SOME EXAMPLES IN ERGODIC THEORY

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Introduction

THE purpose of this paper is to give some counter examples in ergodic theory. Some of the examples answer questions raised previously by some authors, some are counter examples to published theorems later withdrawn and some answer questions entertained privately by the authors of this note.

Let Y be an abstract space, γ a σ-field of subsets of Y and m a measure defined on γ. We shall assume throughout that (Y, γ, m) is a σ-finite, non-atomic measure space. Let T be a $1:1$, ergodic measure preserving transformation of Y on to itself.

Examples 1 and 2 deal with the case where (Y, γ, m) is a finite measure space (in fact it is the ordinary line segment $[0, 1)$. Example 1 shows that one cannot generalize theorems in Diophantine approximation theory like Theorem 2 (**12**, 76) to arbitrary ergodic conservative systems on $[0, 1)$.

Example 2 answers completely the questions raised by Halmos (**7**) and more recently by Standish (**13**) concerning non-homogeneous ergodic theorems.

We put
$$M_n(T, m, A) = \frac{1}{n}\sum_{i=0}^{n-1} m(T^i A)$$
and
$$M_n(T, A, y) = \frac{1}{n}\sum_{i=0}^{n-1} f_A(T^i y)$$
where $f_A(y)$ is the characteristic function of A.

Examples 3, 4, and 5 and Theorems 1 and 2 deal with the case where $m(Y)$ is infinite and study the relative behaviour of two ergodic measure preserving transformations T_1 and T_2. Example 3 shows that contrary to some expectations (**1, 2**) the sequence
$$M_n(A, T_1, T_2, y) = M_n(T_2, A, y)/M_n(T_1, A, y)$$
need not converge p.p. for sets of finite measure A.

Example 4 and the remark following it show that $M_n(A, T_1, T_2, y)$ need not converge p.p. even if $T_2 = T_1^{-1}$. Thus the past and future of *infinite* conservative ergodic systems are relatively independent of one another. It is known however (**10**, 53) that $\lim_{n\to\infty} M_n(T_1, A, y) = \lim_{n\to\infty} M_n(T_2, A, y) = 0$

for every set A of finite measure but example 5 shows that

$$M_n(T_1, A, y) - M_n(T_2, A, y)$$

need *not* converge to zero for every measurable set A.

Theorem 1 shows that despite the possible wildness of behaviour of $M_n(A, T_1, T_2, y)$, this behaviour is independent of the set of finite measure A and depends only on T_1 and T_2. Theorem 2 disproves Theorem 3 of (**1**).

We now change our assumptions about T by not only dropping the assumption that T is measure preserving but assuming T to be a measurable, non-singular, and ergodic transformation of Y on to itself which preserves *no finite* measure equivalent to m. Example 6 studies the relative behaviour of two normalized equivalent measures m_1 and m_2 (both equivalent to m). W. Hurewicz conjectured that

$$M_n(T, m_1, m_2, A) = M_n(T, m_2, A)/M_n(T, m_1, A) \underset{n}{\to} 1.$$

It is known (**4**) that $M_n(T, m_2, A) - M_n(T, m_1, A) \underset{n}{\to} 0$. Example 6 shows that Hurewicz's conjecture is not true.

The unifying idea in the construction of most of the examples is the observation that every ergodic system can be represented as a system 'lifted' from an induced system in a subset by means of a skyscraper structure. This representation is inspired by ideas introduced by Kakutani (**11**) and used previously by various authors (e.g. (**3, 6, 14**)). The various ergodic systems needed for the various examples are obtained by means of varying the heights of the 'storeys' in the skyscrapers, the transformations in the base and the measures in the 'storeys'.

All transformations considered are assumed to be 1:1 and an ergodic measure preserving transformation will be denoted by 'e.m.p. transformation'. The reader may consult (**8**) for further definitions.

1. Induced and lifted transformations

1.1. Let (Y, γ, m) be a measure space, T a measurable ergodic non-singular transformation of Y on to itself and X a measurable subset of Y of positive measure. Then X and Y can be decomposed as follows: let

$$E_1 = \{x \mid x \in X,\ Tx \in X\},$$

$$E_h = \{x \mid x \in X,\ T^h x \in X,\ T^i x \notin X,\ i = 1, \dots, h-1\},$$

$h = 2, 3, \dots$. Then $\{E_h\}$ is a sequence of mutually disjoint measurable sets and

$$X = \bigcup_{h=1}^{\infty} E_h \cup Z_1, \qquad Y = \bigcup_{h=1}^{\infty} \bigcup_{i=0}^{h-1} T^i E_h \cup Z_2,$$

where $m(Z_1) = m(Z_2) = 0$ and all the sets appearing in the double union above are mutually disjoint. (Cf. (**11**), (**3**).)

Some of the sets in the sequence $\{E_h\}$ may be zero sets. Let $\{h(k)\}$ be the subsequence of those integers satisfying $m(E_{h(k)}) > 0$. Let Z be the smallest invariant set containing Z_2, all sets of the sequence E_h which are zero sets and all points $x \in X$ such that $T^i x \notin X$ for $i = -1, -2, \dots$. Let

$$A_k = E_{h(k)} - Z.$$

Then (1) $X = \bigcup_k A_k \cup Z \wedge X$ and (2) $Y = \bigcup_k \bigcup_{i=0}^{h(k)-1} T^i A_k \cup Z$ where all sets appearing in each of the unions (1) and (2) are mutually disjoint.

Let S be defined on X by putting $Sx = T^{h(k)}x$ if $x \in A_k$ and $S_x = x$ if $x \in Z \wedge X$. Then it is known (cf. (**11**), (**3**)) that S is a $1:1$ measurable ergodic non-singular transformation of X on to itself and that if T is measure preserving then so is S. S is called the transformation *induced* on X by T. (X, S) will be called an induced system of (Y, T).

1.2. We can reverse the process and start with a measure space (X, β, m) and a measurable non-singular transformation S of X on to itself. Let $\{A_k\}$ be a finite or infinite sequence of mutually disjoint measurable sets of positive measure such that $X = \bigcup_k A_k$. Let I denote the set of non-negative integers and let $h(k)$ be a sequence of positive integers. Let Y be the subset of $X \times I$ defined by $Y = \bigcup_k \bigcup_{i=0}^{h(k)-1} A_k \times i$. $X \times I$ becomes a measure space by the following procedure: Let $m_0 = m$, $m_i \sim m$, $i = 1, 2, \dots$. Let β_i be the σ-ring of sets of the form $A \times i$, $A \in \beta$, $i = 0, 1, \dots$. If $C \in \beta_i$, i.e. $C = A \times i$, $A \in \beta$, put $\mu(C) = m_i(A)$. Let α be the smallest σ-ring of subsets of $X \times I$ containing β_i, $i = 0, 1, \dots$. Then there is a unique measure m defined on α coinciding with μ on β_i. Then $(X \times I, \alpha, m)$ is a measure space. Let (Y, γ, m) be the restriction of $(X \times I, \alpha, m)$ to Y. With no loss of generality $X \times 0$ can be identified with X and (X, β, m) can be considered as a sub-measure space of (Y, γ, m). (Y, γ, m) is a σ-finite measure space and it is finite if and only if $\sum_k \sum_{i=0}^{h(k)} m_i(A_k) < \infty$. Let $y = (x, i)$, $x \in A_k$,

$$0 \leqslant i \leqslant h(k) - 1.$$

Let $Ty = (x, i+1)$ if $i < h(k) - 1$ and $Ty = (Sx, 0)$ if $i = h(k) - 1$. Then T is a $1:1$ measurable, non-singular transformation of Y on to itself. If S is ergodic then so is T. If $m_i = m$, $i = 1, 2, \dots$ and S is measure preserving then so is T. T will be called the *lifted* transformation of S on Y and (Y, T) will be called a lifted system of (X, S). If $m_i = m$, $i = 1, 2, \dots$, T is completely determined by $\{X, \beta, m, S, A_k, h(k)\}$. Thus if (X, β, m) and S are given, various lifted systems of (X, S) can be obtained by varying $\{A_k\}$ and $h(k)$. It is clear that S is the transformation induced by T on X. And

$$Y = \bigcup_k \bigcup_{i=0} T^i A_k.$$

2. Finite measure spaces

2.1. Consider the ordinary Lebesgue measure space (X, β, m) where $X = [0, 1)$. Let S be defined in X by putting

$$Sx = x + \tfrac{1}{2}, \quad \text{if} \quad 0 \leqslant x < \tfrac{1}{2},$$

$$Sx = x + \frac{3}{2^{k+1}} - 1, \quad \text{if} \quad \sum_{i=1}^{k} (\tfrac{1}{2})^i \leqslant x < \sum_{i=1}^{k+1} (\tfrac{1}{2})^i.$$

This transformation is known to be ergodic, measure preserving and to have the property that every dyadic interval $[k/2^n, (k+1)/2^n)$, $k = 0, 1, \ldots, 2^{n-1}$, is a periodic set of period 2^n, $n = 1, 2, \ldots$. We shall refer to this transformation as the *dyadic* transformation on $[0, 1)$ and will denote it by D.

2.2. Let $X = [0, 1)$ and let S be an e.m.p. transformation of X on to itself. Let B_n be a decreasing sequence of measurable subsets of X such that $\bigwedge_{n=1}^{\infty} B_n = \varnothing$. Let $f(n, x)$ be defined as the first non-negative integer j such that $S^j x \in B_n$. Then

LEMMA 2.1. *There exists a sequence of integers $f(n)$ such that for almost every $x \in [0, 1)$, $f(n, x) \leqslant f(n) - 1$ if $n > N = N(x)$ where $N(x)$ is some integer depending on x.*

Proof. $\bigcup_{i=0}^{\infty} S^{-i} B_n = X - Z_n$ where $m(Z_n) = 0$, $n = 1, 2, \ldots$. Let

$$C_r = \bigcup_{i=0}^{r-1} S^{-i} B_n$$

and let $f(n)$ be an integer satisfying $C_{f(n)} > 1 - (\tfrac{1}{2})^n$. Then

$$m\Big(\bigwedge_{n=k}^{\infty} C_{f(n)}\Big) > 1 - (\tfrac{1}{2})^{k-1}$$

and hence if

$$R = \bigcup_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} C_{f(n)}$$

then $m(R) = 1$. Let $x \in R$ then $\exists$ an integer $N = N(x)$ such that $x \in C_{f(n)}$ for all $n \geqslant N$. But $x \in C_{f(n)} \Rightarrow S^i x \in B_n$ for some i, $0 \leqslant i \leqslant f(n) - 1$. Hence $f(n, x) \leqslant f(n) - 1$.

LEMMA 2.2. *Let $m(B_n) = b_n$, $n = 1, 2, \ldots$, and let $\sum_{n=1}^{\infty} b_n < \infty$. Then for almost all $x \in [0, 1)$ there exists an integer $N = N(x)$ such that $f(n, x) > n - 1$. for $n > N$.*

Proof. Let $C_r = \bigcup_{i=0}^{r-1} S^{-i}B_n$. Then

$$\bigcup_{n=k}^{\infty} C_n = C_k \cup \bigcup_{n=k}^{\infty} S^{-n}B_{n+1}.$$

Hence $$m\Big(\bigcup_{n=k}^{\infty} C_n\Big) \leqslant km(B_k) + \sum_{n=k+1}^{\infty} m(B_n) = kb_k + \sum_{n=k+1}^{\infty} b_n.$$

But $kb_k + \sum_{n=k+1}^{\infty} b_n \underset{k}{\to} 0$ since $b_n \downarrow_n 0$ and $\sum_{n=1}^{\infty} b_n < \infty$. Thus if $Z = \bigwedge_{k=1}^{\infty} \bigcup_{n=k}^{\infty} C_n$ then $m(Z) = 0$. Let $x \notin Z$, then there exists an integer $N = N(x)$ such that $x \notin C_n$ if $n > N$. But $x \notin C_n \Rightarrow S^i x \notin B_n$, for $0 \leqslant i \leqslant n-1$, i.e.

$$f(n, x) > n-1.$$

EXAMPLE 1. LEMMA 2.3. *Let $b_1 = 1$, $\{b_n\}$ a strictly decreasing sequence of positive numbers such that $b_n \to 0$ and let $B_n = [1-b_n, 1)$. Let $f(n)$ be an increasing sequence of positive integers. Then there exists an e.m.p. transformation T of $[0, 1)$ on to itself such that for almost all $x \in [0, 1)$, $f(n, x) > f(n)$ if $n > N = N(x)$ where $N(x)$ is some integer depending on x.*

Proof. Let $\{a_n\}$ be a sequence of positive numbers such that

$$\sum_{n=k}^{\infty} f(n)a_n = b_k, \qquad k = 1, 2,\ldots .$$

Let $$X' = \Big[0, \sum_{n=1}^{\infty} a_n\Big), \quad A_k = \Big[\sum_{n=1}^{k-1} a_n, \sum_{n=1}^{k} a_n\Big), \quad k = 2, 3,\ldots, \quad A_1 = [0, a_1).$$

Let S be any e.m.p. transformation of X' on to itself. Let (Y, T') be the lifted conservative system determined by X', S, A_k, and $h(k) = f(k)$.

Let L be a 1:1 measure preserving transformation of Y on to $[0, 1)$ defined by mapping $\bigcup_{i=0}^{f(k)-1} T^i A_k$ in a natural way on $\Big[\sum_{n=1}^{k-1} f(n)a_n, \sum_{n=1}^{k} f(n)a_n\Big)$. $\bigcup_{k=n}^{\infty} \bigcup_{i=0}^{f(k)-1} T^i A_k$ is mapped by L on $B_n = \Big[\sum_{k=1}^{n-1} f(k)a_k, \sum_{k=1}^{\infty} f(k)a_k = 1\Big)$ and thus $B_n = [1-b_n, 1)$, $n = 1, 2,\ldots$. Clearly $T = L^{-1}T'L$ is a 1:1 e.m.p. transformation of $[0, 1)$ on to itself. Moreover T has the required properties. Indeed let $C_n = \bigcup_{i=0}^{f(n)-1} T^{-i}B_n$. Then $\bigcup_{n=k}^{\infty} C_n = \bigcup_{n=k}^{\infty} \bigcup_{i=0}^{f(n)-1} T^{-i}B_n$. Thus

$$m\Big(\bigcup_{n=k}^{\infty} C_n\Big) \leqslant 2\sum_{n=k}^{\infty} f(n)a_n = 2b_k.$$

Hence if $C = \limsup_{n\to\infty} C_n$ then $m(C) = 0$. Let $x \in [0, 1) - C$ then $x \notin C_n$ for all $n > N = N(x)$. But $x \notin C_n \Rightarrow T^i x \notin B_n$ for $i = 0, 1,\ldots, f(n)-1$ and hence $f(n, x) > f(n)-1$.

2.3. Let X, S, $\{B_n\}$ and $f(n,x)$ be as in 2.2.

LEMMA 2.4. *Let k_n be a sequence of integers satisfying $k_n \to \infty$ as $n \to \infty$, then $f(n,x)/k_n \nrightarrow 1$ as $n \to \infty$ a.e.*

Proof. Let $C_n = \{x \mid \frac{3}{4}k_n \leqslant f(n,x) \leqslant 1\frac{1}{4}k_n\}$ and let $D_k = \{x \mid f(n,x) = k\}$. Then $T^k D_k \subseteq B_n$, $T^i D_k \wedge B_n = \emptyset$ for $i = 0, 1, \ldots, k-1$. Hence $T^i D_k$, $i = 0, 1, \ldots, k-1$ are mutually disjoint and $m(D_k) \leqslant 1/(k+1)$. It follows that $m(C_n) \leqslant 1/(l_1+1)+1/(l_1+2)+\ldots+1/(l_1+s)$ where l_1 is the largest integer in $\frac{3}{4}k_n$ while l_1+s is the largest integer in $\frac{4}{5}k_n$. Thus $m(C_n) < \frac{2}{3}$ if n is large enough. Let $C = \{x \mid f(x,n)/k_n \to 1 \text{ as } n \to \infty\}$. Then $C \subseteq \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} C_n$ and hence $m(C) < \frac{2}{3}$. C however is invariant since

$$f(n,Sx)/k_n = (f(n,x)-1)/k_n$$

if n is large enough. Since S is ergodic we obtain $m(C) = 0$.

COROLLARY 2.5. *$m\{y \mid f(n,y)/f(n,x) \to 1$ as $n \to \infty\} = 0$ for every $x \in X$.*

Proof. Let $x \in X$ and $f(n,x) = k_n$, then $k_n \to \infty$ as $n \to \infty$ and the required result follows directly from Lemma 2.4.

LEMMA 2.6. *Let $f(-n,x) = f(T,-n,x)$ denote the first non-negative integer j satisfying $T^{-j}x \in B_n$ and let $r(n,x) = f(n,x)/f(-n,x)$. Then $r(n,x) \nrightarrow 1$ as $n \to \infty$ a.e. in X.*

Proof. Let l be a fixed integer and let B_l, $[0,1)$ and E_{lh} play the roles of X, Y, and E_h respectively in 1.1. Then discarding a set of measure zero we have $B_l = \bigcup_{h=1}^{\infty} E_{lh}$ and $[0,1) = \bigcup_{h=1}^{\infty} \bigcup_{i=0}^{h-1} T^i E_{lh}$. Let $x \in T^i E_{lh}$ then $r(l,x) = \dfrac{h-i}{i}$ if $0 < i < h$ and we make the convention that $r(l,x) = 0$ if $i = 0$. Let $D_l = \{x \mid r(l,x) < 2\}$. $D_l = B_l \cup C_l$ where $C_l = \bigcup T^i E_{lh}$, where the union is taken over all pairs (i,h) such that $0 < i < h$ and $\dfrac{h-i}{i} < 2$ (i.e. $i > h/3$). Now

$$m(Y-D_l) = m(Y-(B_l \cup C_l)) \geqslant m\Big[\bigcup_{h=4}^{\infty} \bigcup_{i=1}^{[h/3]} T^i E_{lh}\Big]$$
$$\geqslant \tfrac{1}{6} m\Big(Y - \bigcup_{h=1}^{3} \bigcup_{i=0}^{h-1} T^i E_{lh}\Big) \geqslant \tfrac{1}{6}[1-3m(B_l)].$$

Hence $m(Y-D_l) > \frac{1}{10}$ if l is large enough. Let $D = \liminf_{l\to\infty} D_l$, then $m(D) < \frac{9}{10}$. But $D = \Big\{x \mid \limsup_{l\to\infty} r(l,x) < 2\Big\}$ and can be shown to be invariant (possibly discarding a set of measure zero) and hence since T is ergodic $m(D) = 0$. This completes the proof of Lemma 2.6.

Remark. We notice that any positive constant K can be substituted for

2 in the above proof and hence one can prove that $\limsup_{n\to\infty} r(n,x) = \infty$. Since T and T^{-1} play symmetric roles in Lemma 2.6 it would follow that

$$\liminf_{n\to\infty} r(n,x) = 0.$$

2.4. LEMMA 2.7. *Let S be an e.m.p. transformation of $[0,1)$ on to itself. Then for every $\epsilon > 0$ and N there exists a set A such that A, SA,..., $S^{N-1}A$ are mutually disjoint and $m(A) = (1-\epsilon)/N$.*

Proof. Let B be a measurable set with $0 < m(B) < \epsilon/N$. Let B, $[0,1)$ and A_k play the roles of X, Y, and A_k in 1.1. Then discarding a set of measure zero, we have $X = \bigcup_k \bigcup_{i=0}^{h(k)-1} T^i A_k$, $B = \bigcup_k A_k$. Let $l(k)$ be the largest non-negative integral multiple of N in $h(k)$ and let p be the smallest integer k for which $l(k) > 0$. Let

$$A' = \bigcup_{k=p}^{\infty} [A_k \cup S^N A_k \cup S^{2N} A_k \cup \ldots \cup S^{l(k)-N} A_k].$$

Then clearly A', SA',..., $S^{N-1}A'$ are mutually disjoint and

$$m\Big(\bigcup_{i=0}^{N-1} S^i A'\Big) \geqslant 1 - N.m(B) > 1-\epsilon,$$

i.e. $m(A') > (1-\epsilon)/N$. Since m is non-atomic A' contains a measurable set A with $m(A) = (1-\epsilon)/N$. Clearly A, SA,..., $S^{N-1}A$ are mutually disjoint.

LEMMA 2.8. *Let $g(x)$ and $f(x)$ be two measurable functions on X with $m(X) < \infty$ and let $|g(x)| > K$ a.e. Then for every integer l there exists a constant α, $\frac{1}{2} \leqslant \alpha \leqslant 1$, such that $|\alpha g(x) + f(x)| > K/4l$ on a set A with $m(A) > (1-1/l)m(X)$.*

Proof. Let $\alpha_i = \frac{1}{2} + i/2l$, $i = 0, 1, \ldots, l$. Then

$$|\alpha_i g(x) + f(x) - \alpha_j g(x) - f(x)| = |(\alpha_i - \alpha_j) g(x)| \geqslant (1/2l)|g(x)| > K/2l \text{ a.e.}$$

Let $A_i = \{x \mid \alpha_i g(x) + f(x) \leqslant K/4l\}$, $i = 0, 1, \ldots$. Then $A_i \wedge A_j = \varnothing$ if $i \neq j$. Hence for some i, $0 \leqslant i \leqslant l$, $m(A_i) \leqslant (1/l+1)m(X)$,

$$|\alpha_i g(x) + f(x)| > K/4l$$

in $X - A_i$ and $m(X-A_i) > (1-1/l)m(X)$.

EXAMPLE 2. LEMMA 2.9. *Let (X,S) be a conservative ergodic system, $X = [0,1)$, and let $\{c_i\}$ be a sequence of non-negative numbers satisfying $\sum_{i=1}^{\infty} c_i = \infty$. Then there exists a bounded measurable function $f(x)$ on X such that $\int f(x) = 0$ and $\sum_{i=1}^{\infty} c_i f(S^i x)$ does not converge in measure on any subset of X of positive measure.*

Proof. We assume that $c_i \to 0$; otherwise the result is trivial. Let n_k be a sequence of positive integers with n_k divisible by k and with n_k tending to ∞ so rapidly that $n_k > 2n_{k-1}$ and $\sum_{i=1}^{m_k} c_i > 2^k n_{k-1}^2$ where $m_k = n_k/k$. Then we have $(n_k/k) \to \infty$ and $\sum_{i=k+1}^{\infty} (1/n_i) < 2/n_{k+1}$.

By Lemma 2.7 there is a sequence $\{A_k\}$ of sets such that, for each k, A_k, $SA_k, \ldots$, $S^{n_k-1}A_k$ are mutually disjoint and $m(A_k) = (1-2^{-k})/n_k$. Let $f_k(x) = 2^{-k}$ if $x \in B_k = \bigcup_{i=0}^{n_k-1} S^i A_k$ and $f_k = -(1-2^{-k})$ if $x \in X - B_k$. Then clearly $\int f_k(x) = 0$. Let α_i be a sequence of numbers with $\frac{1}{2} \leqslant \alpha_i \leqslant 1$, and satisfying additional conditions to be stated below. Let $r_i = \alpha_i/n_{i-1}$ and let $f(x) = \sum_{i=1}^{\infty} r_i f_i(x)$. For each k we can write $f(x)$ as the sum of three functions, $f(x) = f_a(x)+f_b(x)+f_c(x)$ where

$$f_a(x) = \sum_{i=1}^{k-1} r_i f_i(x), \qquad f_b(x) = r_k f_k(x), \qquad f_c(x) = \sum_{i=k+1}^{\infty} r_i f_i(x).$$

Writing again $m_k = n_k/k$ we have

$$(1) \quad \Big|\sum_{i=1}^{m_k} c_i f_c(S^i x)\Big| \leqslant \Big(\sum_{j=k+1}^{\infty} r_j\Big) \sum_{i=1}^{m_k} c_i \leqslant \sum_{j=k}^{\infty} (1/n_j) \sum_{i=1}^{m_k} c_i \leqslant (2/km_k) \sum_{i=1}^{m_k} c_i,$$

which since $c_i \underset{i}{\to} 0$, will tend to zero as $k \to \infty$. Let $D_k = \bigcup_{i=0}^{l_k-1} S^i A_k$ where $l_k = (1-1/k)n_k$. Then $m(D_k) = (1-1/k)(1-2^{-k})$. For each $x \in D_k$, $f_b(S^i x) = r_k 2^{-k}$ for $i = 0, 1, \ldots, m_k$, and therefore

$$\sum_{i=1}^{m_k} c_i f_b(S^i x) = \alpha_k (n_{k-1})^{-1} 2^{-k} \sum_{i=1}^{m_k} c_i.$$

Writing $g_k(x) = (2^k n_{k-1})^{-1} \sum_{i=1}^{m_k} c_i$ we have

$$(2) \qquad \sum_{i=1}^{m_k} c_i f_b(S^i x) = \alpha_k g_k(x) > n_{k-1}/2.$$

We write $h_k(x) = \sum_{i=1}^{m_k} c_i f_a(S^i x)$ and observe that $h_k(x)$ depends on $\alpha_1, \ldots, \alpha_{k-1}$. Then, by Lemma 2.8, α_k can be chosen so that

$$(3) \qquad |\alpha_k g_k(x) + h_k(x)| > n_{k-1}/8k$$

on a subset of D_k of measure bigger than $(1-1/k)m(D_k)$. (If $\alpha_1 = 1$, then $\{\alpha_k\}$ are thus defined by induction.) Thus since $n_k/k \to \infty$ it follows from (1) and (3) that $\sum c_i f(S^i x)$ does not converge in measure on any subset of X of positive measure.

COROLLARY. *With the same notation and conditions as in Lemma* 2.9 *we have that* $\sum c_i f(S^i x)$

(i) *diverges a.e. in* X, *and*

(ii) *does not converge in* L_p *for* $p \geqslant 1$.

3. Infinite measure spaces

3.1. Let $X = [0, 1)$ and D be the dyadic transformation on X. Let $A_k = [1-1/2^{k-1},\ 1-1/2^k)$ and $B_k = [1-1/2^k,\ 1)$, $k = 1, 2, \ldots$. Let $h(k)$ be an increasing sequence of integers and let (Y, T) be the conservative system determined by X, D, A_k and $h(k)$ (cf. 1.2).

LEMMA 3.1. *Let* $f_X(x)$ *be the characteristic function of* X. *Then*

$$\sum_{i=0}^{m-1} f_X(T^i x) = n$$

implies that, for almost all x, $m \leqslant n\,h(n)$ *for* $n > N(x)$.

Proof. By Lemma 2.1 there exists an $N(x)$ for almost all x such that if $n > N(x)$ then $D^j x \notin B_n$ for $j = 0, 1, \ldots, n-1$. For such x then $D^j x \notin A_l$ for $j = 0, 1, \ldots, n-1$ and $l > n$. Hence as x is iterated by D in X n times, it enters only the sets A with the number $h(l)$ of layers above them not exceeding $h(n)$ and hence x is iterated by T in Y at most $n\,h(n)$ times; i.e.

$$m = m(x, n) \leqslant n\,h(n).$$

LEMMA 3.2. *Let* (Y, T) *be as in Lemma* 3.1, *then* $\sum_{i=0}^{m-1} f_X(T^i x) = 2^n$ *implies that* $m \geqslant h(n-1)$.

Proof. Let $x \in X$ and let $l \leqslant n$. Then there exists a j, $0 \leqslant j \leqslant 2^n$, such that $D^j x \in A_l$, i.e. as X is iterated by D 2^n times in X, it enters every one of the sets A_l, $l = 1, 2, \ldots, n$, at least once. Hence if $D^{2^n} x \in A_n$ then

$$m \geqslant \sum_{i=1}^{n-1} h(i) \geqslant h(n-1).$$

Otherwise $m \geqslant h(n) > h(n-1)$.

3.2. Let (Y, T) be as defined above in Lemmas 3.1 and 3.2 and let Y have infinite measure. Let L be a measure preserving transformation from Y on to $[0, \infty)$ defined by mapping $X \times 0$ on to X naturally ($(x, 0) \xrightarrow{L} x$), ordering the sets $B_k \times i$, $k = 1, 2, \ldots$, $i = 1, 2, \ldots, h(k)-1$, in a sequence and mapping these sets in a natural way one after the other on consecutive intervals congruent to B_k along the line $[0, \infty)$. Let $T' = L^{-1} T L$. Then it is clear that T' is an e.m.p. transformation of $[0, \infty)$ on to itself, and moreover we have that

$$\sum_{i=0}^{m-1} f_X(T^i x) = \sum_{i=0}^{m-1} f_{[0,1)}(T'^i x).$$

In Lemmas 3.3, 3.4, and below we shall assume that (Y, T) is already defined on $[0, \infty)$ as described.

3.3. Example 3. Lemma 3.3. *Let T_i, $i = 1, 2$, be e.m.p. transformations of $[0, \infty)$ on to itself as defined above with $h(k) = 2^{l_i}$, $i = 1, 2$, where $l_1 = k$ and $l_2 = 2^{2^k}$. Let*

$$M_m(X, T_1, T_2, y) = \sum_{i=0}^{m-1} f_X(T_2^i y) \Big/ \sum_{i=0}^{m-1} f_X(T_1^i y).$$

Then $\liminf_{m\to\infty} M_m(X, T_1, T_2, y) = 0$ *a.e. in* Y.

Proof. Let x be outside the exceptional set of Lemma 3.1 and $N(x)$ as in Lemma 3.1. Let $m = m(x, n)$ be such that $\sum_{i=0}^{m-1} f_X(T_1^i x) = 2^n$. Then by Lemma 3.2 $m \geqslant h_1(n-1) = 2^{2^{2^{n-1}}}$. If $\sum_{i=0}^{m-1} f_X(T_2^i x) = r$, then

$$m \leqslant r h_2(r) = r \,.\, 2^r \leqslant 2^{2r}.$$

Thus $2r \geqslant 2^{2^{n-1}}$ and hence

$$M_m(X, T_1, T_2, x) \leqslant 2^{n+1}/2^{2^{n-1}}$$

which approaches zero as $n \to \infty$. But $m = m(x, n) \to \infty$ as $n \to \infty$ and the lemma is proved for almost all $x \in X$. But since T_1 is ergodic and hence has no wandering sets of measure > 0 we have $\sum_{i=1}^{\infty} f_X(T_1^i y) = \infty$ a.e. in Y and hence $\liminf_{m\to\infty} M_m(X, T_1, T_2, y)$ is invariant in Y which completes the proof.

Example 4. Lemma 3.4. *Let T be an e.m.p. transformation of $[0, \infty)$ on to itself defined by $X = [0, 1)$, D, A_k, and $h(k) = 2^{2^k}$ as described in* 3.1 *and* 3.2. *Write $M_m(X, T, T^{-1}, y) = \sum_{i=0}^{m-1} f_X(T^i y) \Big/ \sum_{i=0}^{m-1} (T^{-i} y)$. Then*

$$M_m(X, T, T^{-1}, y) \not\to 1$$

as $n \to \infty$ a.e. in Y.

Proof. By Lemma 2.6 $f(n, x)/f(-n, x) \not\to 1$ as $n \to \infty$, $x \in X - Z$ where $m(Z) = 0$ and $f(n, x) = f(D, n, x)$, $f(-n, x) = f(D, -n, x)$. Let $x \in X - Z$. We prove that given $n > N(x)$ there exists an $m = m(n, x)$ such that

$$M_m(X, T, T^{-1}, x) = (f(n, x) + 1)/f(-n, x).$$

Indeed write $n_1 = f(n, x)$, $n_2 = f(-n, x)$, m_1 $(= m_1(n, x))$ = the first positive integer j such that $T^{j-1} x \in A_n$ and $m_2 = m_2(n, x)$ = the first j such that $T^{-(j-1)} x \in A_k \times (h(k) - 1)$ for some $k > n$. Then $n_1 \leqslant 2^n$, $n_2 \leqslant 2^n$, the induced transformation of T^{-1} on X is D^{-1}, $\sum_{i=0}^{m_1-1} f_X(T^i x) = n_1 + 1$, and

$$\sum_{i=0}^{m_2} f_X(T^{-i} x) = n_2.$$

Then clearly $m_i \leqslant n_i . h(n) \leqslant 2^n . 2^{2^{n-1}}$, $i = 1, 2$. By the construction of (Y, T) we have that $\sum_{i=0}^{m_1+l-1} f_X(T^i x) = n_1+1$ for $0 \leqslant l < 2^{2^n}$ and

$$\sum_{i=0}^{m_2+r} f_X(T^{-i}x) = n_2 \quad \text{for} \quad 0 \leqslant r < 2^{2^n}.$$

Hence there exist two integers r_1 and l_1 such that $1 \leqslant r_1$, $l_1 \leqslant 2^{2^n}$ and $m_1+l_1-1 = m_2+r_2$. Let m $(= m(n,x)) = m_1+l_1-1$, then

$$M_m(X, T, T^{-1}, x) = (f(n,x)+1)/f(-n,x).$$

Since $m(n,x) \to \infty$ and $f(-n,x) \to \infty$ as $n \to \infty$, by Lemma 2.6

$$M_m(X, T, T^{-1}, x) \nrightarrow 1$$

as $m \to \infty$ a.e. in X. As in Lemma 3.3 it can easily be seen that

$$\limsup_{m\to\infty} M_m(X, T, T^{-1}, y)$$

is invariant in Y and hence $M_m(X, T, T^{-1}, y) \nrightarrow 1$ as $m \to \infty$ a.e. in Y.

Remark. It follows from the remark following Lemma 2.6 that

$$\limsup_{m\to\infty} M_m(X, T, T^{-1}, y) = \infty \text{ a.e. in } Y.$$

Let us write
$$M_m(T, A, x) = (1/m) \sum_{i=0}^{m-1} f_A(T^i x).$$

EXAMPLE 5. LEMMA 3.5. *Let T_1 be an e.m.p. transformation of $[0,\infty)$ on to itself defined by $X = [0,1)$, D, A_k* and *$h(k) = 2l(k)$ where $l(k) = 2^{2^k}$. Let $A = \bigcup_{k=1}^{\infty} \bigcup_{i=0}^{l(k)-1} T_1^i A_k$. Let T_2 be defined by putting $T_2(y) = T_1^{-1}(y)$ if $y \notin X$* and *$T_2 y = (Dx, 2l(k)-1)$ if $y = (x, 0)$, $x \in A_k$. Then*

$$\liminf_{n\to\infty} M_m(T_1, A, y) = \tfrac{1}{2}$$

while $\liminf_{n\to\infty} M_m(T_2, A, y) = 0$ a.e. in Y.

Proof. Let q, m be respectively the first non-negative integers such that $D^q x \in B_N$, $T_1^m x \in B_N$. Then $0 \leqslant q \leqslant 2^N$ and if $D^n x \in A_s$ with $0 \leqslant n < q$ then $s \leqslant N$ and hence $h(s) \leqslant 2 . 2^{2^N}$. Hence $0 \leqslant m \leqslant 2^{N+1} . 2^{2^N}$. Let $T_1^m x \in A_{N+k}$, then $k \geqslant 1$. Let $p = m + h(N+k)$. Then

$$M_p(T_1, A, x) \leqslant (2 . 2^{2^{N+k}})^{-1} . (2^{N+1} . 2^{2^N} + 2^{2^{N+k}}) \leqslant \tfrac{1}{2} + 2^{N+1}(2 . 2^{2^N})^{-1}.$$

Now $p \to \infty$ as $N \to \infty$ and hence $\liminf_{m\to\infty} M_m(T_1, A, x) \leqslant \frac{1}{2}$. On the other hand, let m be any integer and let j be the largest integer $\leqslant m$ such that $T_1^j x \in X$. Then $T_1^j(x) \in A_r$ for some integer r. Now clearly $m-j \leqslant h(r)$. If $m-j \geqslant l(r)$ then

$$M_m(T_1, A, x) \geqslant (j+h(r))^{-1} . ((j/2)+l(r)) \geqslant (j+h(r))^{-1} . (j+h(r))/2 = \tfrac{1}{2}.$$

If $m-j < l(r)$ then

$$M_m(T_1, A, x) > (1/m).((j/2)+m-j) = (j+2(m-j))/2(j+m-j)$$
$$= \tfrac{1}{2}+(m-j)/m \geqslant \tfrac{1}{2}.$$

Thus
$$\liminf_{m\to\infty} M_m(T_1, A, x) = \tfrac{1}{2}.$$

As for T_2 we have for the same x

$$M_{m+l(N+k)}(T_2, A, x) \leqslant 2^{N+1}.2^{2^N}.(m+2^{2^{N+1}})^{-1}$$

which approaches zero as $N \to \infty$. Hence $\liminf_{m\to\infty} M_m(T_2, A, x) = 0$. Clearly $\liminf_{m\to\infty} M_m(T_i, A, y)$ is invariant a.e. in Y for $i = 1, 2$. This completes the proof of the lemma.

Remark. By the same method employed in Lemma 3.5 we obtain

$$\limsup_{m\to\infty} M_m(T_i, A, y) = \begin{cases} 1 & \text{if} \quad i = 1 \\ \tfrac{1}{2} & \text{if} \quad i = 2 \end{cases} \text{a.e. in } Y.$$

Remark. Notice that the induced transformations on X of both T_1 and T_2 in examples 3 and 5 coincide and equal D. It should also be remarked here that with the help of Lemma 2.1 it is possible to construct an example satisfying the results of Lemma 3.5 substituting for D above any measure preserving transformation S of X on to itself.

THEOREM 1. *Let T_1 and T_2 be two e.m.p. transformations of a σ-finite measure space (Y, γ, m) on to itself and let $f(y)$ and $g(y)$ be integrable functions on Y with $\int f \neq 0$, $\int g \neq 0$. Let*

$$R_m(T_1, T_2, f, g, y) = M_m(f, T_1, T_2, y)/M_m(g, T_1, T_2, y),$$

where $M_m(f, T_1, T_2, y) = \sum_{i=0}^{m-1} f(T_1^i y) \Big/ \sum_{i=0}^{m-1} f(T_2^i y)$ and $M_m(g, T_1, T_2, y)$ is defined analogously. Then $R_m(T_1, T_2, f, g, y) \to 1$ as $m \to \infty$ a.e. in Y.

Proof. We apply Hopf's theorem ((cf. **10**)) to each of T_1 and T_2 and obtain $\sum_{i=0}^{m-1} f(T_j^i y) \Big/ \sum_{i=0}^{m-1} g(T_j^i y) \to f_j^*(y)$ a.e. in Y where $j = 1, 2$, and $f_j^*(y)$ is a constant a.e. which is equal to $\int f / \int g$. But

$$R_m(T_1, T_2, f, g, y) = \Big[\sum_{i=0}^{m-1} f(T_1^i y) \Big/ \sum_{i=0}^{m-1} g(T_1^i y)\Big]\Big[\sum_{i=0}^{m-1} g(T_2^i y) \Big/ \sum_{i=0}^{m-1} f(T_2^i y)\Big]$$

and hence $R_m(T_1, T_2, f, g, y) \to 1$ as $n \to \infty$ a.e. in Y. Combining now Lemmas 3.3 and 3.4, the remark following Lemma 3.4 and Theorem 1 we obtain

THEOREM 2. *There exist two e.m.p. transformations of* $[0,\infty)$ *on to itself such that (with the same notation as in Theorem* 1)

$$\limsup_{m\to\infty} M_m(f, T_1, T_2, y) = \infty$$

for all integrable functions f *with* $\int f \neq 0$ *and almost all* y. *Moreover* T_2 *can be chosen to be* T_1^{-1}.

Notice that it follows from Theorem 1 that $\limsup_{m\to\infty} M_m(f, T_1, T_2, y)$ and $\liminf_{m\to\infty} M_m(f, T_1, T_2, y)$ are independent of f and g within the required restrictions and hence are functions of T_1 and T_2 only.

4. A counter example to a conjecture of W. Hurewicz

Let (Y, γ, m) be a measure space with $m(Y) = 1$. Let T be an ergodic transformation of Y on to itself. Let us recollect our notation

$$M_n(T, m_1, m_2, A) = \sum_{i=0}^{n-1} m_1(T^i A) \Big/ \sum_{i=0}^{n-1} m_2(T^i A)$$

and $M_n(T, m, A) = (1/n)\sum_{i=0}^{n-1} m(T^i A)$ where m_1, m_2, and m are measures defined on γ.

EXAMPLE 6. LEMMA 4.1. *Let the ergodic transformation* T *be such that there exists no finite invariant measure* $\mu \sim m$. *Then there exist two normalized measures* μ_1 *and* μ_2 *such that* $\mu_1 \sim \mu_2 \sim m$ *and a set* $B \in \gamma$ *such that* $M_n(T, \mu_1, \mu_2, B) \nrightarrow 1$ *as* $n \to \infty$.

Proof. Under the conditions stated above there exists a set $X \in \gamma$ such that $m(X) > 0$ and $M_n(T, m, X) \to 0$ as $n \to \infty$ (cf. (**5**)). Let X and Y be decomposed as in 1.1. Then, with the same notation as in 1.1 and discarding a set of measure zero we have $X = \bigcup_k A_k$ and $Y = \bigcup_k B_{k-1}$ where $B_{k-1} = \bigcup_{i=0}^{h(k)-1} T^i A_k$. Let μ_1 be the measure determined on γ by putting

$$\mu_1(A) = [m(A)/m(T^i A_k)] . (2^{h(1)+h(2)+\ldots h(k-1)+i+1})^{-1}$$

for every $A \in \gamma$, $A \subseteq T^i A_k$, $0 \leqslant i < h(k)$, $k = 1, 2, \ldots$. It is clear that $\mu_1 \sim m$ and that $\mu_1(Y) = 1$. Moreover $M_n(T, \mu_1, X) \to 0$ as $n \to \infty$ (cf. (**4**), Theorem 5). Hence we also have $M_n\left(T, \mu_1, \bigcup_{i=0}^{p-1} T^i X\right) \to 0$ as $n \to \infty$ for any finite integer p.

Let $B = \bigcup_s D_{s-1}$ where $D_{s-1} = \bigcup_{i=0}^{t_s-1} T^i A_{k(s)}$, where $t_s = t(s) = [h(k(s))/2]$ and $k(s)$ is a sequence of positive integers chosen as follows:

Let $k(1) = 1$ and let $C_1 = \bigcup_{i=0}^{t_1-1} T^i A_1 = D_0$. Suppose $k(1)$, $k(2)$,..., $k(s)$ are defined and let $C_s = \bigcup_{j=1}^{s} D_{j-1}$. Now $C_s \subseteq \bigcup_{i=0}^{s-1} T^i X$ and hence there exists an integer $N(s)$ such that $M_n(T, \mu_1, C_s) < 2^{-100s}$ for every $n > N(s)$.

Let $k(s+1)$ be any integer satisfying

$$\text{(i) } k(s+1) > 2N(s), \quad \text{and} \quad \text{(ii) } 2^{-k(s+1)} < 2^{-100(s+1)}.$$

Thus the sequence $\{k(s)\}$ is defined by induction.

Let μ_2 be the measure defined on γ by putting

$$\mu_2(A) = (\tfrac{1}{2}M).[m(A)/m(T^i A_{k(s)})].h^{-k(s)}.2^{-(s-1)}$$

for every measurable set A such that $A \subseteq T^i A_{k(s)}$, $0 \leqslant i < h(k(s))$, $s = 1, 2, \ldots$, where $M = \bigcup_s B_{k(s)-1}$ and $\mu_2(A) = m(A)$ if $A \in \gamma$ and

$$A \wedge \bigcup_s B_{k(s)-1} = \varnothing.$$

For every $s > 1$, $B = C_{s-1} \cup D_{s-1} \cup F_{s-1}$ where C_{s-1} and D_{s-1} have been defined and $F_{s-1} = \bigcup_{j=s+1}^{\infty} D_{j-1}$. Now

$$M_{t(s)}(T, \mu_1, B) = M_{t(s)}(T, \mu_1, C_{s-1}) + M_{t(s)}(T, \mu_1, D_{s-1}) + M_{t(s)}(T, \mu_1, F_{s-1}).$$

Since $h(k(s)) \geqslant k(s)$, $s > 1$ we have

$$M_{t(s)}(T, \mu_1, C_{s-1}) < 2^{-100(s-1)},$$

$$\begin{aligned} M_{t(s)}(T, \mu_1, D_{s-1} \cup F_{s-1}) &\leqslant \mu_1(D_{s-1} \cup F_{s-1}) \\ &\leqslant \sum_{i=0}^{\infty} (\tfrac{1}{2}) 2^{-(h_1+h_2+\ldots h_{k(s-1)}+i)} = 2^{-(h_1+\ldots h_{k(s-1)})} \\ &\leqslant 2^{-h_{k(s-1)}} \leqslant 2^{-k(s-1)} < 2^{-100(s-1)}; \end{aligned}$$

on the other hand

$$\begin{aligned} M_{t(s)}(T, \mu_2, B) &\geqslant M_{t(s)}(T, \mu_2, D_{s-1}) \\ &= \frac{1}{t(s)} \sum_{i=0}^{t_s-1} t_s \frac{1}{h(k(s))} 2^{-(s-1)} \tfrac{1}{2}M \geqslant 2^{-(s-1)} \tfrac{1}{2}M \tfrac{1}{4}. \end{aligned}$$

Thus
$$M_{t(s)}(T, \mu_1, \mu_2, B) \leqslant \frac{8}{M.2^{99(s-1)}}$$

and since $t(s) \to \infty$ as $s \to \infty$ we obtain $\liminf M_{t(s)}(T, \mu_1, \mu_2, B) = 0$.

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