{-td^{(1/2)+\varepsilon}\}\,M[\exp\{Ct\}]\,.$$

From (2.9) and (2.10), we obtain

$$(2.11a)\qquad \left.\begin{matrix} P(C \geqq d^{(1/2)+\varepsilon}) \\ P(C \leqq -d^{(1/2)+\varepsilon}) \end{matrix}\right\} \leqq \exp\{d(t(\cos\varphi/2)^{2s} + t^2) - td^{(1/2)+\varepsilon}\}\,.$$

Thus for any t with $0 \leqq t < 1$ we have

$$(2.11b)\qquad P(|C| > d^{(1/2)+\varepsilon}) \leqq 2\exp\{t(d(\cos\varphi/2)^{2s} - d^{(1/2)+\varepsilon}) + dt^2\}\,.$$

Let us now choose t so as to minimize the right-hand side of (2.11b); that is, choose $t = \frac{1}{2}(d^{\varepsilon-1/2} - (\cos\varphi/2)^{2s})$. We obtain

$$P(|C| \geqq d^{(1/2)+\varepsilon}) \leqq 2\exp\{-\tfrac{1}{4}[d^{\varepsilon} - \sqrt{d}(\cos\varphi/2)^{2s}]^2\} . \tag{2.12}$$

By the inequality $(\cos\varphi/2)^2 \leqq \exp\{-\varphi^2/16\}$, valid for $|\varphi| \leqq \pi$, and in view of (2.4) and (2.5), we have $\sqrt{d}(\cos\varphi/2)^{2s} \leqq \sqrt{d}\,\exp\{-s\varphi^2/16\} \leqq 1 \leqq d^{\varepsilon}/2$, and thus we have from (2.12)

$$P(|C| \geqq d^{(1/2)+\varepsilon}) \leqq 2\exp\{-d^{2\varepsilon}/16\} ; \tag{2.13}$$

thus our Lemma is proved.

3. A Class of Random Lacunary Fourier Series

We shall prove the following

THEOREM. *There exist real, even functions $f(\vartheta)$ defined in the interval $[-\pi, \pi]$ that have the following four properties:*

(a) *$f(\vartheta)$ belongs to the class $L^2(-\pi, \pi)$;*

(b) *the Fourier series of $f(\vartheta)$ is of the form*

$$f(\vartheta) \sim \sum_{j=1}^{\infty} a_j \cos\lambda_j\vartheta , \tag{3.1}$$

where the λ_j are positive integers such that

$$\lim_{j\to+\infty}(\lambda_{j+1} - \lambda_j) = +\infty ; \tag{3.2}$$

(c) *$f(\vartheta)$ is bounded for $\delta \leqq |\vartheta| \leqq \pi$ for every $\delta > 0$;*

(d) *$f(\vartheta)$ does not belong to any class $L^q(-\pi, \pi)$ with $q > 2$.*

PROOF. Put

$$C_k(\vartheta) = \sum_{j=1}^{d_k} \cos(n_k + m_{kj} + \nu_{kj})\vartheta , \tag{3.3}$$

and

$$f(\vartheta) \sim \sum_{k=4}^{\infty} \frac{C_k(\vartheta)}{d_k^{(1/2+\varepsilon_k)}} , \tag{3.4}$$

where n_k, d_k, and m_{kj} are positive integers, defined as follows: $n_1 = 4$, $n_{k+1} = 2^{n_k}$ $(k = 1, 2, \cdots)$; $d_k = n_k/n_{k-1}^5$ $(k = 4, 5, \cdots)$ [evidently d_k is an integer for each $k \geqq 4$]; and $m_{kj} = (j-1)n_{k-1}^4$ $(j = 1, \cdots, d_k;\ k = 4, 5, \cdots)$. Let ε_k be defined by

$$\varepsilon_k = \frac{n_{k-2}}{n_{k-1}} \quad \text{for } k = 4, 5, \cdots .$$

Suppose also that ν_{kj} $(k = 1, 2, \cdots; j = 1, 2, \cdots, d_k)$ are independent random variables having the distribution

$$P(\nu_{kj} = l) = \binom{2s_k}{s_k + l}\frac{1}{2^{2s_k}} \quad (l = 0, \pm 1, \cdots, \pm s_k)\,, \tag{3.5}$$

where $s_k = n_{k-1}^3$. Clearly, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(\vartheta)\,d\vartheta = \sum_{k=4}^{\infty}\frac{d_k}{d_k^{1+4\varepsilon_k}} = \sum_{k=4}^{\infty}\frac{1}{\exp\{\log 2(4n_{k-2}^2 - 20n_{k-2/n_{k-1}})\}}\,. \tag{3.6}$$

Since $\lim_{k\to+\infty} n^2_{k-2/n_{k-1}} = 0$, it follows that $f(\vartheta) \in L^2(-\pi, \pi)$. Let us put

$$N_k = n_k^{3/2} = 2^{3\cdot 2^{n_{k-2}-1}} \tag{3.7}$$

and consider the polynomial $C_k(\vartheta)$ for $\vartheta = \vartheta_h = \pi h/N_k$, where h is any integer such that $N_k\delta_k \leqq |h| \leqq N_k$ and $\delta_k = 1/n_{k-1}$. Clearly, $d_k^{\varepsilon_k} \geqq 2$. Since $\pi^2 > 8$, we have $8 \log d_k/\vartheta^2 \leqq s_k$ for $k \geqq 4$. Thus we may apply our Lemma and obtain

$$P(|C_k(\vartheta_h)| \geqq d_k^{(1/2)+\varepsilon_k}) \leqq 2\exp\{-d_k^{2\varepsilon_k}/16\} \tag{3.8}$$

for $N_k\delta_k \leqq |h| \leqq N_k$, and therefore

$$P(\max_{N_k\delta_k \leqq |h| N_k} |C_k(\vartheta_h)| \geqq d_k^{(1/2)+\varepsilon_k}) \leqq 4N_k \exp\{-d_k^{2\varepsilon_k}/16\}\,. \tag{3.9}$$

Now evidently for $|\vartheta_h - \vartheta| \leqq \pi/N_k$ we have, in view of $m_{kd_k} + s_k < n_k$,

$$|C_k(\vartheta) - C_k(\vartheta_h)| \leqq \frac{\pi}{N_k}\max_{\vartheta}|C_k'(\vartheta)| \leqq \frac{2\pi d_k n_k}{N_k} < d_k^{1/2}\,, \tag{3.10}$$

and thus we obtain

$$P(\max_{\pi\delta_k \leqq |\vartheta| \leqq \pi}|C_k(\vartheta)| \geqq \tfrac{1}{2}d_k^{(1/2)+\varepsilon_k}) \leqq 4N_k\exp\{-d_k^{2\varepsilon_k}/16\}\,. \tag{3.11}$$

As the series $\sum_{k=4}^{\infty} N_k \exp\{-d_k^{2\varepsilon_k}/16\}$ is clearly convergent, it follows by the Borel-Cantelli lemma that with probability 1 the inequalities

$$\max_{\pi\delta_k \leqq |\vartheta| \leqq \pi}|C_k(\vartheta)| < \tfrac{1}{2}d_k^{(1/2)+\varepsilon_k} \tag{3.12}$$

are satisfied except for a finite number of values of k; i.e., inequality (3.12) holds for almost all series of the form (3.2) (equally for almost all choices of the random integers ν_{kj}).

It follows, in view of the convergence of the series $\sum 1/d_k^{\varepsilon_k}$ and of $\delta_k \to 0$, that the series (3.2) is convergent for every ϑ with $0 < |\vartheta| \leqq \pi$, also that the series is uniformly convergent in every closed subinterval of $[-\pi, \pi]$ which does not contain the point $\vartheta = 0$, and that its sum is bounded in every such interval. Thus we have already shown that almost all functions $f(\vartheta)$ defined by (3.2) satisfy conditions (a) and (c) of our Theorem. It is easy to see that condition (b) is always satisfied; as a matter of fact, the gaps of $C_k(\vartheta)$ are all at least equal to $n_{k-1}^4 - 2n_{k-1}^3$, and the gap between the greatest exponent of $C_{k-1}(\vartheta)$ and the least exponent of $C_k(\vartheta)$ is at least

$n_k - 2n_{k-1}^3$, and thus (3.2) is evidently satisfied. We even have $\lambda_j - \lambda_{j-1} > c\log^4\lambda_j$ $(c > 0)$. It remains to show that $f(\vartheta)$ also has property (d).

To prove this let us consider $f(\vartheta)$ for $\pi/n_k \leqq \vartheta \leqq 5\pi/4n_k$. For such values of ϑ we obviously have

$$\sum_{j=1}^{k-1} \frac{C_j(\vartheta)}{d_j^{(1/2)+2\varepsilon_j}} < kd_{k-1}^{1/2} < n_{k-1}^{1/2} . \tag{3.13}$$

We have also in view of $\pi/n_k \geqq \pi\delta_{j+1} = \pi/n_j$ for $j \geqq k$, with probability 1 for k sufficiently large,

$$\sum_{j=k+1}^{\infty} \frac{C_j(\vartheta)}{d_j^{(1/2)+2\varepsilon_j}} < \sum_{j=k+1}^{\infty} \frac{1}{d_j^{\varepsilon_j}} < 1 . \tag{3.14}$$

Finally, we have for $\pi/n_k \leqq \vartheta \leqq 5\pi/4n_k$ and for sufficiently large k

$$|C_k(\vartheta)| \geqq \frac{d_k}{\sqrt{2}} - 2\sum_{j=1}^{d_k} |\sin(m_{kj} + \nu_{kj})\vartheta| \geqq \frac{d_k}{\sqrt{2}} - \frac{20\pi}{n_{k-1}} \geqq \frac{d_k}{2} . \tag{3.15}$$

Thus we obtain from (3.13)-(3.15)

$$|f(\vartheta)| \geqq \tfrac{1}{2} d_k^{(1/2)-2\varepsilon_k} - 1 - n_{k-1}^{1/2} > \tfrac{1}{4} d_k^{(1/2)-2\varepsilon_k} \tag{3.16}$$

for $\pi/n_k \leqq \vartheta \leqq 5\pi/4n_k$ and sufficiently large k.

Now let $q > 2$ be arbitrary but fixed. It follows that for sufficiently large k one has $q(\frac{1}{2} - 2\varepsilon_k) > 1 + \rho$, where $\rho = (q-2)/4 > 0$, and thus

$$\int_{\pi/n_k}^{5\pi/4n_k} |f(\vartheta)|^q \, d\vartheta > \frac{\pi n_k^{\rho}}{4^{q+1} n_{k-1}^{5(1+\rho)}} . \tag{3.17}$$

As the right-hand side of (3.16) tends to $+\infty$ for $k \to +\infty$, it follows that

$$\int_{-\pi}^{\pi} |f(\vartheta)|^q \, d\vartheta = +\infty. \quad \text{Q.E.D.}$$

One can even prove somewhat more. As a matter of fact, it follows from (3.16) that if $\alpha > 9$,

$$\lim_{k\to+\infty} \int_{\pi/n_k}^{5\pi/4n_k} f^2(\vartheta)\log^{\alpha}|f(\vartheta)| \, d\vartheta = +\infty .$$

Thus not even $f^2(\vartheta)\log^{9+\varepsilon}|f(\vartheta)|$ is integrable if $\varepsilon > 0$.

4. Some Remarks on Additional Problems

It seems that our method is not applicable in the case $1 < q < 2$. As a matter of fact, to settle this case one has to consider series of the form (1.1) with $\sum(a_k^2 + b_k^2) = +\infty$, and in this case if we choose the exponents at random we cannot even be sure of obtaining a Fourier series.

Another open question is how rapidly the exponents of a series of the form (1.1) can increase so that the series will still have properties (a)-(d). As is well known (see [6]), if $\lambda_{k+1}/\lambda_k \geqq \lambda > 1$, that is, if (1.1) is a series with Hadamard gaps, then if $f(\vartheta)$ belongs to L^2 (i.e., $\sum(a_k^2 + b_k^2) < +\infty$), it also belongs to L^q for every q. Thus certainly λ_k cannot increase exponentially. By modifying our construction, we could get series having properties (a) — (d) with $\lambda_{j+1} - \lambda_j > (\log \lambda_j)^A$ with arbitrarily large A. Our method is not applicable in the case where $\lambda_{j+1} - \lambda_j > \lambda_j^B$ with $0 < B < 1$. It may be true, however, that if the function is bounded in a subinterval (α, β), then it belongs to $L^q(-\pi, \pi)$ for some $q > 2$ if $\lambda_{j+1} - \lambda_j > \lambda_j^\alpha$ for some $\alpha > 0$. If $\lambda_{j+1} - \lambda_j > \lambda_j^\alpha$ for every $\alpha < 1$ if $j > j_0(\alpha)$, then perhaps it belongs to $L^q(-\pi, \pi)$ for every q.

Finally, we may ask whether for every function $\omega(x)$ tending monotonically to $+\infty$ for $x \to +\infty$ there exists a function $f(\vartheta)$ which has properties (a), (b), and (c) and is such that $f^2(\vartheta)\omega(|f(\vartheta)|)$ is not integrable. As we mentioned at the end of Sec. 3, our functions $f(\vartheta)$ are such that

$$\int_{-\pi}^{\pi} f^2(\vartheta) \log^\alpha |f(\vartheta)| \, d\vartheta$$

is divergent for $\alpha > 9$. By modifying the construction, the value 9 could be replaced by a smaller one, but our method is not suitable to deal with the case of arbitrarily slowly increasing functions $\omega(x)$.

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REFERENCES

[1] PÓLYA, G., Lücken und Singularitäten von Potenzreihen, *Math. Zeit.*, **29** (1929), 549-640.

[2] WIENER, N., A Class of Gap Theorem, *Annali di Pisa*, **3** (1934), 367-72.

[3] INGHAM, A. E., Some Trigonometric Inequalities with Applications to the Theory of Series, *Math. Zeit.*, **41** (1936), 367-69.

[4] TURÁN, P., On an Inequality, *Annal. Univ. Sci. R. Eötvös*, Budapest, **1** (1958), 3-6.

[5] MARCZINKIEWICZ, J., and A. ZYGMUND, Proof of a Gap Theorem, *Duke Math. J.*, **4** (1938), 469-72.

[6] ZYGMUND, A., *Trigonometric Series*. Cambridge: Cambridge Univ. Press, 1959.

[7] ERDÖS, P., and A. RÉNYI, On Singular Radii of Power Series, *Pub. Math. Inst. Hung. Acad. Sci.*, **3** (1958), 159-70.

[8] ERDÖS, P., and A. RÉNYI, Probabilistic Approach to Some Problems of Diophantine Approximation, *Illinois J. Math.*, **1** (1957), 303-15.

[9] BERNSTEIN, S., *Teoria Veroiatnostei* (Theory of probability), 4th ed. Moscow: 1946.

[10] NORDLANDER, A., Studies in Exponential Polynomials (Ph.D. Dissertation), Univ. Uppsala, Sweden, 1961.