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MATHEMATICS

ON THE IRRATIONALITY OF CERTAIN SERIES

BY

P. ERDÖS

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Extending previous results of CHOWLA I [1]) proved that for every integer $t > 1$ the series

$$\sum_{n=1}^{\infty} \frac{d(n)}{t^n} \text{ and } \sum_{n=1}^{\infty} \frac{r(n)}{t^n}$$

are irrational, where $d(n)$ denotes the number of divisors of n and $r(n)$ denotes the number of solutions of $n = x^2 + y^2$. In my above paper I remarked that I cannot prove that any of the series

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{t^n}, \quad \sum_{n=1}^{\infty} \frac{\sigma(n)}{t^n}, \quad \sum_{n=1}^{\infty} \frac{\nu(n)}{t^n}$$

are irrational, where $\varphi(n)$ is Euler's φ function, $\sigma(n)$ the sum of the divisors of n and $\nu(n)$ the number of distinct prime factors of n. On the other hand by the methods used in the above paper I can prove without difficulty that the two series

$$\sum_{n=1}^{\infty} \frac{1}{t^{n+\nu(n)}}, \quad \sum_{n=1}^{\infty} \frac{1}{t^{n-\nu(n)}}$$

are irrational, but I failed to prove the same for the two series

$$\sum_{n=1}^{\infty} \frac{1}{t^{n+d(n)}}, \quad \sum_{n=1}^{\infty} \frac{1}{t^{n-d(n)}}$$

The main difficulty seems to be that I cannot prove that for infinitely many n

$$(1) \qquad \max_{m \leq n} (m + d(m < \min_{m > n}(m + d(m)).$$

(1) can be proved with $\nu(m)$ instead of $d(m)$ (2). I cannot prove anything about the series

$$\sum_{n=1}^{\infty} \frac{1}{t^{n+\varphi(n)}}, \quad \sum_{n=1}^{\infty} \frac{1}{t^{n+\sigma(n)}}, \quad \sum_{n=1}^{\infty} \frac{1}{t^{n+p_n}}$$

where p_n is the greatest prime factor of n (if in (1) $d(m)$ is replaced by $\varphi(n)$, $\sigma(n)$ or p_n (1) becomes false).

[1]) Indian Journal of Math. 12, 63–66 (1948).

[2]) In fact this is essentially contained in 1).

Quoting LANDAU [1]) I remark that all these statements do not yet justify writing a note. But I can (and will) prove that the two series

$$\sum_{n=1}^{\infty} \frac{1}{t^{\varphi(n)}}, \sum_{n=1}^{\infty} \frac{1}{t^{\sigma(n)}}$$

are irrational.

Denote $\sigma_k(n) = \sum_{d|n} d^k$. KAC and I [2]) conjectured that

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n!}$$

is irrational for every integer $k>0$. We proved this for $k=1$ and $k=2$, for $k>2$ the proof seems to present great difficulties.

STRAUS and I [3]) proved that if $n_1 < n_2 < \dots$ is a sequence of integers satisfying limsup $\log n_k / \log k = \infty$, then $\sum_{k=1}^{\infty} \frac{1}{t^{n_k}}$ is transcendental. By a modification of our method used there I can prove that if limsup $n_k/k^l = \infty$, then $\sum_{k=1}^{\infty} \frac{1}{t^{n_k}}$ does not satisfy an algebraic equation with integer coefficients of degree not exceeding l. I do not know to what extent this theorem can be improved, I do not know if a series $\sum_{k=1}^{\infty} \frac{1}{t^{n_k}}$ satisfying limsup $n_k/k = \infty$ can be an algebraic number. On the other hand I cannot even prove that if $n_k > ck^2$ then $\left(\sum_{k=1}^{\infty} \frac{1}{t^{n_k}}\right)^2$ is always irrational.

Theorem 1. The series

$$\sum_{n=1}^{\infty} \frac{1}{t^{\varphi(n)}} \text{ and } \sum_{n=1}^{\infty} \frac{1}{t^{\sigma(n)}}$$

are irrational.

First we prove three Lemmas.

Lemma 1. *Let a_k, $k=1, 2, \dots$ be a sequence of non-negative integers such that*

$$\lim\sup \frac{1}{n} \sum_{k=1}^{n} a_k < \infty. \tag{2}$$

Denote by $f(n)$ the number of k's $1 \leqslant k \leqslant n$ for which $a_k > 0$. Assume that $f(n) \to \infty$ and liminf $f(n)/n = 0$. *Then*

$$\sum_{k=1}^{\infty} \frac{a_k}{t^k}$$

is irrational.

[1]) Math. Zeitschrift 30, 610 (1929).

[2]) This was a problem in Amer. Math. Monthly 1, 264, (1954), for $k=2$ solution by R. BREUSCH, for $k=1$ solution by J. B. KELLY 60, 557, (1953).

Elemente der Math. 9, 18 Problem 154, (1954).

The Lemma is known[1]). I do not give the proof, since Lemma 4 will contain it essentially as a special case.

Lemma. 2. *The number of integers n for which* $\varphi(n) \leqslant x$ *holds is less than c x. The same holds for* $\sigma(n)$.

Since $\sigma(n) \geqslant n$ the Lemma obviously holds for $\sigma(n)$ with $c=1$. For $\varphi(n)$ the Lemma is known[2]) but for completeness I give the simple proof. We have

$$\sum_{m=1}^{x}\left(\frac{m}{\varphi(m)}\right)^2 = \sum_{m=1}^{x}\prod_{p|m}\left(1+\frac{1}{p}+\dots\right)^2 \leqslant \sum_{m=1}^{x}\prod_{p|m}\left(1+\frac{6}{p}\right) \leqslant$$

$$\leqslant \sum_{m=1}^{x}\sum_{d|m}\frac{6^{\nu(d)}}{d} < x\sum_{d=1}^{\infty}\frac{6^{\nu(d)}}{d^2} = c_1 x.$$

Thus clearly the number of integers $m<x$ with $m/\varphi(m) > r$ is less than $c_1 \cdot x/r^2$ where c_1 is an absolute constant independent of x and r. Thus the number of integers not exceeding $2^{k+1}x$ for which $m/\varphi(m) > 2^k$ is less than

$$\frac{c_1\, 2^{k+1}x}{2^{2k}} = \frac{c_1 x}{2^{k-1}} \tag{3}$$

But if $\varphi(m) \leqslant x$, then if $m > x$ we must have for some k, $k=0, 1, \dots$ $2^k x < m \leqslant 2^{k+1}x$ and $m/\varphi(m) > 2^k$. Thus by (3) the number of integers satisfying $\varphi(m) \leqslant x$ is less than

$$x\left(1+\sum_{k=0}^{\infty}\frac{c_1}{2^{k-1}}\right) < cx$$

which proves the Lemma.

Lemma 3. *The number of integers* $n \leqslant x$ *for which one of the equations* $\varphi(k)=n$ *or* $\sigma(k)=n$ *is solvable is* $o(x)$.

Lemma 3 is also known[3]), but for sake of completeness we give the proof. It will be more conveniant to prove the Lemma separately for $\varphi(k)$ and $\sigma(k)$. We want to prove that for every ε there exists an x_0 so that for $x > x_0$ the number of integers $n \leqslant x$ for which $\varphi(k)=n$ is solvable, is less than $\varepsilon\, x$. Choose first r so that $2^r > 2/\varepsilon$. If k has r or more distinct prime factors then $\varphi(k) \equiv 0 \pmod{2^r}$, hence the number of $n \leqslant x$ of the form $\varphi(k)$, where k has at least r distinct prime factors is less than $x/2^r < \varepsilon x/2$. If k has fewer than r prime factors, the $\varphi(k) > k/r$, thus since $\varphi(k) \leqslant x$ we can assume $k < r \cdot x$. But a well known theorem of LANDAU[4]) states that

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[1]) This was a problem in the Amer. Math. Monthly proposed by me **62**, **261**, **(1954)** solution by LORENTZ. The proof of lemma 4 will be similar to the proof of LORENTZ.

[2]) In fact TURÁN and I proved that the number of solutions of $\varphi(n) \leqslant x$ is $cx+o(x)$, (P. ERDÖS, Bull. Amer. Math. Soc. **51**, 543–544, (1945).

[3]) For $\varphi(k)$ this is due to SIVASANKARANARAYANA PILLAI and his proof easily applies for $\delta(k)$. For sharper results see P. ERDÖS Quarterly Journal **6**, 205–213, (1935). See also a recent paper by H. J. KANOLD, Journal Reine und Angew. Math. **195**, 180–195, (1955).

[4]) E. LANDAU, Handbuch der Lehre von der Verteilung der Primzahlen, Volume 1, page 211.

the number of integers not exceeding y having fewer than r distinct prime factors is less than

$$c\frac{y(\log\log y)^{r-1}}{(r-1)!\log y} \tag{4}$$

Thus for $x>x_0$ the number of $k<r\cdot x$, $\nu(k)<r$ is less than $\varepsilon x/2$, which completes the proof of the Lemma for $\varphi(k)$.

To prove the Lemma for $\sigma(k)$, we first observe that because of $\sigma(k)>k$, we can assume $k\leqslant x$. Write $k=a^2b$ where b is squarefree. If b has r or more prime factors then $\sigma(k)\equiv 0 \pmod{2^r}$. The number of integers $k\leqslant x$ with $a^2>16/\varepsilon^2$ is less than $x\sum_{a>4/\varepsilon}\frac{1}{a^2}<\frac{\varepsilon x}{4}$, and finally the number of integers $k=a^2b\leqslant x$ with $a\leqslant 4/\varepsilon$ and b having fewer than r prime factors is $o(x)$, by (4). Thus finally the number of integers $n\leqslant x$ for which $\sigma(k)=n$ is solvable is less than

$$\frac{\varepsilon}{2}x+\frac{\varepsilon}{4}x+o(x)<\varepsilon x,$$

which proves Lemma 3 for $\sigma(k)$.

The proof of Theorem 1 now follows easily. Denote by a_k the number of solutions of $\varphi(l)=k$ and by a' the number of solutions of $\sigma(l)=k$. We have

$$\sum_{n=1}^{\infty}\frac{1}{t^{\varphi(n)}}=\sum_{k=1}^{\infty}\frac{a_k}{t^k},\quad \sum_{n=1}^{\infty}\frac{1}{t^{\sigma(n)}}=\sum_{k=1}^{\infty}\frac{a'_k}{t^k}.$$

By Lemma 2 (2) is satisfied and by Lemma 3 $f(n)/n\to 0$ for both a_k and a'_k which completes the proof of Theorem 1.

Clearly the conclusion of Theorem 1 holds for the more general multiplicative functions considered by KANOLD [1]), but I expect that it will hold for a much more general class of multiplicative functions, but I have not yet succeeded in showing this.

Theorem 2. *Let $1<n_1<n_2<\dots$ be an infinite sequence of integers satisfying* $\limsup n_k/k^l=\infty$, *then*

$$\sum_{k=1}^{\infty}\frac{1}{t^{n_k}}$$

does not satisfy an algebraic equation with integer coefficients of degree not exceeding l.

First we prove

Lemma 4. *Let a_k and b_k be two sequences of non negative integers, the sequence of a's is supposed to be infinite. Denote by $f(n)$ and $g(n)$ the number of k's $1\leqslant k\leqslant n$ satisfying $a_k>0$, respectively $b_k>0$. Assume that there exists an s so that for all sufficiently large k*

$$a_k<k^s,\ b_k<k^s \tag{5}$$

[1] See foregoing page, note [3]).

and that there exists an infinite sequence m_i for which

(6) $$\sum_{k=1}^{m_i} (a_k+b_k) < c_1 m_i,\ f(m_i)=o(m_i),\ g(m_i)=o(m_i/\log m_i).$$

Further assume the following condition (C): There exists an absolute constant c_2 so that if i_1 and i_2 are two consecutive indices with $b_{i_1}>0$ and $b_{i_2}>0$, then for every x satisfying $i_1+c_2x<i_2$ there exists an index k satisfying $a_k>0$ and $i_1+x<k<i_1+c_2x$. Then

$$\sum_{k=1}^{\infty} \frac{a_k+\varepsilon_k b_k}{t^k},\ \varepsilon_k=\pm 1$$

is irrational.

Clearly Lemma 1 is a special case of Lemma 4. In Lemma 1 all the b's are 0 and $m_i=i$; $a_k>k^s$ is satisfied in Lemma 1 for every $s>1$ (because of (2)).

Put

$$A_k=\frac{a_k}{t}+\frac{a_{k+1}}{t^2}+\ldots,\ B_k=\frac{b_k}{t}+\frac{b_{k+1}}{t^2}+\ldots$$

To prove Lemma 4 we first have to show that for every $\varepsilon>0$ there are j's satisfying

(7) $$A_j+B_j<\varepsilon,\quad A_j>B_j.$$

Assume that we already proved (7), then we prove Lemma 4 as follows: If Lemma 4 would not hold we would have (u and v are integers)

(8) $$\frac{u}{v}=\sum_{k=1}^{\infty}\frac{a_k+\varepsilon_k b_k}{t^k}.$$

Choose $\varepsilon<\frac{1}{v}$. By (8) $vt^{j-1}\sum_{k=1}^{\infty}\frac{a_k+\varepsilon_k b_k}{t^k}$ is an integer. But by (7)

$$I=vt^{j-1}\sum_{k=1}^{\infty}\frac{a_k+\varepsilon_k b_k}{t^k}=I'+v(A_j+\vartheta B_j)\ \ (I,\ I' \text{ are integers, } |\vartheta|\leqslant 1),$$

an evident contradiction, since by (7) $0<v\,(A_j+\vartheta B_j)<1$, which proves the lemma.

Thus we only have to prove (7). Denote by α_i the number of indices $k\leqslant\frac{m_i}{2}$ for which

(9) $$A_k+B_k\geqslant\varepsilon$$

and by β_i the number of indices $k\leqslant\frac{m_i}{2}$ for which

(10) $$A_k>B_k$$

First we show that

(11) $$\alpha_i=o(m_i)$$

and that for a certain constant c,

(12) $$\beta_i > c_3\, m_i\,.$$

Clearly (11) and (12) imply (7). Thus it will suffice to prove (11) and (12).

We split the indices $k \leqslant m_i/2$ which satisfy (9) into two classes. In the first class are the indices k for which there exists a j such that $k \leqslant j \leqslant k+l$ and for which $a_j + b_j > 0$. It follows from (6) that the number of indices of the first class is not greater than

(13) $$(l+1)\,(f(m_i) + g\,(m_i)) = o(m_i).$$

For the indices k of the second class [1]) we have by (5) and (6) (the dash in $\sum'$ indicates that the summation is extended over the $k \leqslant m_i/2$ of the second class):

$$\sum{}' (A_k + B_k) \leqslant \sum_{r=1}^{m_i} (a_r + b_r)\left(\frac{1}{t^l} + \frac{1}{t^{l+1}} + \dots\right) + \sum_{L > m_i} \Big(\sum_{r=1}^{L} (a_r + b_r)/t^{L-m_i/2}\Big) <$$

$$< 2\, c_1\, m_i/t^l + \sum_{L > m_i} 2\, \frac{L^{s+1}}{t^{L-m_i/2}} = 2\, c_1\, m_i/t^l + o(m_i) < \eta\, m^i$$

for all η if l is sufficiently large. Thus the number of k's of the second class which satisfy (9) is less than

(14) $$\frac{\eta}{\varepsilon}\, m_i = o(m_i)$$

since η can be chosen arbitrarily small. (13) and (14) clearly imply (11).

Now we prove (12). Let $a_k > 0$ and $i \geqslant k$ be the smallest index for which $b_i > 0$. Assume that $i > k + c_4 \log k$ where c_4 is a sufficiently large absolute constant. Then $A_k > B_k$. This is almost obvious, since by (5) if c_4 is sufficiently large

$$A_k - B_k \geqslant \frac{1}{t} - \sum_{i > k + c_4 \log k} \frac{b_i}{t^{i-k}} > \frac{1}{t} - \sum_{i > k + c_4 \log k} \frac{i^s}{t^{i-k}} \geqslant$$

$$\geqslant \frac{1}{t} - \left(\frac{(k + c_4 \log k)^s}{t^{c_4} \log k}\right)\left(1 + \frac{2}{3} + \frac{4}{9} + \dots\right) > 0$$

(i.e. the terms of $\sum\limits_{i > k + c_4 \log k} \frac{i^s}{t^{i-k}}$ drop off faster than a geometric series of quotient 2/3).

Thus if the above holds for k and $j \leqslant k$ is such that there is no $b_r > 0$ with $j \leqslant r \leqslant k$, then we have

(15) $$A_j > B_j$$

Let now j and j' be the indices of two consecutive positive b's (i.e. $b_j > 0$, $b_{j'} > 0$ and $b_k = 0$ for $j < k < j'$). Clearly from (6)

$$\sum{}' (j' - j) = o(m_i)$$

[1]) For the k of the second clan we have $a_k = a_{k+1} = \dots = a_{k+l} = b_k = b_{k+1} = \dots = b_{k+l} = 0$.

where the dash indicates that $j'-j \leqslant 2c_4 \log m_i$ and $j < m_i/2$. Thus

(16) $$\sum'' (j'-j) = \tfrac{1}{2} m_i + o(m_i)$$

where the double dash indicates that $j'-j > 2c_4 \log m_i$, $j < m/ /2$ (if $j < m_i/2 < j'$, then we put $j' = \frac{m_i}{2}$). Let now $j'-j > 2c_4 \log m_i$. Let $k_1 > j$ be the largest index for which $a_{k_1} > 0$ and $k_1 \leqslant (j+j')/2$. By (C) we have

(17) $$k_1 > j + (j'-j)/2\, c_2 \text{ or } k_1 - j > (j'-j)/2\, c_2$$

By (15) we have for $j < k < k_2$

(18) $$A_k > B_k.$$

(16 and (17) implies that

(19) $$\sum'' (k_1 - j) > (\tfrac{1}{2} m_i + o\,(m_i))/2\, c_2 > c_3\, m_i.$$

(18 and (19) clearly imply (12) and thus the proof of lemma 4. is complete.

With a little more trouble I can prove the following sharper

Lemma 4'. *Let a_k and b_k be two sequences of non negative integers. The a-s are supposed to be infinite. Assume that*

$$\limsup\, (a_k + b_k)^{1/k} < t,$$

and that there exist an infinite sequence m_i for which

$$\sum_{k=1}^{m_i} (a_k + b_k) < c_1\, m_i,\ f(m_i) = o(m_i),\ g(m_i) = o(m_i).$$

Further assume that (C) holds. Then

$$\sum_{k=1}^{\infty} \frac{a_k + \varepsilon_k b_k}{t^k},\ \varepsilon_k = \pm 1$$

is irrational.

The proof is very similar to that of lemma 4, only the proof of $\beta_i > c_3\, m_i$ is a bit more troublesome here.

Now we can prove Theorem 2. Put $\alpha = \sum\limits_{k=1}^{\infty} \frac{1}{t^{n_k}}$, and assume that

(20) $d_0\, \alpha^{l_1} + d_1\, \alpha^{l_1 - 1} + \ldots + d_{l_1} = 0,\ l_1 \leqslant l,\ d_0 > 0$, the $d'\,s$ are integers.

First of all we can assume that for a certain c_5

(21) $$n_{k+1} < c_5\, n_k,\ 1 \leqslant k < \infty.$$

For if (21) does not hold then $\limsup n_{k+1}/n_k = \infty$, and therefore

$$\alpha - \sum_{i=1}^{k} \frac{1}{t^{n_i}} = \alpha - \frac{u_k}{t^{n_k}} < \frac{2}{t^{n_{k+1}}} = 2\left(\frac{1}{t^{n_k}}\right)^{n_{k+1}/n_k},$$

thus α is a Liouville number and therefore transcendental, which contradicts (20).

Expanding by the polynomial theorem we obtain

$$d_0 \alpha^{l_1} = \sum_{k=1}^{\infty} \frac{a_k}{t^k}, \; d_1 \alpha^{l_1-1} + \ldots + d_{l_1} = \sum_{k=1}^{\infty} \frac{\varepsilon_k b_k}{t^k}.$$

(5) is clearly satisfied with $s = l_1 + 1$. Further since $\lim \sup n_k / k^l = \infty$, there exists a sequence n_{k_i} for which $\lim n_{k_i} / k_i^l = \infty$. Now $a_k > 0$ if and only if k is the sum of l_1 n's, and $b_k > 0$ implies that k is the sum of $l_1 - 1$ or fewer n's. Thus by a simple argument

$$f(n_{k_i}) \leqslant k_i^{l_1} = o(n_{k_i}), \; g(n_{k_i}) \leqslant k_i^{l_1-1} = o(n_{k_i}^{1-1/l_1}).$$

Further by a simple argument

$$\sum_{j=1}^{n_{k_i}} (a_j + b_j) < c_5 \, k_i^{l_1} = o(n_{k_i}).$$

Thus (6) is satisfied with $m_i = n_{k_i}$. To show that (C) is satisfied we observe that if $b_k > 0$ then k is the sum of say r n's, $r < l_1$. Thus all the integers $k + (l_1 - r) n_i$, $i = 1, 2, \ldots$ are the sum of l_1 n's. Thus

$$a_{k+(l_1-r)n_i} > 0, \quad i = 1, 2, \ldots$$

Thus in view of (21) (C) is satisfied with $c_2 = l_1 c_4$. Hence by lemma 4

$$d_0 \alpha^{l_1} + d_1 \alpha^{l_1-1} + \ldots + d_{l_1} = \sum_{k=1}^{\infty} \frac{a_k + \varepsilon_k b_k}{t^k}$$

is irrational, which contradicts (20), and thus Theorem 2 is proved.