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MATHEMATICS

On Sets Which Are Measured by Multiples of Irrational Numbers

by

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The frequency of naturals n satisfying a condition Φ is defined as the limit

$$\operatorname{fr}\{n: n \text{ satisfies } \Phi\} = \lim_{N\to\infty} \frac{1}{N} \overline{\overline{\{n: n \text{ satisfies } \Phi,\ n \leqslant N\}}}$$

provided this limit exists. ($\overline{\overline{A}}$ denotes the power of A).

We say that a set A ($A \subset [0, 1)$) belongs to *the class* Ξ if for every irrational ξ the frequency $\operatorname{fr}\{n: n\xi \in A (\mathrm{mod}\,1)\}$ exists and does not depend on the choice of ξ. It is well-known that every Jordan measurable set belongs to Ξ and, moreover, the frequencies $\operatorname{fr}\{n: n\xi \in A(\mathrm{mod}\,1)\}$ are equal to the measure of A. Further, it is easy to verify that every Hamel base (mod 1) belongs to Ξ, which shows that sets belonging to Ξ may be Lebesgue non-measurable.

We say that a class Ξ_0 is *the base* of the family Ξ if for every $A \in \Xi$ there exists a set $B \in \Xi_0$ such that

$$\operatorname{fr}\{n: n\xi \in A \dot{-} B (\mathrm{mod}\,1)\} = 0 \text{ *)}$$

for any irrational ξ.

We say that a class Ξ_1 is *the weak base* of the family Ξ if for every $A \in \Xi$ there exists a set $B \in \Xi_1$ such that

$$\operatorname{fr}\{n: n\xi \in A \dot{-} B\ (\mathrm{mod}\,1)\} = 0$$

for almost all ξ.

The purpose of this note is the investigation of Lebesgue measurability of sets belonging to a base or to a weak base of the family Ξ. Namely, we shall prove with the aid of the axiom of choice

*) $A \dot{-} B$ denotes the symmetric difference of the sets A and B.

THEOREM 1. *Every base of the family Ξ contains $2^{2^{\aleph_0}}$ Lebesgue non-measurable sets.*

THEOREM 2. *Every weak base of the family Ξ contains at least $2^{\aleph_1}$ Lebesgue non-measurable sets.*

COROLLARY. *Under assumption of the continuum hypothesis every weak base of the family Ξ contains $2^{2^{\aleph_0}}$ Lebesgue non-measurable sets.*

Before proving the Theorems, we shall prove three Lemmas. Let us introduce the following notations

$$\mathcal{U}_k = \{x : x \text{ rational}, \ k(k-1) \leqslant x < k^2\} \quad (k=1,2,\ldots),$$

$$(1) \qquad \mathcal{W}_+ = \bigcup_{k=1}^{\infty} \mathcal{U}_k, \quad \mathcal{W}_- = \{x : -x \in \mathcal{W}_+\}, \quad \mathcal{W} = \mathcal{W}_+ \cup \mathcal{W}_- .$$

LEMMA 1. *For every rational number $r \neq 0$ the equality*

$$\operatorname{fr}\{n : nr \in \mathcal{W}\} = \tfrac{1}{2}$$

is true.

Proof. It is sufficient to prove that, for every positive rational number r, the equality $\operatorname{fr}\{n : nr \in \mathcal{W}_+\} = \frac{1}{2}$ holds.

Let $I^{(k)}(r)$ denote the number of such naturals n that $nr \in \mathcal{U}_k$.

Obviously,

$$(2) \qquad \left[\frac{k}{r}\right] - 1 \leqslant I^{(k)}(r) \leqslant \left[\frac{k}{r}\right] + 1,$$

where $[x]$ denotes the greatest integer $\leqslant x$.

$I_N(r)$ will denote the number of such naturals n ($n \leqslant N$) that $nr \in \mathcal{W}_+$. If $k \leqslant [\sqrt{Nr}]$ and $nr \in \mathcal{U}_k$ then, in view of (1), $nr < k^2 \leqslant [\sqrt{Nr}]^2 \leqslant Nr$, which implies the inequality $n < N$. Hence, we obtain the inequality

$$I_N(r) \geqslant \sum_{k=1}^{[\sqrt{Nr}]} I^{(k)}(r) \quad (N = 1, 2, \ldots).$$

Consequently, taking into account (2), we have the inequality

$$(3) \qquad I_N(r) \geqslant \sum_{k=1}^{[\sqrt{Nr}]} \left[\frac{k}{r}\right] - [\sqrt{Nr}] \quad (N = 1, 2, \ldots).$$

Further, if $k > [\sqrt{Nr}] + 1$ and $nr \in \mathcal{U}_k$ then, in view of (1), $nr \geqslant k(k-1) > Nr$, which implies the inequality $n > N$. Hence, we obtain the following inequality

$$I_N(r) \leqslant \sum_{k=1}^{[\sqrt{Nr}]+1} I^{(k)}(r) \quad (N = 1, 2, \ldots).$$

Consequently, taking into account (2), we have the inequality

$$I_N(r) \leqslant \sum_{k=1}^{[\sqrt{Nr}]+1} \left[\frac{k}{r}\right] + [\sqrt{Nr}]+1 \quad (N = 1, 2, \ldots).$$

Hence, and from (3), it follows that

$$I_N(r) = \sum_{k=1}^{[\sqrt{Nr}]} \left[\frac{k}{r}\right] + o(N) \quad (N = 1, 2, \ldots). \tag{4}$$

Setting $r = \frac{p}{q}$, $[\sqrt{Nr}] = d_N p + s_N$ $(0 \leqslant s_N < p)$, where p, q, d_N and s_N are integers we obtain by simple reasoning

$$\sum_{k=1}^{[\sqrt{Nr}]} \left[\frac{k}{r}\right] = \frac{1}{2} pq d_N(d_N - 1) + d_N \sum_{j=1}^{p} \left[\frac{j}{r}\right] + \sum_{j=1}^{s_N} \left[\frac{j}{r}\right] + q d_N s_N$$

$$= \frac{1}{2} pq d_N^2 + o(N) = \frac{1}{2} N + o(N).$$

Hence, in virtue of (4), we obtain the equality $I_N(r) = \frac{1}{2} N + o(N)$. The Lemma is thus proved.

By γ we denote the first ordinal number of the power continuum. Let us consider a Hamel base $x_0 = 1, x_1, x_2, \ldots, x_a, \ldots$ $(a < \gamma)$. Every irrational number x may be represented as a linear combination with rational coefficients $x = r_0 + r_1 x_{a_1} + \ldots + r_n x_{a_n}$, where $1 \leqslant a_1 < a_2 < \ldots < a_n$, $r_1 \neq 0$. In the sequel we shall use the notations $r(x) = r_1$, $a(x) = a_1$.

Let $\mathfrak{B}$ be the class of all subsets of the set of all positive ordinals less than γ. Obviously,

$$\overline{\overline{\mathfrak{B}}} = 2^{2^{\aleph_0}}. \tag{5}$$

For every $V \in \mathfrak{B}$ we define the set

$$A_V = \{x : x \text{ irrational}, \ 0 < x < 1, \ r(x) \in \mathfrak{W}, \ a(x) \in V\} \cup$$
$$\{x : x \text{ irrational}, \ 0 < x < 1, \ r(x) \operatorname{non} \in \mathfrak{W}, \ a(x) \operatorname{non} \in V\}.$$

LEMMA 2. *For every* $V \in \mathfrak{B}$ $A_V \in \Xi$. *Moreover,*

$$\operatorname{fr}\{n : n\xi \in A_V (\operatorname{mod} 1)\} = \tfrac{1}{2}$$

for each irrational ξ.

Proof. Since $r(n\xi) = nr(\xi)$ and $a(n\xi) = a(\xi)$ we have the following equality

$$\{n : n\xi \in A_V (\operatorname{mod} 1),\ n \leqslant N\} = \begin{cases} \{n : nr(\xi) \in \mathfrak{W},\ n \leqslant N\} & \text{if} \quad a(\xi) \in V, \\ \{n : nr(\xi) \operatorname{non} \in \mathfrak{W},\ n \leqslant N\} & \text{if} \quad a(\xi) \operatorname{non} \in V. \end{cases}$$

Hence, according to Lemma 1, for every irrational ξ, we obtain the equality $\mathrm{fr}\{n : n\xi \in A_V (\mathrm{mod}\,1)\} = \frac{1}{2}$, which was to be proved.

LEMMA 3. *Let D $(D \subset [0, 1))$ be a set satisfying the equality*

$$\mathrm{fr}\{n : n\xi \in A_V \dot{-} D (\mathrm{mod}\,1)\} = 0 \quad (V \in \mathfrak{B}) \tag{6}$$

for almost all ξ. Then, D is Lebesgue non-measurable.

Proof. Suppose the contrary, i. e. that D is Lebesgue measurable. First we shall prove that, for every interval U $(U \subset [0, 1))$ and for almost all ξ,

$$\mathrm{fr}\{n : n\xi \in A_V \cap U (\mathrm{mod}\,1)\} = \tfrac{1}{2}|U|, \tag{7}$$

where $|U|$ denotes the measure of U.

For brevity, we shall use the notations

$$\mathcal{W}^0 = \mathcal{W}, \quad \mathcal{W}^1 = \mathcal{W}', \quad V^0 = V \quad \text{and} \quad V^1 = V',$$

where $\mathcal{W}'$ denotes the complement of the set $\mathcal{W}$ to the set of all rationals and V' denotes the complement of the set V to the set of all positive ordinal numbers less than γ.

For every rational r $(r \neq 0)$ we denote by $k_n^{(i)}(r)$ $(n = 1, 2, \ldots)$ the sequence of naturals n such that $nr \in \mathcal{W}^i$ $(i = 0, 1)$.

It is well-known ([2], p. 344-346) that, for every sequence of integers $k_1 < k_2 < \ldots$ and for every interval U $(U \subset [0, 1))$,

$$\mathrm{fr}\{n : k_n\xi \in U (\mathrm{mod}\,1)\} = |U|$$

for almost all ξ. Consequently, for almost all ξ and for every rational r $(r \neq 0)$, the equality

$$\mathrm{fr}\{n : k_n^{(i)}(r)\xi \in U (\mathrm{mod}\,1)\} = |U| \quad (i = 0, 1). \tag{8}$$

From the definitions of the set A_V and the sequences $k_n^{(i)}(r)$ it follows directly that

$$\overline{\overline{\{n : n\xi \in A_V \cap U (\mathrm{mod}\,1),\, n \leqslant N\}}} = \overline{\overline{\{n : k_n^{(i)}(r(\xi))\xi \in U (\mathrm{mod}\,1),\, k_n^{(i)}(r(\xi)) \leqslant N\}}}$$

and

$$\overline{\overline{\{n : k_n^{(i)}(r(\xi)) \leqslant N\}}} = \overline{\overline{\{n : nr(\xi) \in \mathcal{W}^i,\, n \leqslant N\}}}$$

if $a(\xi) \in V^i$ $(i = 0, 1)$. Hence,

$$\frac{1}{N}\overline{\overline{\{n : n\xi \in A_V \cap U (\mathrm{mod}\,1),\, n \leqslant N\}}} =$$

$$= \frac{1}{N}\overline{\overline{\{n : nr(\xi) \in \mathcal{W}^i,\, n \leqslant N\}}} \cdot \frac{\overline{\overline{\{n : k_n^{(i)}(r(\xi))\xi \in U (\mathrm{mod}\,1),\, k_n^{(i)}(r(\xi)) \leqslant N\}}}}{\overline{\overline{\{n : k_n^{(i)}(r(\xi)) \leqslant N\}}}}$$

if $\alpha(\xi) \in V^i$ $(i = 0, 1)$, which implies, in view of (8) and Lemma 1, the equality

$$\operatorname{fr}\{n : n\xi \in A_V \cap U(\operatorname{mod} 1)\} =$$
$$= \operatorname{fr}\{n : nr(\xi) \in \mathcal{W}^i\} \operatorname{fr}\{n : k_n^{(i)}(r(\xi))\xi \in U(\operatorname{mod} 1)\} = \tfrac{1}{2}|U|.$$

The formula (7) is thus proved.

From (6) and (7) it follows directly that, for every interval U and for almost all ξ, the following equality holds

$$\operatorname{fr}\{n : n\xi \in D \cap U(\operatorname{mod} 1)\} = \tfrac{1}{2}|U|. \tag{9}$$

Further, in view of a Theorem of Raikov ([1], p. 377),

$$\lim_{N\to\infty} \int_0^1 \left| \frac{1}{N} \sum_{n=1}^{N} \chi(n\xi) - |D \cap U| \right| d\xi = 0,$$

where χ is the characteristic function of $D \cap U$ extended on the line with the period 1. Hence, and from (9), for every interval U, we obtain the equality $|D \cap U| = \frac{1}{2}|U|$, which contradicts the Lebesgue density theorem. The Lemma is thus proved.

Proof of Theorem 1. Let $V \in \mathfrak{V}$. By B_V we denote a set belonging to the base of the family Ξ such that

$$\operatorname{fr}\{n : n\xi \in A_V \dot{-} B_V(\operatorname{mod} 1)\} = 0$$

for each irrational ξ. (According to Lemma 2 the sets A_V $(V \in \mathfrak{V})$ belong to Ξ). Applying Lemma 3 we find that the sets B_V are Lebesgue non-measurable. Since the power of the base is $\leqslant 2^{2^{\aleph_0}}$, then, to prove the Theorem, it suffices to show, in virtue of (5), that the function $V \to B_V$ establishes a one-to-one correspondence between sets V and sets B_V. Suppose $V_1 \neq V_2$. There is then an irrational ξ_0 such that $\alpha(\xi_0) \in V_1 \dot{-} V_2$. Taking into account the definition of A_V, we have $n\xi_0 \in A_{V_1} \dot{-} A_{V_2}$ (mod 1) $(n = 1, 2, \ldots)$. Hence, $\operatorname{fr}\{n : n\xi_0 \in A_{V_1} \dot{-} A_{V_2}$ (mod 1)$\} = 1$, which implies $\operatorname{fr}\{n : n\xi_0 \in B_{V_1} \dot{-} B_{V_2}(\operatorname{mod} 1)\} = 1$. Consequently, $B_{V_1} \neq B_{V_2}$.

Theorem 1 is thus proved.

Proof of Theorem 2. By $\mathfrak{V}_0$ we denote the class of all subsets of the set of all denumerable ordinal numbers. Obviously, $\mathfrak{V}_0 \subset \mathfrak{V}$ and $\overline{\overline{\mathfrak{V}}}_0 = 2^{\aleph_1}$. By C_V $(V \in \mathfrak{V}_0)$ we denote a set belonging to the weak base of the family Ξ such that

$$\operatorname{fr}\{n : n\xi \in A_V \dot{-} C_V(\operatorname{mod} 1)\} = 0$$

for almost all ξ. According to Lemma 3, the sets C_V are Lebesgue non-measurable. To prove the Theorem it suffices to show that the function: $V \to C_V$ $(V \in \mathfrak{V}_0)$ establishes a one-to-one correspondence between the sets V and the sets C_V. Suppose $V_1 \neq V_2$ $(V_1, V_2 \in \mathfrak{B}_0)$. Similarly to the preceding proof we find that

$$\operatorname{fr}\{n: n\xi \in C_{V_1} \dot{-} C_{V_2} (\operatorname{mod} 1)\} = 1$$

for almost all ξ satisfying the condition $\alpha(\xi) \in V_1 \dot{-} V_2$. Obviously, to prove the inequality $C_{V_1} \neq C_{V_2}$ it is sufficient to show that the outer Lebesgue measure of the set $S = \{\xi: \alpha(\xi) \in V_1 \dot{-} V_2\}$ is positive. Suppose the contrary, i. e.

$$|S| = 0 . \tag{10}$$

Let η be the first ordinal number belonging to $V_1 \dot{-} V_2$. It is easy to verify that the real line R may be represented as the denumerable union of sets congruent to S

$$R = \bigcup_{\substack{r_1, \ldots, r_n \\ \alpha_1, \ldots, \alpha_n \leqslant \eta}} \Big\{x + \sum_{i=1}^{n} r_i x_{\alpha_i} : x \in S\Big\},$$

where $r_1, \ldots, r_n$ are rationals $(n = 1, 2, \ldots)$. Hence, and from (10), it follows that $|R| = 0$, which is impossible. The Theorem is thus proved.

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