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Covering space with convex bodies

by

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1. A few years ago Rogers [1] showed that, if K is any convex body in n-dimensional Euclidian space, there is a covering of the whole space by translates of K with density less than

$$n\log n + n\log\log n + 5n\,,$$

provided $n \geqslant 3$. However the fact that the covering density is reasonably small does not imply that the maximum multiplicity is also small. In the natural covering of space by closed cubes, the density is 1, but each cube vertex is covered 2^n times. Our object in this note is to prove that, provided n is sufficiently large, there is, for each convex body K, a covering with density less than

$$n\log n + n\log\log n + 4n\,,$$

and such that no point is covered more than

$$e\{n\log n + n\log\log n + 4n\}$$

times. By dimension theory, some points must be covered $n+1$ times.

2. In this section we take K to be a Lebesgue measurable set with finite positive measure V. Further let Λ be the lattice of all points with integral coordinates, and suppose all the distinct translates of K by the vectors of Λ are disjoint.

We suppose that N points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ are chosen at random in the cube C of points $\mathbf{x}$ with

$$0 \leqslant x_1 < 1,\ 0 \leqslant x_2 < 1,\ \dots,\ 0 \leqslant x_n < 1\,,$$

and investigate the average density of the set of points covered exactly k times by the system of sets

$$K + \mathbf{x}_i + \mathbf{g} \quad (1 \leqslant i \leqslant N,\ \mathbf{g} \in \Lambda)\,. \tag{1}$$

Let $\varrho(x)$ be the characteristic function of the set K. Then

$$\sigma_i(\mathbf{x}) = \sum_{g \in \Lambda} \varrho(\mathbf{x} - \mathbf{x}_i - \mathbf{g})$$

is the characteristic function of the set

$$\Sigma_i = \bigcup_{g \in \Lambda} \{K + \mathbf{x}_i + \mathbf{g}\},$$

for $1 \leqslant i \leqslant N$.

Suppose $0 \leqslant k \leqslant N$ and consider the function

$$\tau_k(\mathbf{x}) = \sum_{h=k}^{N} \frac{h!(-1)^{h-k}}{k!(h-k)!} \sum_{i_1,\dots,i_h}' \Bigl(\prod_{s=1}^{h} \sigma_{i_s}(\mathbf{x})\Bigr),$$

where the sum $\sum'$ is taken over all selections of h distinct integers from $1, 2, \dots, N$. Suppose that $\mathbf{x}$ is a point of space, which belongs to just r of the sets (1). Then $\mathbf{x}$ belongs to just r of the sets $\Sigma_1, \Sigma_2, \dots, \Sigma_N$. Thus, if $r < k$, we have $\tau_k(\mathbf{x}) = 0$, and, if $r \geqslant k$, we have

$$\begin{aligned} \tau_k(\mathbf{x}) &= \sum_{h=k}^{r} \frac{h!(-1)^{h-k}}{k!(h-k)!} \cdot \frac{r!}{h!(r-h)!} \\ &= \frac{r!}{k!(r-k)!}(1-1)^{r-k} \\ &= 1, \quad \text{if} \quad r = k, \\ &= 0, \quad \text{if} \quad r > k. \end{aligned}$$

Thus $\tau_k(\mathbf{x})$ is the characteristic function of the set E_k of points belonging to just k of the sets (1). Since E_k is periodic with period 1 in each coordinate, its density is equal to the measure of $E_k \cap C$, and so is given by

$$\delta(E_k) = \int_C \tau_k(\mathbf{x})\, d\mathbf{x}.$$

The mean value $\mathcal{M}(\delta(E_k))$ of this density, taken over all choices of the points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ in C is

$$\mathcal{M}(\delta(E_k)) = \int_C \dots \int_C \Bigl\{\int_C \tau_k(\mathbf{x})\, d\mathbf{x}\Bigr\} d\mathbf{x}_1 \dots, d\mathbf{x}_N.$$

But, using Fubini's theorem,

$$\begin{aligned} &\int_C \dots \int_C \Bigl\{\int_C \prod_{s=1}^{h} \sigma_{i_s}(\mathbf{x}) d\mathbf{x}\Bigr\} d\mathbf{x}_1 \dots d\mathbf{x}_N \\ &= \int_C \Bigl\{\int_C \sigma_1(\mathbf{x})\, d\mathbf{x}_1\Bigr\}^h d\mathbf{x} \\ &= \int_C \Bigl\{\int_C \sum_{g \in \Lambda} \varrho(\mathbf{x} - \mathbf{x}_1 - \mathbf{g})\, d\mathbf{x}_1\Bigr\}^h d\mathbf{x} \\ &= \int_C \Bigl\{\int \varrho(\mathbf{x} - \mathbf{x}_1)\, d\mathbf{x}_1\Bigr\}^h d\mathbf{x} \\ &= V^h. \end{aligned}$$

Hence, the mean value of the density of E_k reduces to

$$\mathcal{M}(\delta(E_k)) = \sum_{h=k}^{N} \frac{h!(-1)^{h-k}}{k!(h-k)!}\cdot\frac{N!}{h!(N-h)!} V^h$$
$$= \frac{N!}{k!(N-k)!} V^k(1-V)^{N-k}.$$

3. In this section we take K to be a convex set in n-dimensional space of volume V, and establish the existence of a covering of space by translates of K, which has both its density and its maximum multiplicity reasonably small. By a result of Rogers and Shephard [2] there is a lattice Λ_1 of determinant $4^n V$ such that the distinct translates of K by the vectors of Λ_1 are disjoint. Thus, after applying a suitable linear transformation to K, we may suppose that the volume V of K is 4^{-n} and that the distinct translates of K, by the vectors of the lattice Λ of points with integral coordinates, are disjoint.

Now take N to be the integer nearest to

$$4^n\{n\log n + n\log\log n + 4n\},$$

and h to be the integer nearest to

$$e\{n\log n + n\log\log n + 4n\}.$$

If F_h is the set of points covered by h or more of the sets of the system

$$K + \mathbf{x}_i + \mathbf{g} \quad (1 \leqslant i \leqslant N,\ \mathbf{g} \in \Lambda),$$

it follows from § 2, that

$$\mathcal{M}(\delta(F_h)) = \sum_{k=h}^{N} \frac{N!}{k!(N-k)!} V^k(1-V)^{N-k}$$
$$= \frac{N!}{h!(N-h)!} V^h(1-V)^{N-h} \sum_{t=0}^{N-h} \frac{h!(N-h)!}{(h+t)!(N-h-t)!}\left(\frac{V}{1-V}\right)^t.$$

In this sum the ratio of the $(t+1)$-st term to the t-th term is

$$\frac{N-h-t}{h+t+1}\cdot\frac{V}{1-V} \leqslant \frac{N-h}{h+1}\cdot\frac{V}{1-V} < 1.$$

Hence

$$\mathcal{M}(\delta(F_h)) \leqslant \frac{N!}{h!(N-h)!} V^h(1-V)^{N-h} \sum_{t=0}^{\infty}\left(\frac{N-h}{h+1}\cdot\frac{V}{1-V}\right)$$
$$= \frac{N!}{h!(N-h)!} V^h(1-V)^{N-h} \frac{(h+1)(1-V)}{(h+1)-(N+1)V}.$$

So, using Stirling's formula,

$$\begin{aligned}\log\mathcal{M}\big(\delta(F_h)\big) \leqslant (N-h)\log\left(1+\frac{h}{N-h}\right)-h\log\frac{h}{VN}+ \\ +(N-h)\log(1-V)- \\ -\tfrac{1}{2}\log\left(1-\frac{h}{N}\right)-\log\left(1-\frac{(N+1)V}{h+1}\right)+O(1)\,.\end{aligned}$$

By the choice of V, N and h, this yields

$$\log\mathcal{M}\big(\delta(F_h)\big)\leqslant -n\log n-n\log\log n-4n+O(1)\,.$$

Similarly, if

$$\eta=\frac{1}{2n\log n}\,,$$

the mean value of the density of the set E_0' of points belonging to no set of the system

$$(1-2\eta)K+\mathbf{x}_i+\mathbf{g} \quad (1\leqslant i\leqslant N,\ \mathbf{g}\in\Lambda)\,,$$

is

$$\mathcal{M}\big(\delta(E_0')\big)=\big(1-(1-2\eta)^n V\big)^N;$$

so that

$$\begin{aligned}\log\mathcal{M}\big(\delta(E_0')\big) &= N\log\left(1-4^{-n}\left(1-\frac{1}{n\log n}\right)^n\right) \\ &\leqslant -N4^{-n}\left(1-\frac{1}{n\log n}\right)^n \\ &= -N4^{-n}\left\{1-\frac{1}{\log n}+O\left(\left(\frac{1}{\log n}\right)^2\right)\right\} \\ &= -n\log n-n\log\log n-3n+O\left(\frac{n\log\log n}{\log n}\right).\end{aligned}$$

Provided n is sufficiently large, we have

$$\log\mathcal{M}\big(\delta(F_h)\big)<-n\log n-n\log\log n-n\log 8-\log 2\,,$$

$$\log\mathcal{M}\big(\delta(E_0')\big)<-n\log n-n\log\log n-n\log 8-\log 2\,.$$

This ensures that

$$\mathcal{M}\big(\delta(F_h)\big)+\mathcal{M}\big(\delta(E_0')\big)<\left(\frac{1}{8n\log n}\right)^n=\eta^n V\,.$$

Thus we can choose the points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ so that

(2) $$\delta(F_h)+\delta(E_0')<\eta^n V\,.$$

We prove that the system of sets

$$(1-\eta)K+\mathbf{x}_i+\mathbf{g} \quad (1 \leqslant i \leqslant N,\ \mathbf{g} \in \Lambda), \tag{3}$$

forms a covering of the whole of space with the required properties.

Let $\mathbf{x}$ be any point of space. Consider the system of sets

$$-\eta K+\mathbf{x}+\mathbf{g} \quad (\mathbf{g} \in \Lambda).$$

No two distinct sets of the system have any common point. So the density of the system is

$$\eta^n V > \delta(E_0').$$

Hence there will be a point of a set of the system which does not belong to E_0'. Thus, for some i with $1 \leqslant i \leqslant N$, for some points $\mathbf{k}_1, \mathbf{k}_2$ of K, and for some points $\mathbf{g}_1, \mathbf{g}_2$ of Λ, we have

$$-\eta\mathbf{k}_1+\mathbf{x}+\mathbf{g}_1 = (1-2\eta)\mathbf{k}_2+\mathbf{x}_i+\mathbf{g}_2.$$

Consequently

$$\mathbf{x} = (1-2\eta)\mathbf{k}_2+\eta\mathbf{k}_1+\mathbf{x}_i+(\mathbf{g}_2-\mathbf{g}_1)$$

belongs to the set

$$(1-\eta)K+\mathbf{x}_i+(\mathbf{g}_2-\mathbf{g}_1)$$

of the system (3). This shows that the system (3) covers the whole of space.

Now suppose that a point $\mathbf{x}$ of space was covered h or more times by the system of sets (3). Then each point of the set

$$\eta K+\mathbf{x}$$

is covered at least h times by the sets of the system (1). So F_h contains the union

$$\bigcup_{g \in \Lambda} \{\eta K+\mathbf{x}+\mathbf{g}\}.$$

But this set has density $\eta^n V$. Hence $\delta(F_n) \geqslant \eta^n V$ contrary to (2). This shows that no point of space is covered by the system (3) with multiplicity h or more, and completes the proof.

References

[1] C. A. Rogers *A note on coverings*, Mathematika 4 (1957) , pp. 1-6.

[2] C. A. Rogers and G. C. Shephard, *The difference body of a convex body*, Archiv der Mathematik, 8 (1957), pp. 220-233, § 5.

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