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Distributions of the values of some arithmetical functions

by

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§ 1. Y. Wang and A. Schinzel proved, by Brun's method, the following theorem ([3]):

For any given sequence of h non-negative numbers $a_1, a_2, \ldots, a_h$ and $\varepsilon > 0$, there exist positive constants $c = c(a, \varepsilon)$ and $x_0 = x_0(a, \varepsilon)$ such that the number of positive integers $n \leqslant x$ satisfying

$$\left|\frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i\right| < \varepsilon \quad (1 \leqslant i \leqslant h)$$

is greater than $cx/\log^{h+1}x$, whenever $x > x_0$.

They also proved the analogous theorem for the function σ.

Shao Pin Tsung, also using Brun's method, extended this result to all multiplicative positive functions $f_s(n)$ satisfying the following conditions ([4]):

I. *For any positive integer l and prime number p:*

$$\lim_{p\to\infty} \left(f_s(p^l)/p^{ls}\right) = 1 \quad (p \text{ denotes primes}).$$

II. *There exists an interval $\langle a, b\rangle$, $a = 0$ or $b = \infty$, such that for any integer $M > 0$ the set of numbers $f_s(N)/N^s$, where $(N, M) = 1$, is dense in $\langle a, b\rangle$.*

(This formulation is not the same but equivalent to the original one.)

In this paper we shall show without using Brun's method that *if we replace the condition* I *by the condition*

$$\sum \frac{\left(f_s(p)-p^s\right)^2}{p^{2s+1}} < \infty$$

(but preserving condition II*) then there exist more than $C(a, \varepsilon)x$ posi-*

tive integers $n \leqslant x$ *for which*

$$\left|\frac{f_s(n+i)}{f_s(n+i-1)} - a_i\right| < \varepsilon \quad (i = 1, 2, \ldots, h).$$

This theorem follows easily from the following stronger theorem.

THEOREM 1. *Let* $f(n)$ *be an additive function, satisfying the following conditions*

1. $\sum_p (\|f(p)\|^2/p)$ *is convergent, where* $\|f\|$ *denotes* $f(p)$ *for* $|f(p)| \leqslant 1$ *and* 1 *for* $|f(p)| > 1$.

2. *There exists a number* c_1 *such that, for any integer* $M > 0$, *the set of numbers* $f(N)$, *where* $(N, M) = 1$ *is dense in* (c_1, ∞).

Then, for any given sequence of h *real numbers* $a_1, a_2, \ldots, a_h$ *and* $\varepsilon > 0$, *there exist more than* $C(a, \varepsilon)x$ *positive integers* $n \leqslant x$ *for which*

$$(1) \qquad |f(n+i) - f(n+i-1) - a_i| < \varepsilon \quad (i = 1, 2, \ldots, h);$$

$C(a, \varepsilon)$ *is a positive constant, depending on* ε *and* a_i.

LEMMA. *There exists an absolute constant* c *such that the number of the integers of the form* $pq > x$ *for which one can find* $n \leqslant x$ *satisfying* $n \equiv b \pmod{a}$, $n \equiv 0 \pmod{p}$ *and* $n+1 \equiv 0 \pmod{q}$ *is for* $x > x_0(a)$ *less than* cx/a.

Proof. Let $c_1, c_2, \ldots$ denote absolute constants. Assume $p > x^{1/2}$ ($q > x^{1/2}$ can be dealt similarly). Denote by $A_l(x)$ the number of integers of the form pq satisfying

$$pq > x, \quad x^{1-1/2^l} \leqslant p < x^{1-1/2^{l+1}}, \quad n \equiv b \pmod{a}, \quad p|n,\ q|n+1,$$
$$\text{for some } n,\ 1 \leqslant n \leqslant y,$$

and by $A_l'(x)$ the number of integers pq for which

$$x^{1-1/2^l} \leqslant p < x^{1-1/2^{l-1}}, \quad q > x^{1/2^{l+1}}, \quad n \equiv b \pmod{a}, \quad p|n, \quad q|n+1,$$
$$\text{for some } n, \quad 1 \leqslant n \leqslant x.$$

Clearly $A_l'(x) \geqslant A_l(x)$ and it will suffice to prove that for $x > x_0(a)$, $\sum_{l=1}^{\infty} A_l'(x) < cx/a$.

Define positive integer l_x by the inequality

$$2^{l_x} \geqslant \frac{1}{a}\log x > 2^{l_x-1}.$$

The number k of integers n satisfying

$$(2) \quad n \leqslant x, \quad n \equiv b \pmod{a}, \quad n \equiv 0 \pmod{p}, \quad x^{1-1/2^l} < p < x^{1-1/2^{l+1}}$$

for an $l \geqslant l_x$ does not exceed $\sum_{x \geqslant p > x^{1-2^{-l_x}}} \left(\left[\frac{x}{pa}\right]+1\right)$, thus by theorems of Mertens and Chebyshev

$$k < \frac{c_1 x}{a 2^{l_x}} + \frac{c_2 x}{\log x}$$

and by the definition of l_x

$$k < \frac{c_3 x}{\log x}.$$

Denote the numbers satisfying (2) for an $l \geqslant l_x$ by $a_1 < a_2 < \ldots < a_k \leqslant x$. Since for all $y \leqslant x$, $\nu(y) < c_4 \log x / \log\log x$ (from the prime number theorem or from more elementary results), we have

$$\sum_{l \geqslant l_x} A_l'(x) \leqslant \sum_{i=1}^{k} \nu(a_i) < \frac{c_3 x}{\log x} \cdot \frac{c_4 \log x}{\log\log x} < \frac{c_5 x}{a} \tag{3}$$

for $x > x_1(a)$.

For $l < l_x$ denote numbers satisfying (2) by $a_1^{(l)} < a_2^{(l)} < \ldots < a_{k_l}^{(l)}$. Similarly as for k we have for k_l the inequality

$$k_l < \frac{c_6 x}{a 2^{l-1}} + \frac{c_2 x}{\log x}$$

hence by $l < l_x$

$$k_l < \frac{c_7 x}{a \cdot 2^l}. \tag{4}$$

We shall prove that for $l < l_x$ and sufficiently large x

$$A_l'(x) = \sum_{i=1}^{k_l} \nu_l(a_i^{(l)}+1) < \frac{c_8 x}{a \cdot l^2} \tag{5}$$

where $\nu_l(m)$ denotes the number of prime factors $> x^{1/2^{l+1}}$ of m.

For this purpose, we split the summands of the sum (5) into two classes. In the first class are the integers $a_i^{(l)}$ for which $\nu_l(a_i^{(l)}+1) \leqslant 2^l/l^2$. From (4) it follows that the contribution of these integers $a_i^{(l)}$ to (5) is less than $c_7 x/al^2$. The integer in the second class satisfy $\nu_l(a_i^{(l)}+1) > 2^l/l^2$. Thus these integers are divisible by more than $2^l/l^2$ primes $q > x^{1/2^{l+1}}$. Thus the number of integers of the second class is less than

$$\frac{x\left(\sum_{x^{1/2^{l+1}} < p \leqslant x} \frac{1}{q}\right)^{[2^l/l^2]}}{[2^l/l^2]!} + \left[\frac{2^l}{l^2}\right]! < \frac{x(c_9 l)^{[2^l/}}{a[2^l/l^2]!} + \left[\frac{2l^{l_x}}{l_x^2}\right]!$$
$$< \frac{x}{a \cdot 5^l} + \left[\frac{2\log x}{a \log\log x}\right]! < \frac{x}{a \cdot 4^l}$$

for $l > c_{10}$, $x > x_2(a)$. By definition, $\nu_l(a_i^{(l)}+1) < 2^{l+1}$. Thus, for $l > c_{10}$, the contribution of the numbers of the second class to (5) is $< x/a\cdot 2^{l-1}$; for $l \leqslant c_{10}$ the contribution is clearly $< 2^{c_{10}+1}x$. Thus, for $l < l_x$, $x > x_2(a)$,

$$A_l'(x) < c_8 x/al^2$$

and in view of (3) we have for $x > x_0(a)$

$$\sum_{l=1}^{\infty} A_l'(x) < \frac{c_5 x}{a} + \sum_{l<l_x} \frac{c_8 x}{al^2} < \frac{cx}{a}$$

which proves the Lemma.

Proof of the theorem. Let ε be a positive number and let a sequence a_i $(i = 1, 2, \ldots, h)$ be given.

By condition 2 we can find positive integers $N_0, N_1, \ldots, N_h$ such that

(6) $\big(N_i, (h+1)!\big) = 1 \quad (i = 0, 1, \ldots, h), \quad (N_i, N_j) = 1 \quad (0 \leqslant i < j \leqslant h),$

$$f(N_0) > c_1 + \max_{1\leqslant i\leqslant h}\Big\{f(i+1) - \sum_{j=1}^{i} a_j\Big\}$$

and

$$\Big|f(N_i) - \Big\{f(N_0) - f(i+1) + \sum_{j=1}^{i} a_j\Big\}\Big| < \tfrac{1}{4}\varepsilon \quad (1 \leqslant i \leqslant h);$$

hence

(7) $$|f((i+1)N_i) - f(iN_{i-1}) - a_i| < \tfrac{1}{2}\varepsilon \quad (1 \leqslant i \leqslant h).$$

Let k_1 be the greatest prime factor of $N_0 N_1 \ldots N_h$. Put $\mu = \varepsilon/\sqrt{96hc}$ (c is the constant of the Lemma). By condition 1, $\sum_{|f(p)|\geqslant\mu} (1/p)$ is convergent. Since $\sum_p (1/p^2)$ is also convergent, there exists a k_2 such that

(8) $$\sum_{\substack{|f(p)|\geqslant\mu\\ p>k_2}} \frac{1}{p} + \sum_{p>k_2} \frac{1}{p^2} < \frac{1}{3(h+1)}.$$

Finally by condition 1 there exists a k_3 such that

(9) $$\sum_{\substack{|f(p)|<\mu\\ p>k_3}} \frac{f(p)^2}{p} < \frac{\varepsilon^2}{48h}.$$

Let us put

$$k = \max(k_1, k_2, k_3), \quad N = N_1 N_2 \ldots N_h, \quad P = \prod_{\substack{p \leqslant k \\ p \nmid N}} p, \;\; Q = (h+1)!\, N^2 P$$

and let us consider the following system of congruences

$$n \equiv 1 \pmod{(h+1)!P}, \quad n \equiv -i + N_i \pmod{N_i^2}, \quad 0 \leqslant i \leqslant h.$$

By (6) and the Chinese Remainder Theorem there exists a number n_0 satisfying these congruences.

It is easy to see that

(10) *for every integer t the numbers $(Qt+n_0+i)/(i+1)N_i$ $(i = 1, 2, \ldots, h)$ are integers which are not divisible by any prime $\leqslant k$;*

(11) *the number of terms not exceeding x of the arithmetical progression $Qt+n_0$ is $x/Q+O(1)$.*

In order to prove Theorem 1 we shall estimate the number of integers n of the progression $Qt+n_0$ which satisfy the inequalities

$$(12) \quad n \leqslant x, \quad \sum_{i=1}^{h} \big(f(n+i) - f(n+i-1) - f((i+1)N_i) + f(iN_{i-1})\big)^2 > \tfrac{1}{4}\varepsilon^2.$$

We divide the set of integers $n \equiv n_0 \pmod{Q}$ for which the inequalities (12) hold into two classes. Integers n such that $n(n+1)\ldots(n+h)$ is divisible by a prime $p > k$ with $|f(p)| \geqslant \mu$, or by p^2, $p > k$, are in the first class and all other integers are in the second class.

(13) *The number of integers $n \leqslant x$, $n \equiv r \pmod{Q}$ which are divisible by a given integer $d > 0$ is equal to $x/dQ+O(1)$ for $(d, Q) = 1$,*

hence the number of integers $n \leqslant x$, $n \equiv n_0 \pmod{Q}$ of the first class is less than

$$(h+1)\frac{x}{Q}\Big(\sum_{\substack{p>k \\ |f(p)| \geqslant \mu}} \frac{1}{p} + \sum_{p>k} \frac{1}{p^2}\Big) + O\Big(\sum_{p \leqslant x+h} 1 + \sum_{p^2 \leqslant x+h} 1\Big).$$

By the inequality (8) and the definition of k this number is less than $\frac{1}{3}x/Q + o(x)$.

For the integers of the second class, by remark (10) we have

$$\sum_{n}'' \sum_{i=1}^{h} \big(f(n+i) - f(n+i-1) - f((i+1)N_i) + f(iN_{i-1})\big)^2$$
$$= S = \sum_{n}'' \sum_{i=1}^{h} \Big\{\sum_{\substack{p \mid n+i \\ p>k}} f(p) - \sum_{\substack{p \mid n+i-1 \\ p>k}} f(p)\Big\}^2,$$

where $\sum''$ means that the summation runs through the integers of the second class. In view of remark (13), since $(Q, p) = 1$ we have

$$S \leqslant \sum_{\substack{n\equiv n_0 (\mathrm{mod}\, Q)\\ n\leqslant x}} \sum_{i=1}^{h} \Big\{ \sum_{\substack{p|n+i\\ p>k, |f(p)|<\mu}} f(p) - \sum_{\substack{p|n+i-1\\ p>k, |f(p)|<\mu}} f(p) \Big\}^2$$

$$= \sum_{\substack{x+h\geqslant p>k\\ |f(p)|<\mu}} f^2(p)\left(\frac{2hx}{Qp}+O(1)\right)+$$

$$+ \sum_{\substack{n\equiv n_0 (\mathrm{mod}\, Q)\\ n\leqslant x}} \sum_{i=1}^{h} \Big\{ 2 \sum_{\substack{pq|n+i,\, q>p>k,\\ |f(p)|<\mu, |f(q)|<\mu}} f(p)f(q) +$$

$$+2 \sum_{\substack{pq|n+i-1,\, q>p>k\\ |f(p)|<\mu, |f(q)|<\mu}} f(p)f(q) - 2 \sum_{\substack{p|n+i,\, q|n+i-1,\, q>k\\ p>k, |f(p)|<\mu, |f(q)|<\mu}} f(p)f(q) \Big\}$$

$$\leqslant \frac{2hx}{Q} \sum_{p>k, |f(p)|<\mu} \frac{f^2(p)}{p} + \sum_{\substack{n\equiv n_0 (\mathrm{mod}\, Q)\\ n\leqslant x}} \sum_{i=1}^{h} 2 \sum_{\substack{p|n+i,\, q|n+i-1\\ pq\geqslant x,\, p>k,\, q>k\\ |f(p)|<\mu, |f(q)|<\mu}} |f(p)f(q)| +$$

$$+O\Big(\sum_{\substack{p\leqslant x+h\\ |f(p)|<\mu}} f^2(p) + \sum_{\substack{p>q>k,\, pq\leqslant x+h\\ |f(p)|<\mu, |f(q)|<\mu}} |f(p)f(q)| \Big).$$

Thus finally from (9), Lemma, the equality $\mu^2 = \varepsilon^2/96hc$ and from the fact that the number of integers of the form pq not exceeding $x+h$ is $o(x)$, we get

$$S < \frac{\varepsilon^2}{12}\cdot\frac{x}{Q} + o(x).$$

Thus the number of integers of the second class is less than $\frac{1}{3}x/Q + o(x)$.

Hence there exist less than $\frac{2}{3}x/Q + o(x)$ positive integers $n \leqslant x$, $n \equiv n_0 \,(\mathrm{mod}\, Q)$ for which

$$\sum_{i=1}^{h} \Big(f(n+i) - f(n+i-1) - f\big((i+1)N_i\big) + f(iN_{i-1})\Big)^2 > \tfrac{1}{4}\varepsilon^2.$$

Therefore by (11) there exist more than $\frac{1}{3}x/Q + o(x)$ positive integers $n \leqslant x$, for which

$$\sum_{i=1}^{h} \Big(f(n+i) - f(n+i-1) - f\big((i+1)N_i\big) + f(iN_{i-1})\Big)^2 \leqslant \tfrac{1}{4}\varepsilon^2$$

and then

$$|f(n+i)-f(n+i-1)-f((i+1)N_i)+f(iN_{i-1})| \leqslant \tfrac{1}{2}\varepsilon \quad (i = 1, 2, \ldots, h).$$

In view of (7), the proof is complete.

THEOREM 2. *Let $f(n)$ be an additive function satisfying the conditions of Theorem 1 and such that partial sums of $\sum(\|f(p)\|/p)$ are bounded:*

$$A > |S_k|, \quad S_k = \sum_{p \leqslant k} \frac{\|f(p)\|}{p}. \tag{14}$$

Then for any given natural number h there exists a number c_h such that for any $\varepsilon > 0$ and every sequence of h numbers: $a_1, a_2, \ldots, a_h \geqslant c_h$, there exist more than $C(a, \varepsilon)x$ positive integers $n \leqslant x$, for which

$$|f(n+i)-a_i| < \varepsilon \quad (i = 1, 2, \ldots, h). \tag{15}$$

$C(a, \varepsilon)$ is a positive constant, depending on ε and a_i.

Proof. Let ε be a positive number, $c_h = c_1 + \max f(i)$ and let a sequence $a_i \geqslant c_h$ $(i = 1, 2, \ldots, h)$ be given.

By condition 2 we can find positive integers $N_1, N_2, \ldots, N_h$ such that

$$(N_i, h!) = 1 \quad (i = 1, 2, \ldots, h), \quad (N_i, N_j) = 1 \quad (1 \leqslant i < j \leqslant h) \tag{16}$$

and

$$|f(N_i) - a_i + f(i)| < \tfrac{1}{2}\varepsilon \quad (i = 1, 2, \ldots, h). \tag{17}$$

Let k_1 be the greatest prime factor of $N_1 N_2 \ldots N_h$. Let C be an absolute constant such that

$$\sum_{y \leqslant p < z} \frac{1}{p} < C \log \frac{\log z}{\log y} \quad \text{for all} \quad z > y > 1.$$

Put $\mu = \varepsilon / 7C\sqrt{h}$. By condition 1, $\sum\limits_{|f(p)| \geqslant \mu} (1/p)$ is convergent. Since $\sum (1/p^2)$ is also convergent, there exists a k_2 such that

$$\sum_{|f(p)| \geqslant \mu,\, p > k_2} \frac{1}{p} + \sum_{p > k_2} \frac{1}{p^2} < \frac{1}{3h}. \tag{18}$$

By condition 1 there exists also a k_3 such that

$$\sum_{p > k_3,\, |f(p)| < \mu} \frac{f(p)^2}{p} < \frac{\varepsilon^2}{24h}. \tag{19}$$

Put $\eta = \varepsilon / \sqrt{96h}$, $B = A + 1/3h$ and denote by I_ν the interval

$$[\nu\eta - \tfrac{1}{2}\eta, \nu\eta + \tfrac{1}{2}\eta], \quad \nu = 0, \pm 1, \pm 2, \ldots, \pm [B/\eta + 1]$$

and let k_ν be the least integer $k > \max(k_1, k_2, k_3)$ such that $\sum_{p \leqslant k, |f(p)| k\mu} (f(p)/p) \in I_\nu$ if such integers k exist, otherwise let $k_\nu = 1$.

Now if $\sum_{p \leqslant x+h, |f(p)| < \mu} (f(p)/p) \in I_{\nu_x}$ —by the condition (14) and by (18) such ν_x certainly exists—we put $k_{\nu_x} = k$ and then we get

$$\left|\sum_{\substack{x+h \geqslant p > k \\ |f(p)| < \mu}} \frac{f(p)}{p}\right| < \eta, \quad k \leqslant \max_{|\nu| \leqslant [B/\eta]+1} k_\nu = \bar{k}. \tag{20}$$

Let $\sum'$ denote that the summation runs through all primes p, q satisfying conditions $p > q > k$, $pq \leqslant x-h$, $|f(p)| < \mu$, $|f(q)| < \mu$. From (20) we get

$$2\sum{}' \frac{f(p)f(q)}{pq} \leqslant \Big(\sum_{\substack{x+h \geqslant p > k \\ |f(p)| < \mu}} \frac{f(p)}{p}\Big)^2 + \sum_{x-h \geqslant p > \sqrt{x+h}} \frac{\mu}{p} \sum_{x+h \geqslant q > \frac{x+h}{p}} \frac{\mu}{q} \tag{21}$$

$$\leqslant \frac{\varepsilon^2}{96h} + \sum_{l=2}^{\infty} \sum_{(x+h+1)^{1-1/2^l} > p \geqslant (x+h+1)^{1-1/2^{l-1}}} \frac{\mu}{p} \sum_{x+h \geqslant q > \frac{x+h}{p}} \frac{\mu}{q}$$

$$\leqslant \frac{\varepsilon^2}{96h} + \mu^2 C^2 \sum_{l=2}^{\infty} \frac{l}{2^l} = \frac{\varepsilon^2}{96h} + \mu^2 C^2 \frac{3}{2} < \frac{\varepsilon^2}{24h}.$$

Let us put $N = N_1 N_2 \ldots N_h$, $\quad P = \prod_{p \leqslant k, p \nmid N} p$,

$$Q = h!N^2P \leqslant h!N^2 \prod_{p \leqslant \bar{k}, p \nmid N} p = \bar{Q} \tag{22}$$

and let us consider the following system of congruences:

$$n \equiv 0 \,(\mathrm{mod}\, h!P), \quad n \equiv -i + N_i \,(\mathrm{mod}\, N_i^2).$$

By (16) and the Chinese Remainder Theorem there exists a number n_0 satisfying these congruences.

It is easy to see, that

(23) *for every integer t the numbers $\dfrac{Qt+n_0+i}{iN_i}$ $(i = 1, 2, \ldots, h)$ are integers, which are not divisible by any prime $\leqslant k$.*

Analogously, as in the proof of Theorem 1, we shall estimate the number of integers n of the progression $Qt+n_0$, which satisfy the inequalities

$$n \leqslant x, \quad \sum_{i=1}^{h} \big(f(n+i) - f(iN_i)\big)^2 > \tfrac{1}{4}\varepsilon^2. \tag{24}$$

We divide the set of integers $n \equiv n_0 \pmod{Q}$, for which the inequalities (24) hold, into two classes. Integers n such that $(n+1)(n+2)\ldots(n+h)$ is divisible by a prime $p>k$ with $|f(p)| \geqslant \mu$ or by p^2, $p>k$, are in the first class and all others integers are in the second class.

By remark (13) the number of integers $n \leqslant x$, $n \equiv n_0 \pmod{Q}$ of the first class is less than

$$h\frac{x}{Q}\Big(\sum_{p>k,\,|f(p)|\geqslant\mu}\frac{1}{p}+\sum_{p>k}\frac{1}{p^2}\Big)+O\Big(\sum_{p\leqslant x+h}1+\sum_{p^2\leqslant x+h}1\Big).$$

By the inequality (18) and the definition of k this number is less than $\frac{1}{3}x/Q+o(x)$.

For the integers of the second class, by remark (23), we have

$$\sum_{i=1}^{h}\big(f(n+i)-f(iN_i)\big)^2=\sum_{i=1}^{h}\Big(\sum_{p|n+i,\,p>k}f(p)\Big)^2$$

and

$$\sum_{n}{}^{''}\sum_{i=1}^{h}\big(f(n+i)-f(iN_i)\big)^2=\sum_{n}{}^{''}\sum_{i=1}^{h}\Big(\sum_{p|n+i,\,p>k}f(p)\Big),$$

where $\sum_{n}''$ means that the summation runs through the integers of the second class. In view of remark (13), we have

$$\sum_{n}{}^{''}\sum_{i=1}^{h}\big(f(n+i)-f(iN_i)\big)^2\leqslant\sum_{\substack{n\equiv n_0 \pmod{Q}\\ n\leqslant x}}\sum_{i=1}^{h}\Big(\sum_{p|n+i,\,p>k}f(p)\Big)^2$$

$$=\sum_{\substack{x+h\geqslant p>k\\ |f(p)|<\mu}}f^2(p)\Big(\frac{hx}{Qp}+O(1)\Big)+2\sum{}'f(p)f(q)\Big(\frac{hx}{Qpq}+O(1)\Big)$$

$$\leqslant\frac{hx}{Q}\Big(\sum_{p>k,\,|f(p)|<\mu}\frac{f^2(p)}{p}+2\sum{}'\frac{f(p)f(q)}{pq}\Big)+$$

$$+O\Big(\sum_{\substack{p\leqslant x+h\\ |f(p)|<\mu}}f^2(p)+\sum{}'|f(p)f(q)|\Big).$$

Thus, finally from (19), (21) and from the fact that the number of integers of the form pq not exceeding $x+h$ is $o(x)$ we get

$$\sum{}''\sum_{i=1}^{h}\big(f(n+i)-f(iN_i)\big)^2<\frac{\varepsilon^2}{12}\cdot\frac{x}{Q}+o(x).$$

Thus the number of integers of the second class is less than $\frac{1}{3}x/Q+o(x)$.

Hence, there exist less than $\frac{2}{3}x/Q+o(x)$ positive integers $n \leqslant x$, $n \equiv n_0 \pmod{Q}$ for which

$$\sum_{i=1}^{h} (f(bn+i)-f(iN_i))^2 > \tfrac{1}{4}\varepsilon^2.$$

By (11) and (22) there exist, therefore, more than $\frac{1}{3}x/\bar{Q}+o(x)$ positive integers $n \leqslant x$, for which

$$\sum_{i=1}^{h} (f(n+i)-f(iN_i))^2 \leqslant \tfrac{1}{4}\varepsilon^2,$$

and then

$$|f(n+i)-f(iN_i)| \leqslant \tfrac{1}{2}\varepsilon \quad (i = 1, 2, \ldots, h).$$

In view of (16) and (17), this completes the proof.

Theorem 2 is best possible. Assume only that there exists an a and a $c > 0$ so that the number of integers $n \leqslant x$ satisfyng $|f(n)| < a$ is greater than cx.

Then $\sum \frac{\|f(p)\|^2}{p}$ *converges and* $\sum \frac{\|f(p)\|}{p}$ *has bounded partial sums.*

In the paper [2], P. Erdös proved([1]) the following theorem:

If there exist two constants c_1 and c_2 and an infinite sequence $x_k \to \infty$ so that for every x_k there are at least $c_1 x_k$ integers:

$$1 \leqslant a_1 < a_2 < \ldots < a_l \leqslant x_k, \quad l \geqslant c_1 x_k,$$

for which

$$|f(a_i)-f(a_j)| < c_2, \quad 1 \leqslant i < j \leqslant l,$$

then

$$f(n) = a\log n + g(n), \quad \textit{where} \quad \sum \frac{\|g(p)\|^2}{p} < \infty.$$

In our case the conditions of this theorem are clearly satisfied and, in fact, we clearly must have $a = 0$. This implies that

$$\sum \frac{\|f(p)\|^2}{p} < \infty.$$

([1]) The proof of Lemma 8 [2] is not clear and on p. 15 needs more details similar to these given above.

Assume now that $\sum(\|f(p)\|/p)$ does not have bounded partial sums. Let e.g. $\sum_{p<x}(\|f(p)\|/p) = A$, A large. Then by the method of Turán ([5], cf. also [2]) we obtain

$$\sum_{n=1}^{x}\big(f(n)-A\big)^2 < c_3 x$$

which implies that $|f(n)-A| < A-a$ for all but ηx integers $n \leqslant x$, where $\eta = c_3/(A-a)^2$. For sufficiently large A, it contradicts the assumption that $|f(n)| < a$ has cx solutions $n \leqslant x$, thus the proof is complete.

In Theorem 1 one can replace $\sum(\|f(p)\|^2/p) < \infty$ by: *there is an* a *so that if we put* $f(n) - a\log n = g(n)$ *then* $\sum(\|g(p)\|^2/p) < \infty$. We think that here we again have a necessary and sufficient condition, but we cannot prove this. In fact, we conjecture that if there exist an a and an $c > 0$ such that the number of integers $n \leqslant x$ satisfying $|f(n+1)-f(n)| < a$ is $> cx$, then

$$f(n) = a\log n + g(n) \quad \text{with} \quad \sum\frac{\|g(p)\|^2}{p} < \infty.$$

§ 2. The proof of Theorem 2 is very similar to the proof of Lemma 1 of P. Erdös' paper [1]. Using ideas and results from that paper we can prove the following theorem.

THEOREM 3. *Let* $f(n)$ *be an additive function satisfying condition* 1 *of Theorem* 1 *and let* $\sum_{f(p)\neq 0}(1/p)$ *be divergent,* $\sum(\|f(p)\|/p)$ *convergent, then the distribution function of* h*-tuples* $\{f(m+1), f(m+2), \ldots, f(m+h)\}$ *exists, and it is a continuous function.*

Proof. We denote by $N(f; c_1, c_2, \ldots, c_h)$ the number of positive integers m not exceeding n, for which

$$f(m+i) \geqslant c_i, \quad i = 1, 2, \ldots, h,$$

where c_i are given constants.

It is sufficient to consider, as in [1], the special case in which, for any a, $f(p^a) = f(p)$, so that

$$f(m) = \sum_{p|m} f(p).$$

Let us also consider the function $f_k(m) = \sum_{p|m,\, p\leqslant k} f(p)$. We are going to show that the sequence $N(f_k; c_1, c_2, \ldots, c_h)/n$ is convergent. For, if we denote by $A_{i,j}$ $(j \leqslant j_{0,i})$ the squarefree integers whose prime factors are not greater than k, and for which $f_k(A_{i,j}) \geqslant c_i$, we can see that the integers m for which

$$f_k(m+i) \geqslant c_i \quad (i = 1, 2, \ldots, h)$$

are distributed periodically with the period $\prod_{\substack{1\leqslant i\leqslant h\\ 1\leqslant j\leqslant j_{0\cdot i}}} A_{i,j}$. Hence $N(f_k; c_1, c_2, \ldots, c_h)/n$ has a limit.

To prove the existence of a limit of $N(f; c_1, c_2, \ldots, c_h)/n$ it is sufficient to show that for arbitrary $\varepsilon > 0$ there exists k_0 such that for every $k > k_0$ and $n > n(\varepsilon)$

$$|N(f; c_1, c_2, \ldots, c_h) - N(f_k; c_1, c_2, \ldots, c_h)|/n < \varepsilon.$$

To show this, it is enough to prove that the number of integers $m \leqslant n$ for which there exists $i \leqslant h$ such that $f_k(m+i) < c_i$ and $f(m+i) \geqslant c_i$ or $f_k(m+i) \geqslant c_i$ and $f(m+i) < c_i$ is less than εhn. But it is an immediate consequence of the analogous theorem for $h = 1$ proved in [1], p. 123.

In order to prove that the distribution function is continuous we must show that for every $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$\Delta = N(f; c_1 - \delta, c_2 - \delta, \ldots, c_h - \delta) - N(f; c_1 + \delta, c_2 + \delta, \ldots, c_h + \delta) < \varepsilon.$$

Now

$$\Delta = \sum_{i=1}^{h} \{N(f; c_1 + \delta, \ldots, c_{i-1} + \delta, c_i - \delta, \ldots, c_h - \delta) - \\ - N(f; c_1 + \delta, \ldots, c_i + \delta, c_{i+1} - \delta, \ldots, c_h - \delta)\}$$

and by Lemma 2 of [1] each term of this sum is less than ε/h for suitably chosen δ. This completes the proof.

We conclude from Theorems 2 and 3 that if an additive function f satisfies conditions 1, 2, $\sum_{f(p)\neq 0} (1/p)$ is divergent and $\sum(\|f(p)\|/p)$ convergent, then the distribution function of $\{f(m+1), \ldots, f(m+h)\}$ exists, is continuous and strictly decreasing on some half straight-line, thus the sequence of integers n for which inequality (15) holds has a positive density. Similarly we can prove the following:

THEOREM 4. *Assume that* $\sum_{f(p)\neq 0} \frac{1}{p} = \infty$ *and that* $\sum \frac{\|f(p)\|^2}{p} < \infty$ *then* $\{f(n+1) - f(n),\ f(n+2) - f(n+1),\ \ldots,\ f(n+k) - f(n+k-1)\}$ *has a continuous distribution function.*

It is easy to see that condition 2 can be replaced by the conditions

$$\lim_{p\to\infty} f(p) = 0 \quad \text{and} \quad \sum_{p} |f(p)| = \infty.$$

§ 3. Y. Wang proved in [6] that the number N of primes $p < x$ satisfying

$$\left|\frac{\varphi(p+\nu+1)}{\varphi(p+\nu)} - a_\nu\right| < \varepsilon, \quad 1 \leqslant \nu \leqslant k$$

is greater than

$$c(a, \varepsilon)\frac{x}{(\log x)^{k+2}\log\log x}.$$

By our methods we can obtain in that case

$$N > c_1(a, \varepsilon)\frac{x}{\log x}.$$

After having passed to the additive function $\log(\varphi(n)/n)$ the proof is similar to the proof of Theorem 1. We use the fact, that $\log(\varphi(n)/n)$ is always negative, and apply the asymptotic formula for the number of primes in arithmetical progression instead of (11) and the Brun-Titchmarsh theorem instead of (13).

We can also prove that there exists distribution function $N(c_1, c_2, \ldots, c_k)$ defined as

$$\lim_{x\to\infty}\frac{1}{\pi(x)} N\Big(p < x; \frac{\varphi(p+\nu)}{p+\nu} \geqslant c_\nu, \quad \nu = 1, 2, \ldots, k\Big).$$

References

[1] P. Erdös, *On the density of some sequences of numbers, III*, J. London Math. Soc. 13 (1938), pp. 119-127.

[2] P. Erdös, *On the distribution function of additive functions*, Ann. of Math. 47 (1946), pp. 4-20.

[3] A. Schinzel and Y. Wang, *A Note on some properties of the functions $\varphi(n)$, $\sigma(n)$ and $\theta(n)$*, Bull. Acad. Polon. Sci. Cl. III 4 (1956), 207-209 and Ann. Pol. Math. 4 (1958), pp. 201-213.

[4] Shao Pin Tsung, *On the distribution of the values of a class of arithmetical functions*, Bull. Acad. Polon. Sci., Cl. III 4 (1956), pp. 569-572.

[5] P. Turán, *On a theorem of Hardy and Ramanujan*, J. London Math. Soc. 9 (1934), pp. 274-276.

[6] Y. Wang, *A note on some properties of the arithmetical functions $\varphi(n)$, $\sigma(n)$ and $d(n)$*, Acta Math. Sinica, 8 (1958), pp. 1-11.

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