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CONCERNING APPROXIMATION WITH NODES

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This note contains a remark on the subject treated by Paszkowski [1], [2].

Define

$$E_n = \min_{P_n(x)} \max_{-1\leqslant x\leqslant 1} |f(x)-P_n(x)|, \qquad E'_n = \min_{P_n(0)=f(0)} \max_{-1\leqslant x\leqslant 1} |f(x)-P_n(x)|$$

where $P_n(x)$ runs through all polynomials of degree n. Clearly

$$(1) \qquad E_n \leqslant E'_n \leqslant 2E_n.$$

I shall prove that there exists an $f(x)$ satisfying

$$(2) \qquad \overline{\lim_{n=\infty}} E'_n/E_n = 2.$$

Let $n_k \to \infty$ sufficiently fast. Put

$$f(x) = \sum_{k=1}^{\infty} T_{2n_k}(x)/k!,$$

where $T_n(x)$ is the n-th Tchebycheff polynomial. Because of $|T_{2n}(0)| = 1$ we have

$$(3) \qquad E_{2n_k} \leqslant \big(1+o(1)\big)/(k+1)! \qquad \Big(P_n(x) = \sum_{j=1}^{k} T_{2n_j}(x)/j!\Big).$$

Next we show that

$$(4) \qquad E'_{2n_k} \geqslant \big(2+o(1)\big)/(k+1)!.$$

Equality (2) follows from (1), (3) and (4). Thus we only have to show (4).

Let $\Theta_{2n_k}(x)$ be the polynomial of degree $\leqslant 2n_k$ for which

$$\max_{-1\leqslant x\leqslant 1} |f(x)-\Theta_{2n_k}(x)| = E'_{2n_k}.$$

Denote by y the nearest extremum of $T_{2n_{k+1}}(x)$ to 0; clearly $|y| < \pi/n_{k+1}$ and $|T_{2n_{k+1}}(y) - T_{2n_{k+1}}(0)| = 2$. If the n_k tend to ∞ fast enough we clearly have

$$(5) \qquad |f(y) - f(0)| = (2 + o(1))/(k+1)!,$$

i. e. $f(x) = \Sigma_1(x) + \Sigma_2(x) + \Sigma_3(x)$ where

$$\Sigma_1(x) = \sum_{j=1}^{k} T_{2n_j}(x)/j!, \qquad \Sigma_2(x) = T_{2n_{k+1}}(x)/(k+1)!,$$

$$\Sigma_3(x) = \sum_{j=k+2}^{\infty} T_{2n_j}(x)/j!.$$

Now clearly

$$\Sigma_1(y) - \Sigma_1(0) = O\left(\frac{n_k^2}{n_{k+1}}\right) = o\left(\frac{1}{(k+1)!}\right)$$

if $n_k \to \infty$ fast enough, *i. e.* if $|g_n(x)| \leqslant 1$, $g_n(x)$ is a polynomial of degree n, then by Markoff $|g_n'(x)| \leqslant n^2$, $-1 \leqslant x \leqslant 1$,

$$\Sigma_2(y) - \Sigma_2(0) = \frac{2}{(k+1)!}, \qquad \Sigma_3(y) - \Sigma_3(0) = o\left(\frac{1}{(k+1)!}\right).$$

Thus (5) follows.

Now $|\Theta_{2n_k}(x)| \leqslant 2e$ for $-1 \leqslant x \leqslant 1$ (since $|f(x)| \leqslant e$) and since $\Theta_{2n_k}(x)$ is a polynomial of degree at most $2n_k$, we have, by Markoff's theorem $|\Theta'_{2n_k}(x)| \leqslant 8en_k^2$, $-1 \leqslant x \leqslant 1$. Thus

$$(6) \qquad |\Theta_{2n_k}(y) - \Theta_{2n_k}(0)| \leqslant 8en_k^2 y < 8\pi e n_k^2/n_{k+1} = o\left(\frac{1}{(k+1)!}\right)$$

if $n_k \to \infty$ fast enough. Thus from (5) and (6)

$$|f(y) - \Theta_{2n_k}(y)| = (2 + o(1))/(k+1)!;$$

Hence (2) follows and our proof is complete.

By a simple modification of this argument it is easy to construct an $f(x)$ with

$$\overline{\lim}\, E_n'/E_n = 2, \qquad \underline{\lim}\, E_n'/E_n = 1$$

(it suffices to put $f(x) = \sum T_{n_k}(x)/k!$ where $n_{2k} \equiv 0 \pmod 2$, $n_{2k+1} \equiv 1 \pmod 2$ and $n_k \to \infty$ fast enough).

I expect that one can show $\lim E_n'/E_n = 2$ for suitable $f(x)$, but I have not succeeded in doing it.

Note of the Editors. It has been stated by Paszkowski ([2], theorem 5.2) that for the approximation with algebraic polynomials the inequality

$$\overline{\lim_{n\to\infty}}\, \varepsilon_n(\xi;\, T)/\varepsilon_n(\xi) \leqslant 2 \tag{7}$$

holds for an arbitrary continuous function $\xi(t)$ and for an arbitrary system T of nodes the notation being that of [1].

The relation (2) proved here by Erdös shows that (7) cannot be strengthened.

REFERENCES

[1] S. Paszkowski, *On the Weierstrass approximation theorem,* Colloquium Mathematicum 4 (1957), p. 206-210.

[2] — *On approximation with nodes,* Rozprawy Matematyczne 14, Warszawa 1957.

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