ON SETS OF DISTANCES OF n POINTS IN EUCLIDEAN SPACE

by

P. ERDŐS

Let $[P_n^{(k)}]$ be the class of all subsets $P_n^{(k)}$ of the k dimensional space consisting of n distinct points and having diameter 1. Denote by $g_k(n, r)$ the maximum number of times a given distance r can occur among n points of a set $P_n^{(k)}$. Put

$$G_k(n) = \max_r g_k(n, r), \quad g_k(n) = g_k(n, 1)$$

(i. e. $g_k(n)$ denotes the maximum number of times the diameter can occur as a distance among n points of k dimensional space and $G_k(n)$ denotes the maximum number of times the same distance can occur between n suitably chosen points in k dimensional space). It is well known [1] that $g_2(n) = n$ and I [2] proved that

$$(1) \qquad n^{1+c/\log\log n} < G_2(n) < n^{3/2} .$$

Further I conjectured that $G_2(n) < n^{1+\varepsilon}$ for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$. VÁZSONYI conjectured that $g_3(n) = 2n - 2$ and this was proved simultaneously and independently by GRÜNBAUM [3], HEPPES [4] and STRASZEWICZ [5] (all using similar methods). I am going to prove

$$(2) \qquad c_1 n^{4/3} < G_3(n) < c_2 \cdot n^{5/3} .$$

Perhaps $G_3(n) < n^{4/3+\varepsilon}$ holds for all $n > n(\varepsilon)$.

One could have expected that $G_k(n) = o(n^2)$ and $g_k(n) < c_k \cdot n$ for every k. In 1955 LENZ showed that this is not so. In fact LENZ showed that (LENZ's result is unpublished)

$$(3) \qquad g_4(n) \geq \left[\frac{n^2}{4}\right].$$

The proof of LENZ is very simple. Put $s = \left[\frac{n}{2}\right]$ and consider the following n points in four-dimensional space:

$$(x_i, y_i, 0, 0),\ 1 \leq i \leq s,\ (0, 0, x_j, y_j),\ s+1 \leq j \leq n$$

where $0 < x_i, x_j, y_i, y_j < \frac{1}{\sqrt{2}}$, $x_i^2 + y_i^2 = \frac{1}{2}$, $x_j^2 + y_j^2 = \frac{1}{2}$. Clearly all the

$s(n-s) = \left[\frac{n^2}{4}\right]$ distances between the points $(x_i, y_i, 0, 0)$ and $(0, 0, x_j, y_j)$ is 1 (and 1 is the diameter of the set $(x_i, y_i, 0, 0)$; $(0, 0, x_j, y_j)$).

By a slight modification of this method LENZ in fact proved that $g_4(n) > \frac{n^2}{4} + c_3 n$ for a certain $c_3 > 0$. LENZ then asked: what is the limit of $g_k(n)/n^2$ as $n \to \infty$? In this note I am going to prove the following

Theorem. *For every $k \geq 4$*

$$\lim_{n\to\infty} g_k(n)/n^2 = \lim_{n\to\infty} G_k(n)/n^2 = \frac{1}{2} - \frac{1}{2\left[\frac{k}{2}\right]}.$$

Clearly $g_k(n) \leq G_k(n)$ and $g_k(n) \leq g_{k+1}(n)$, $G_k(n) \leq G_{k+1}(n)$. Thus to prove our Theorem it will suffice to show that for every $l \geq 2$

$$\lim_{n\to\infty} g_{2l}(n)/n^2 \geq \frac{1}{2} - \frac{1}{2l} \tag{4}$$

and

$$\lim_{n\to\infty} G_{2l+1}(n)/n^2 \leq \frac{1}{2} - \frac{1}{2l}. \tag{5}$$

The proof of (4) is trivial generalization of the proof of LENZ. For each t, $1 \leq t \leq l$ denote by I_t the group of $\left[\frac{n}{l}\right]$ points whose first $2t-2$ coordinates are 0 the $2t-1$-th and $2t$-th coordinates are x_i, y_i, $1 \leq i \leq \left[\frac{n}{l}\right]$, $x_i, y_i > 0$, $x_i^2 + y_i^2 = \frac{1}{2}$ and the remaining $2l-2t$ coordinates are 0. Clearly for any $t_1 \neq t_2$ the distance between any two points of I_{t_1} and I_{t_2} is 1 and the set $\bigcup_{1 \leq t \leq l} I_t$ has diameter 1. Thus

$$g_{2l}(n) \geq \binom{l}{2}\left[\frac{n}{l}\right]^2 = \frac{n^2}{2}\left(1 - \frac{1}{l}\right) + O(n)$$

which clearly implies (4).

Next we prove (5). If (5) is not true then there exists an $\varepsilon > 0$ so that for a certain $l \geq 2$ and infinitely many n_s

$$G_{2l+1}(n_s) > \left(\frac{1}{2} - \frac{1}{2l} + \varepsilon\right) n_s^2 = A(n_s).$$

In other words there exists a set $P_{n_s}^{(2l+1)}$ in $2l+1$ dimensional space and a distance r which occurs among at least $A(n_s)$ pairs of points of $P_{n_s}^{(2l+1)}$. Connect any two points of $P_{n_s}^{(2l+1)}$ whose distance is r. Thus we obtain a graph of n_s vertices and $A(n_s)$ edges. By a theorem of A. H. STONE and myself[1] [6] this graph contains for sufficiently large $n_s = n_s(\varepsilon)$ a subgraph of $3(l+1)$ vertices $x_i^{(t)}$ $1 \leqq i \leqq 3$, $1 \leqq t \leqq l+1$ so that any two vertices $x_{i_1}^{(t_1)}$ and $x_{i_2}^{(t_2)}$ are connected by an edge if $t_1 \neq t_2$ (in other words the distance between $x_{i_1}^{(t_1)}$ and $x_{i_2}^{(t_2)}$ is r if $t_1 \neq t_2$). But, then a simple geometrical argument shows that the $l+1$ planes $(x_1^{(t)}, x_2^{(t)}, x_3^{(t)})$, $1 \leqq t \leqq l+1$ must be mutually perpendicular, which implies that the dimension of the space spanned by the $x_i^{(t)}$ is at least $2l+2$. This contradiction proves (5) and thus the proof of our Theorem is complete.

By a sharpening which I recently obtained of the result of ***STONE*** and myself I can prove

$$G_k(n) < \left(\frac{1}{2} - \frac{1}{2\left[\frac{k}{2}\right]}\right) n^2 + O(n^{2-\varepsilon_k}) \tag{6}$$

where $\varepsilon_k \to 0$ as $k \to \infty$. I do not know how close (6) is to the true order of magnitude of $G_k(n)$. Perhaps the result of LENZ

$$G_k(n) > \left(\frac{1}{2} - \frac{1}{2\left[\frac{l}{k}\right]}\right) n^2 + c_k n \tag{7}$$

gives the right order of magnitude.

Now we are going to prove (2). First we prove the upper estimate. Let $x_1, x_2, \dots, x_n$ be n points in three dimensional space, assume that there are a_i points at distance r from x_i. Clearly to any three points $x_{j_1}, x_{j_2}, x_{j_3}$ there can be at most two points x_l at distance r. Thus since the total number of triplets $(x_{j_1}, x_{j_2}, x_{j_3})$ is $\binom{n}{3}$ a simple argument gives

$$\sum_{i=1}^{n} \binom{a_i}{3} \leqq 2\binom{n}{3}$$

or

$$\sum_{i=1}^{n} a_i^3 < c_4 n^3 . \tag{8}$$

If $\sum_{i=1}^{n} a_i^3$ is given $\sum_{i=1}^{n} a_i$ is maximal if all the a_i are equal. Thus (8) implies

$$\sum_{i=1}^{n} a_i < c_2 n^{5/2}$$

which proves the upper bound in (2).

---

[1] The theorem in question states as follows: To every ε, $r \geq 2$ and l there exists an $n_0(\varepsilon, r, l)$ so that if $n > n_0(\varepsilon, r, l)$ and G_n is a graph of n vertices and more than $n^2\left(\frac{1}{2} - \frac{1}{2(r-1)} + \varepsilon\right)$ edges then G_n contains rl vertices $x_i^{(t)}$ $1 \leqq i \leqq l$, $1 \leqq t \leqq r$ so that for every $t_1 \neq t_2$, $x_{i_1}^{(t_1)}$ and $x_{i_2}^{(t_2)}$ are connected by an edge for every $1 \leqq i_1, i_2 \leqq l$.

To prove the lower bound in (2) consider the points (x, y, z) of integer coordinates $0 \leq x, y, z \leq [n^{1/3}]$. Clearly the number of these points is less than n but is greater than $n(1 - \varepsilon)$. The square of the distance between two of these points is of the form

$$u^2 + v^2 + w^2, \quad 0 \leq u, v, w \leq n^{1/3} \tag{9}$$

The numbers (9) are all less than or equal $3n^{2/3}$ and since there are more than $\binom{n(1-\varepsilon)}{2}$ such distances, clearly for some r the same distance must occur at least $1/7 n^{4/3}$ times, which completes the proof of (2). From deep number theoretic results it follows that for suitable r the same distance occurs more than $c_5 n^{4/3} \log\log n$ times and this is the **best** lower bound I can get for $G_3(n)$ at the present time.

(Received December 18, 1959.)

REFERENCES

[1] Aufgabe 167. *Jahresbericht der Deutschen Math. Vereinigung* **43** (1934) 114.

[2] ERDŐS, P.: „On sets of distaneos of n points." *Amer. Math. Monthly* **53** (1946) 248—250.

[3] GRÜNBAUM, B.: "A proof of Vázsonyi's conjecture," *Bull. Research Council of Israel* **6A** (1956) 77-78.

[4] HEPPES, A.: „Beweis einer Vermutung von A. Vázsonyi." *Acta Math. Acad. Sci. Hung.* 7 (1957) 463-466.

[5] STRASZEWICZ, S.: "Sur un problème geometrique de P. Erdős." *Bull. Acad. Poll. Sci. Cl III.* 5 (1957) 39—40.

[6] ERDŐS, P. and STONE, A. H.: "On the structure of linear graphs." *Bull. Amer. Math. Soc.* 52 (1946) 1087—1091.

О РАССТОЯНИЯХ МЕЖДУ n ТОЧКАМИ ЭВКЛИДОВА ПРОСТРАНСТВА

P. ERDŐS

Резюме

Пусть $P_n^{(k)}$ есть множество, состоящее из n точек k-мерного пространства, диаметр которого равен 1. Обозначим через $g_k(n, r)$ максимальное число пар точек (x_i, x_j), для которых расстояние x_i и x_j равно r.

$$G_k(n) = \max_{(r)} g_k(n, r)\,; \quad g_k(n) = g_k(n, 1)\,.$$

Раньше автор доказал, что

$$n^{1-c_1/\log\log n} < G_2(n) < n^{3/2}.$$

Было известно, что $g_2(n) = n$. Grünbaum, Heppes и Straszewicz доказали гипотезу Vázsonyi, согласно которой $g_3(n) = 2n - 2$. Lenz доказал, что

$$g_4(n) > \frac{n^2}{4} + c_2 n \, .$$

В настоящей статье автор доказывает, что

$$c_3 n^{4/3} < G_3(n) < c_4 n^{5/3}$$

и, если $k \geqq 4$, то

$$\lim_{n\to\infty} g_k(n)/n^2 = \lim_{n\to\infty} G_k(n)/n^2 = \frac{1}{2} - \frac{1}{2\left[\frac{k}{2}\right]} \, .$$