PROBLEMS AND RESULTS ON THE THEORY OF INTERPOLATION. I

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Let $\begin{matrix} & x_1^{(1)} & \\ x_1^{(2)} & & x_2^{(2)} \\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot \end{matrix}$ be a triangular matrix of real numbers in $(-1,+1)$, $-1 \leqq x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leqq 1$. It is well known that the unique polynomial of degree at most $n-1$, which assumes the values y_k at $x_k^{(n)} (1 \leqq k \leqq n)$, is given by (the upper index n will be omitted if there is no danger of confusion)

$$\sum_{k=1}^{n} y_k l_k(x), \quad l_k(x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)}, \quad \omega(x) = \prod_{i=1}^{n} (x-x_i).$$

Clearly

$$l_k(x_k) = 1, \quad l_k(x_i) = 0 \quad \text{for} \quad 1 \leqq i \leqq n, \quad i \neq k.$$

Let $f(x)$ be a continuous function in $(-1,+1)$. Put

$$L_n(f(x)) = \sum_{k=1}^{n} y_k l_k(x),$$

i. e. $L_n(f(x))$ assumes the values $f(x_k)$ at x_k $(1 \leqq k \leqq n)$.

Let $f(x)$ be continuous in $(-1,+1)$. A well known theorem of HAHN[1] states that the sequence $L_n(f(x_0))$ converges to $f(x_0)$ if and only if

$$\sum_{k=1}^{n} |l_k(x_0)| < c \tag{1}$$

where c is independent of n.

FABER[2] first proved that for every point group there exists a function $f(x)$ continuous in $(-1,+1)$ for which $L_n(f(x))$ does not converge uniformly to $f(x)$ in $(-1,+1)$, FABER in fact proved that for every point group

$$\overline{\lim_{n=\infty}} \max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| = \infty. \tag{2}$$

Actually FABER proved slightly more, he shoved that

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| > c \log n. \tag{3}$$

[1] H. HAHN, Über das Interpolationsproblem, *Math. Zeitschrift*, 1 (1918), pp. 115—142.

[2] G. FABER, Über die interpolatorische Darstellung stetiger Funktionen, *Jahresb. der Deutschen Math. Ver.*, **23** (1914), pp. 190—210.

From (2) and (3) it does not yet follow that to every point group there exists a point x_0 and a continuous function $f(x)$ so that the sequence $L_n(f(x_0))$ does not converge to $f(x_0)$, or what is the same thing (by HAHN's theorem) that there exists an x_0 $(-1 \leqq x_0 \leqq 1)$ so that

$$\overline{\lim_{n=\infty}} \sum_{k=1}^{n} |l_k(x)| = \infty. \tag{4}$$

S. BERNSTEIN[3] proved (4), and in fact he proved that for a certain x_0 and for infinitely many n

$$\sum_{k=1}^{n} |l_k(x_0)| > \left(\frac{2}{\pi} + o(1)\right) \log n. \tag{5}$$

In the case of the Chebyshev abscissae

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)| < \frac{2}{\pi} \log n + O(1). \tag{6}$$

Thus in some sense (5) is best-possible. (5) holds in fact for the Chebyshev abscissae for every x_0 and infinitely many n.

In this paper we shall prove that for every point group (4) holds for almost all x. In fact, we shall prove the following stronger

THEOREM 1. *Let ε and A be any given numbers, and let $n > n_0 = n_0(\varepsilon, A)$. Further let $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$ be any n points. Then the measure of the set in x $(-\infty < x < \infty)$ for which*

$$\sum_{k=1}^{n} |l_k(x)| \leqq A \tag{7}$$

holds, is less than ε.

Theorem 1 clearly implies that (4) holds for almost all x in $(-1, +1)$. It is well known that (4) does not have to hold for all x in $(-1, +1)$, in fact, it is easy to see that there exists a Cantor set S so that for every x in S

$$\sum_{k=1}^{n} |l_k(x)| < c \tag{8}$$

where c does not depend on n or x. To see this it suffices to start with the roots of $T_n(x)$ and push two consecutive roots close together; we suppress the details which can easily be supplied by the reader. By slightly more complicated arguments one can show that the set S can have Hausdorff dimension 1.

[3] S. BERNSTEIN, Sur la limitation des valeurs etc., *Bull. Acad. Sci. de l'URSS*, (1931), No. 8, pp. 1025—1050.

By more complicated arguments we could prove the following stronger

THEOREM 2. *Let* $n > n_0(A, c, \beta, \varepsilon)$ *be sufficiently large,* $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$. *Then the measure of the set in* x *for which*

$$\sum_{k=1}^{n} |l_k(x)| \leqq A$$

holds, is less than $c/\log n$ $(c = c(A))$. *Further if* $\eta = \eta(\varepsilon)$ *is sufficiently small, then the measure of the set in* x *for which*

$$\sum_{k=1}^{n} |l_k(x)| \leqq \eta \log n$$

holds, is less than ε.

The case of the Chebyshev abscissae shows that Theorem 2 is best-possible. We do not give the proof of Theorem 2 since it is similar but more complicated than that of Theorem 1.

In a subsequent paper I shall prove that for every point group there exists a point x_0 $(-1 \leqq x_0 \leqq 1)$ so that for infinitely many n

$$\sum_{k=1}^{n} |l_k(x_0)| > \frac{2}{\pi} \log n - c. \tag{9}$$

In view of (6) (9) is best-possible.

Here the following question could be asked: For which distribution of points $-1 \leqq x_1 < \cdots < x_n \leqq 1$ is

$$\max_{-1 \leqq x \leqq 1} \sum_{k=1}^{n} |l_k(x)|$$

a minimum? It has been conjectured that this point group $x_1, x_2, \ldots, x_n$ is uniquely determined by the fact that all the $n+1$ maxima of $\sum_{k=1}^{n} |l_k(x)|$ in (x_i, x_{i+1}) $(0 \leqq i \leqq n+1,\ x_0 = -1,\ x_{n+1} = +1)$ are equal. Denote the value of this minimum by U_n. We would further conjecture that for every other n points in $(-1, +1)$ $(-1 \leqq x'_1 < x'_2 < \cdots \leqq x'_n \leqq 1)$ at least one of the $n+1$ maxima of $\sum_{k=1}^{n} |l_k(x)|$ in (x'_i, x'_{i+1}) $(0 \leqq i \leqq n+1)$ is less than U_n. The only result I can prove in this direction is that at least one of these maxima is less than $n^{1/2}$.[4]

It might be further worth-while to formulate the following conjecture: Let $|z_i| = 1$ $(1 \leqq i \leqq n)$. Then

$$\max_{|z| \leqq 1} \sum_{k=1}^{n} |l_k(z)|$$

is a minimum if and only if the z_i's are the roots of $z^n = a, |a| = 1$.

[4] P. ERDŐS, Some remarks on polynomials, *Bull. Amer. Math. Soc.*, **53** (1947), pp. 1169—1176, Theorem 2 (p. 1171).

G. GRÜNWALD[5] and J. MARCINKIEWICZ[6] proved simultaneously and independently that there exists a continuous function $f(x)$ so that the sequence of Lagrange interpolatory polynomials $L_n(f(x))$ taken at the Chebyshev abscissae diverge for every x $(-1 \leqq x \leqq 1)$. In fact, they prove that for all x $(-1 \leqq x \leqq 1)$ $\overline{\lim\limits_{n=\infty}} L_n(f(x)) = \infty$. In a subsequent paper I hope to prove the following result: Let $\begin{matrix} & x_1^{(1)} & \\ x_2^{(1)} & & x_2^{(2)} \\ \cdots & \cdots & \cdots \end{matrix}$ be any point group. Then there exists a continuous function $f(x)$ so that for almost all x, $\overline{\lim\limits_{n=\infty}} L_n(f(x)) = \infty$.

On the other hand, it is not true that there always exists a continuous $f(x)$ so that $L_n(f(x))$ diverges at all those x for which (4) holds. In fact, I can construct a point group so that for every continuous function $f(x)$ there are continuum many points x_0 for which (4) holds and nevertheless $L_n(f(x_0)) \to f(x_0)$ as n tends to infinity.

If (4) holds for every x_0 $(-1 \leqq x_0 \leqq 1)$, I can not decide whether there exists an $f(x)$ so that $L_n(f(x))$ diverges for every x $(-1 \leqq x \leqq 1)$.

One final remark. It is not difficult to construct a point group so that for every continuous function $f(x)$ and every x_0 there should exist a sequence n_k (depending on $f(x)$ and x_0) so that

$$\lim_{k=\infty} L_{n_k}(f(x_0)) \to f(x_0). \tag{10}$$

We can, in fact, construct a point group so that for every x_0

$$\underline{\lim} \sum_{k=1}^{n} |l_k(x_0)| = 1 \tag{11}$$

which by HAHN's theorem implies (10).

The proof of (11) is easy, we just outline it. Let $x_1, \ldots, x_i, x_{i+2}, \ldots, x_n$ be $n-1$ roots of $T_n(x)$ and $x_{i+1} - x_i = o\left(\frac{1}{n(\log n)^{1/2}}\right)$. Then a simple computation shows that for $x_i \leqq x \leqq x_{i+1}$

$$\sum_{k=1}^{n} |l_k(x)| = 1 + o(1).$$

Now $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^{1/2}} = \infty$, thus it is easy to see that we can construct a point group whose n^{th} row is the above-modified Chebyshev point group and so

[5] G. GRÜNWALD, Über die Divergenzerscheinungen etc., *Annals of Math.*, **37** (1936), pp. 908—918.

[6] J. MARCINKIEWICZ, Sur la divergence des polynômes d'interpolation, *Acta Sci. Math. Szeged*, **8** (1937), pp. 131—135.

that every x_0 $(-1 \leqq x_0 \leqq 1)$ should be contained infinitely often in a „short" interval (x_i, x_{i+1}) and thus (11) follows.

Now we prove Theorem 1. We put $x_0 = -1, x_{n+1} = +1$. Henceforth $c_1, c_2, \ldots$ will denote positive absolute constants independent of ε, A and n. The points x $(-\infty < x < \infty)$ which satisfy (7) we shall call bad points. Thus we have to prove that the measure of the bad points is less than ε. First we prove the following

LEMMA 1. *The measure of the bad points x which satisfy any of the conditions*

a) *are not in* $(-1, +1)$,

b) *for which* $\min\limits_{1 \leqq k \leqq n} |x - x_i| < \varepsilon/12n$,

c) *which are in intervals* (x_i, x_{i+1}) *satisfying* $x_{i+1} - x_i > c_1 A/\varepsilon n$,

is less than $\varepsilon/2$.

Let x_0 be a bad point not in $(-1, +1)$. By interpolating x^{n-1} on the points x_i $(1 \leqq i \leqq n)$ we obtain for all x

$$x^{n-1} = \sum_{k=1}^{n} x_k^{n-1} l_k(x)$$

or

$$x_0^{n-1} < \sum_{k=1}^{n} |l_k(x_0)|. \tag{12}$$

Now if $|x_0| \geqq 1 + \dfrac{A}{n}$, then $x^{n-1} > A$, or we have from (12) that for $|x| \geqq 1 + A/n$

$$\sum_{k=1}^{n} |l_k(x)| > A,$$

or if x_0 is bad we have $|x_0| < 1 + A/n$. Thus the measure of the bad points satisfying a) is less than $2A/n < \varepsilon/6$ for $n > n_0$.

By similar arguments we can easily show that the measure of the bad points satisfying a) is less than c_2/n^2. In fact, it is easy to show that the measure of the points x outside $(-1, +1)$ for which

$$\max_{1 \leqq k \leqq n} |l_k(x)| < A,$$

is less than c_3/n^2.

Since the number of the points x_i is $n+2$, it is clear that the measure of the points in x satisfying b) is less than $\varepsilon/6$.

Now we have to deal with the bad points satisfying c). Let $-1 \leqq \xi \leqq 1$. It is well known[7] that there exists a polynomial $p_{n-1}(x)$ of degree at most

[7] See e. g. P. ERDŐS and P. TURÁN, On interpolation. II, *Annals of Math.*, **39** (1938), p. 712.

$n-1$ for which

(13) $$p_{n-1}(\xi)=1, \quad |p(x)|<\min\left(c_4, \frac{c_5}{n(x-\xi)}\right) \quad \text{for} \quad -1 \leqq x \leqq 1.$$

Assume now that

(14) $$\min_{1\leqq k\leqq n} |\xi - x_k| > c_5 A/n.$$

Then we obtain from (13) and (14)

(15) $$|p_{n-1}(x_k)| < \frac{1}{A} \qquad (1 \leqq k \leqq n).$$

Thus from (15) and the identity $p_{n-1}(\xi)\equiv\sum_{k=1}^{n} p_{n-1}(x_k) l_k(\xi)$ we obtain

$$\sum_{k=1}^{n} |l_k(\xi)| > A,$$

or a point ξ can be bad only if

(16) $$\min_{1\leqq k\leqq n} |\xi - x_k| \leqq c_5 A/n.$$

The number of intervals (x_i, x_{i+1}) satisfying $x_{i+1}-x_i > c_1 A/\varepsilon n$ is clearly less than $\varepsilon n/c_1 A$ and thus from (16) the measure of the bad points satisfying c) is less than

$$\frac{2c_5 A}{n} \cdot \frac{\varepsilon n}{c_1 A} = \frac{2c_5 \varepsilon}{c_1} < \frac{\varepsilon}{6},$$

if $c_1 > 12 c_5$. Thus the proof of Lemma 1 is complete.

To complete the proof of our theorem we only have to prove that the measure of the bad points not satisfying any of the conditions a), b) and c) is less than $\varepsilon/2$. In other words, we have to prove that the measure of the points x (satisfying $-1\leqq x \leqq 1$) which are in intervals (x_i, x_{i+1}) satisfying $\frac{\varepsilon}{6n} < x_{i+1}-x_i < \frac{c_1 A}{\varepsilon n}$ and for which

(17) $$x_i + \frac{\varepsilon}{12n} < x < x_{i+1} - \frac{\varepsilon}{12n}$$

and which satisfy (7), is less than $\varepsilon/2$.

If the above statement is false there clearly must exist at least $\varepsilon^2 n/2c_1 A$ intervals (x_i, x_{i+1}), $x_{i+1}-x_i < c_1 A/\varepsilon n$, which contain bad points satisfying (17). But then there must exist at least $\varepsilon^2 n/4c_1 A$ such intervals

(18) $$(x_{i_r}, x_{i_r+1}) \qquad \left(1 \leqq r \leqq \left[\frac{\varepsilon^2 n}{4c_1 A}\right]\right)$$

which do not have an endpoint in common and each of which contains bad points satisfying (17). Now we prove

LEMMA 2. *Assume without loss of generality that* $|\omega'(x_i)| \geqq |\omega'(x_{i+1})|$. *Then for any* x *satisfying* (17) *and* $x_{i+1}-x_i \leqq c_1 A/\varepsilon n$ *(i. e. not satisfying* c)*) we have*

$$|l_{i+1}(x)| > c_6 \varepsilon^2/A.$$

It is known[8] (and easy to see) that for $x_i < x < x_{i+1}$

$$l_i(x) + l_{i+1}(x) > 1, \tag{19}$$

or by $|\omega'(x_i)| \geqq |\omega'(x_i)|$ we have for $x_i < x < x_{i+1}$

$$\left|\frac{\omega(x)}{\omega'(x_{i+1})(x-x_i)}\right| + \left|\frac{\omega(x)}{\omega'(x_{i+1})(x-x_{i+1})}\right| > 1. \tag{20}$$

Now since x satisfies (17) and (x_i, x_{i+1}) does not satisfy c) we have

$$|x-x_i| > \frac{12c_1 A}{\varepsilon^2}|x-x_{i+1}|. \tag{21}$$

Thus from (20) and (21)

$$|l_{i+1}(x)|\left(1+\frac{12c_1A}{\varepsilon^2}\right) > 1$$

which proves the lemma.

LEMMA 3. *Let* $y_1, y_2, \ldots, y_t$ *be any* t $(t > t_0)$ *distinct numbers in* $(-1, +1)$ *not necessarily in increasing order. Then for at least one* u $(1 \leqq u \leqq t)$

$$\sum_{i=1}^{u-1}{}' \frac{1}{|y_u - y_i|} > \frac{t \log t}{8}.$$

If the lemma would be false we would have

$$\sum_{1\leqq i<j\leqq t} \frac{1}{y_i'-y_j'} < \frac{t^2 \log t}{8} \tag{22}$$

where the y''s are ordered by size, i. e. $-1 \leqq y_1' < y_2' < \cdots < y_t' \leqq 1$. But it is easy to see that (22) is false. To see this observe that

$$\sum_{i=1}^{t-k}(y_{i+k}'-y_i') \leqq 2k,$$

and using the inequality between the arithmetic and harmonic mean, we obtain

$$\sum_{i=1}^{t-k}\frac{1}{y_{i+k}'-y_i'} \geqq (t-k)\frac{t-k}{2k} = \frac{(t-k)^2}{2k}$$

or

$$\sum_{1\leqq i<j\leqq t}\frac{1}{y_i'-y_j'} = \sum_{k=1}^{t-1}\sum_{i=1}^{t-k}\frac{1}{y_{i+k}'-y_i'} > \sum_{k=1}^{t-1}\frac{(t-k)^2}{2k} > \frac{t^2\log t}{8}$$

for $t > t_0$ which proves Lemma 3.

[8] See P. ERDŐS and P. TURÁN, On interpolation. III, *Annals of Math.*, **41** (1940), Lemma IV, p. 529.

Consider now the intervals (18). By assumption each of them contains bad points satisfying (17). Define

$$y_{i_r}=x_{i_r} \quad \text{if} \quad |\omega'(x_{i_r})|\geqq|\omega'(x_{i_r+1})|, \quad \text{otherwise} \quad y_{i_r}=x_{i_r+1}.$$

Reorder the y_{i_r} as follows:

$$|\omega'(y_1)|\leqq|\omega'(y_2)|\leqq\cdots\leqq|\omega'(y_t)|, \quad t=\left[\frac{\varepsilon^2 n}{4c_1 A}\right]. \tag{23}$$

By Lemma 3 there is a y_u so that

$$\sum_{i=1}^{u}\frac{1}{|y_u-y_i|}>\frac{t\log t}{8}. \tag{24}$$

We have (say) $y_u=x_{i_r}$. Now we show that the interval

$$x_{i_r}+\frac{\varepsilon}{12n}<x<x_{i+1}-\frac{\varepsilon}{12n} \tag{25}$$

does not contain any bad points, and this contradiction will complete the proof of Theorem 1. To see this let x satisfy (25).

We have form (24) and (23) by a simple computation that for the x's satisfying (25) we have for $n>n_0(\varepsilon)$

$$\sum_{i=1}^{u-1}\frac{1}{|x-y_i|}>\frac{t\log t}{16}>\frac{n\log n}{20}. \tag{26}$$

Further clearly

$$\sum_{k=1}^{n}|l_k(x)|>\sum{}'\,|l_k(x)| \tag{27}$$

where the dash indicates that the summation is extended only over those x_k's which are equal to one of the y_i $(1\leqq i\leqq u)$. By (23) we have $(x_{i_r}=y_u)$

$$\sum|l_k'(x)|=\sum{}'\left|\frac{\omega(x)}{\omega'(x_k)(x-x_k)}\right|\geqq\left|\frac{\omega(x)}{\omega'(y_u)(x-y_u)}\right|\sum_{i=1}^{u-1}\left|\frac{x-y_u}{x-y_i}\right|. \tag{28}$$

Now, by Lemma 2, (23) and (26) we have $\left(|x-y_u|>\frac{\varepsilon}{12n}\text{ by (25)}\right)$

$$\left|\frac{\omega(x)}{\omega'(y_u)(x-y_u)}\right|\sum_{i=1}^{u-1}\left|\frac{x-y_u}{x-y_i}\right|>\frac{c_6\varepsilon^2}{A}\cdot\frac{\varepsilon}{12n}\sum_{i=1}^{u-1}\frac{1}{x-y_i}>\frac{c_6\varepsilon^3}{12n}\frac{n\log n}{20}>A \tag{29}$$

which completes the proof of Theorem 1.

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