A LIMIT THEOREM FOR THE MAXIMUM OF NORMALIZED SUMS OF INDEPENDENT RANDOM VARIABLES

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1. **Introduction.** The main purpose of this paper is to prove the following theorem:

THEOREM 1. *Let* $X_1, X_2, \cdots$ *be independent random variables with mean* 0, *variance* 1, *and a uniformly bounded third absolute moment. Put* $S_k = X_1 + X_2 + \cdots + X_k$ *and let*

$$U_n = \max_{1 \le k \le n} \frac{S_k}{k^{1/2}} \cdot \tag{1.1}$$

Then

$$\lim_{n\to\infty} \Pr\left\{U_n < (2 \log \log n)^{\frac{1}{2}} + \frac{\log \log \log n}{2(2 \log \log n)^{1/2}} + \frac{t}{(2 \log \log n)^{1/2}}\right\}$$
$$= \exp\left(-e^{-t}/2(\pi)^{\frac{1}{2}}\right), \qquad -\infty < t < \infty.$$

The corresponding limit theorem for $U_n' = \max_{1 \le k \le n} S_k/n^{\frac{1}{2}}$ is well known [3], but the distribution of U_n is considerably more delicate, mainly because, speaking roughly, $S_k/k^{\frac{1}{2}}$ attains its maximum for a relatively small index, and the usual crude application of the central limit theorem will not work. Indeed, the above theorem is probably false if we drop the condition on the third absolute moment, even in the case of identically distributed X_i.

Theorem 1 solves, in an asymptotic form, the classical optional stopping problem (for example see Robbins [7]). Robbins gave a one-sided inequality for the distribution of $U_n'' = \max_{tn \le k \le n} S_k/k^{\frac{1}{2}}$, $0 < t < 1$, in the case of normally distributed X_i. In the case the X_i satisfy only the central limit theorem, Darling and Siegert [2] found the limiting distribution of U_n'' in terms of a Laplace transform, namely they found an explicit expression for

$$\int_0^\infty e^{-\lambda\sigma} \lim_{n\to\infty} \Pr\left\{\max_{e^{-2\sigma}n \le k \le n} \frac{S_k}{k^{1/2}} < \xi\right\} d\sigma.$$

The evaluation of the limiting distribution of U_n, given by (1.1), is however a qualitatively different matter.

This problem is also closely related to a problem posed by Levy [6] (footnote 19) on the law of the iterated logarithm for the Wiener process. Theorem 1 sheds some new light on the law of the iterated logarithm, and it may be true that the requirement on the third absolute moment could be replaced by the condition that the X_i are such that the law of the iterated logarithm holds.

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2. The method of proof. The idea behind the proof is quite simple though its execution is somewhat devious. We suppose first the X_i are Gaussian, independent, with mean 0 and variance 1. We then show there is a sequence $\{t_k\}$ and a stationary Gaussian stochastic process (the Uhlenbeck-Ornstein process) $X(t)$ such that the sequence $\{S_k/k^{\frac{1}{2}}\}$, $k = 1, 2, \cdots, n$, has the same joint distribution as $X(t_k)$, $k = 1, 2, \cdots, n$. It turns out, because certain machinery is available for stochastic processes, that the limiting distribution of $\max_{0<\tau<t} X(\tau)$ can be computed asymptotically when $t \to \infty$. And it is possible further to show that this limiting distribution is the same as the limiting distribution of $\max_{1 \le k \le n} X(t_k)$ when $n \to \infty$, and so the same for U_n when $n \to \infty$. Next an application of the so-called invariance principle of Erdös-Kac will conclude the proof.

In Section 3 below, the first part of this program is carried out by proving Theorem 1 in the special case of Gaussian random variables. In Section 4 the invariance principle is applied. In Section 5 we conclude with a few additional remarks, stated without proof.

3. Proof of Theorem 1 for Gaussian variables. For simplicity we have organized the exposition of this section in a series of 10 lemmas which culminate in the proof of Theorem 1 for Gaussian variables.

Let $X_1, X_2, \cdots$ be independent Gaussian random variables with means 0, variance 1; put $S_k = X_1 + X_2 + \cdots + X_k$. Let $X(t)$, $0 \le t < \infty$, be the Uhlenbeck stochastic process, that is, $X(t)$ is Gaussian, stationary, Markoffian with mean 0 and covariance $E(X(s)X(t)) = \exp(-|t-s|)$; $X(0)$ has its stationary distribution. Let U_n be as in (1.1).

We define

$$t_k = \tfrac{1}{2} \log k. \tag{3.1}$$

LEMMA 3.1. *The sequences* $\{S_k/k^{\frac{1}{2}}\}$, $\{X(t_k)\}$, $k = 1, 2, \cdots, n$, *have the same joint distribution.*

Proof. Both are Gaussian chains with mean 0 and

$$E\left(\frac{S_k}{k^{1/2}} \frac{S_j}{j^{1/2}}\right) = \frac{\min(k, j)}{(kj)^{1/2}} = \exp\left(-\left|\tfrac{1}{2}\log j - \tfrac{1}{2}\log k\right|\right) = E(X(t_k)X(t_j)).$$

We define

$$N(\alpha) = \text{integer } N \text{ such that } X(t_k) < \alpha, \qquad k = 1, 2, \cdots, N-1 \tag{3.2}$$
$$X(t_N) \ge \alpha, \qquad N = 1, 2, \cdots.$$

That is, $N(\alpha)$ is the smallest positive integer such that $X(t_N) \ge \alpha$.

LEMMA 3.2. $\Pr\{U_n < \alpha\} = \Pr\{N(\alpha) > n\}$.

Proof. Obvious.

We define

$$T(\alpha) = \sup\{t \mid X(\tau) < \alpha, \quad 0 \le \tau \le t\}, \tag{3.3}$$

$$K(\alpha) = \text{integer } K \text{ such that } t_{K-1} \le T(\alpha) < t_K. \tag{3.4}$$

These last two definitions make sense since the process $X(t)$ is continuous with probability one.

We define

$$L(\alpha) = t_{K(\alpha)} - t_{K(\alpha)-1}. \tag{3.5}$$

Using (3.1) to (3.5) the following relationships are easily established:

$$K(\alpha) = [e^{2T(\alpha)}] + 1 = e^{2T(\alpha)} + \delta, \qquad 0 \le \delta \le 1 \tag{3.6}$$

$$L(\alpha) = \tfrac{1}{2}\log K(\alpha) - \tfrac{1}{2}\log(K(\alpha) - 1) \le \frac{1}{K(\alpha)}, \qquad K(\alpha) \ge 2;$$

$$L(\alpha) \le e^{-2T(\alpha)}; \tag{3.7}$$

$$N(\alpha) \ge K(\alpha). \tag{3.8}$$

Denote by $T_x(\alpha)$ a random variable defined the same as $T(\alpha)$ in (3.3) but in place of having $X(0)$ with its stationary distribution we have $X(0) = x$.

LEMMA 3.3.

$$E\{\exp[-\xi T_x(\alpha)]\} = \frac{D_{-\xi}(-x)e^{x^2/4}}{D_{-\xi}(-\alpha)e^{\alpha^2/4}}, \qquad x \le \alpha, \qquad \xi \ge 0,$$

where $D_\nu(z)$ is the Weber function, cf. Whittaker and Watson [8; 347].

Proof. This is given in Darling-Siegert [2].

We define

$$\mu(\alpha) = \frac{(2\pi)^{\frac{1}{2}}}{\alpha} e^{\alpha^2/2}. \tag{3.9}$$

LEMMA 3.4. $\lim_{\alpha\to\infty} \Pr\{T(\alpha) > \mu(\alpha)y\} = e^{-y}$, $0 \le y < \infty$, *where $T(\alpha)$, $\mu(\alpha)$ are given in* (3.3), (3.9).

Proof. Let

$$\phi_\alpha(\xi) = \frac{D_{-\xi/\mu(\alpha)}(-x)e^{x^2/4}}{D_{-\xi/\mu(\alpha)}(-\alpha)e^{\alpha^2/4}} = E\left\{\exp\left[-\xi\frac{T_x(\alpha)}{\mu(\alpha)}\right]\right\}.$$

Now $D_0(t) = e^{-t^2/4}$ and since $\mu(\alpha) \to \infty$, $\alpha \to \infty$ the numerator in $\phi_\alpha(\xi)$ approaches 1 when $\alpha \to \infty$ for any x. We use the asymptotic expansion of $D_{-s}(-\alpha)\, e^{\alpha^2/4}$ for $\alpha \to \infty$, $0 \le s < M$ given in [8; 348] as follows:

$$D_{-s}(-\alpha)e^{\alpha^2/4} \sim e^{-s\pi i}\alpha^{-s}A + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(s)} e^{\alpha^2/2}\alpha^{s-1}B$$

$$A = 1 - \frac{-s(-s-1)}{2\alpha^2} + \cdots, \qquad B = 1 + \frac{(-s+1)(-s+2)}{2\alpha^2} + \cdots$$

and since $1/\Gamma(s) = s + O(s^2)$, $s \to 0$, we get from (3.9)

$$D_{-\xi/\mu(\alpha)}(-\alpha)e^{\alpha^2/4} = 1 + \frac{(2\pi)^{\frac{1}{2}}e^{\alpha^2/2}}{\alpha}\alpha^{\xi/[\mu(\alpha)]}\left(\frac{\xi}{\mu(\alpha)} + o\left(\frac{1}{\mu(\alpha)}\right)\right)$$

$$= 1 + \xi + o\left(\frac{1}{\mu(\alpha)}\right).$$

Hence $\phi_\alpha(\xi) \to 1/(1+\xi)$, $\alpha \to \infty$ for all x. Since $(1+\xi)^{-1}$ is the Laplace transform of e^{-y} we obtain the lemma.

LEMMA 3.5.

$$\lim_{\alpha\to\infty} \Pr\left\{T(\alpha) > \frac{\mu(\alpha)}{\alpha}\right\} = 1.$$

Proof. Follows easily from Lemma 3.4.

LEMMA 3.6. *If* $\epsilon = \epsilon(\alpha)$ *approaches* 0 *sufficiently slowly,* $\lim_{\alpha\to\infty} \Pr\{N(\alpha) \leq K(\alpha+\epsilon)\} = 1$. *It will suffice to take* $\epsilon = 1/\alpha^2$.

Proof. We calculate the conditional probability p

$$p = \Pr\left\{N(\alpha) > K(\alpha+\epsilon) \mid T(\alpha) > \frac{\mu(\alpha)}{\alpha}\right\}.$$

Now $T(\alpha) > \mu(\alpha)/\alpha$ means by (3.7) that

$$L(\alpha) \leq \exp\left[-\frac{2\mu(\alpha)}{\alpha}\right] = \exp\left[-\frac{2(2\pi)^{\frac{1}{2}}}{\alpha}e^{\alpha^2/2}\right] = \chi(\alpha),$$

and we denote the two right members by $\chi(\alpha)$. The event $N(\alpha) > K(\alpha+\epsilon)$ implies that $X(t)$ having reached the value $\alpha+\epsilon$ has decreased to a value less than α within a time interval less than $\chi(\alpha)$. Then, recalling the stationarity of $X(t)$,

$$p \leq \Pr\{X(\chi(\alpha)) < \alpha \mid X(0) = \alpha+\epsilon\}.$$

The conditional distribution of $X(\chi(\alpha))$, given $X(0) = \alpha+\epsilon$ is Gaussian with mean and variance, respectively,

$$m(\alpha) = (\alpha+\epsilon)e^{-\chi(\alpha)}$$
$$\sigma^2(\alpha) = (1-e^{-2\chi(\alpha)}) \leq 2\chi(\alpha).$$

Hence

$$p \leq \frac{e^{-\xi^2/2}}{(2\pi)^{1/2}\xi}$$

where

$$\begin{aligned}\xi &= \frac{(\alpha+\epsilon)e^{-\chi(\alpha)}}{\sigma(\alpha)} \geq \frac{(\alpha+\epsilon)e^{-\chi(\alpha)}}{(2\chi(\alpha))^{1/2}}\\ &= \frac{-\alpha(1-e^{-\chi(\alpha)})}{(2\chi(\alpha))^{1/2}} + \frac{\epsilon e^{-\chi(\alpha)}}{(2\chi(\alpha))^{1/2}}\\ &\geq -\frac{\alpha}{2}[\chi(\alpha)]^{\frac{1}{2}} + \frac{\epsilon}{(2\chi(\alpha))^{1/2}} - \frac{\epsilon}{2^{1/2}}[\chi(\alpha)]^{\frac{1}{2}}\\ &\geq \frac{\epsilon}{2[\chi(\alpha)]^{1/2}}, \qquad \alpha > \alpha_0,\end{aligned}$$

where α_0 is independent of ϵ, $\epsilon < 1$, say. Consequently, if $\epsilon(\alpha)$ is such that $\epsilon/\chi^{\frac{1}{2}}(\alpha) \to \infty$ then $p \to 0$. It is amply sufficient to have $\epsilon = 1/\alpha^2$ from the above definition of $\chi(\alpha)$. Using Lemma 3.5 we thus obtain $\Pr\{N(\alpha) > K(\alpha + \epsilon)\} \to 0$, $\alpha \to \infty$ for such an ϵ, and Lemma 3.6 is proved.

LEMMA 3.7. *If* $\epsilon = \epsilon(\alpha)$ *approaches zero sufficiently rapidly*, $T(\alpha + \epsilon) - T(\alpha) \to 0$ *in probability. It will suffice to take* $\epsilon = 1/\alpha^2$.

Proof. It is sufficient to prove $E\{\exp[-s(T(\alpha + \epsilon) - T(\alpha))]\} \to 1$, $s > 0$. By the same formula used in Lemma 3.3 and the same expansion used in the proof of Lemma 4,

$$E\{\exp\,[-s(T(\alpha + \epsilon) - T(\alpha))]\} = \frac{D_{-s}(-\alpha)e^{\alpha^2/4}}{D_{-s}(-\alpha - \epsilon)e^{(\alpha+\epsilon)^2/4}}$$

$$\sim \frac{e^{-s\pi i}\alpha^{-s}A + \dfrac{(2\pi)^{1/2}}{\Gamma(s)}\exp\,(\alpha^2/2)\alpha^{s-1}B}{e^{-s\pi i}(\alpha + \epsilon)^{-s}A + \dfrac{(2\pi)^{1/2}}{\Gamma(s)}\exp\,[(\alpha + \epsilon)^2/2](\alpha + \epsilon)^{s-1}B} \sim e^{\frac{1}{2}(\alpha^2 - (\alpha+\epsilon)^2)}$$

$$= e^{-\alpha\epsilon + \epsilon^2/2}$$

and hence if $\epsilon = o(1/\alpha)$ the last expression approaches 1, and the lemma is proved.

LEMMA 3.8. $\log N(\alpha) - 2T(\alpha) \to 0$ *in probability.*

Proof. Using (3.8) and Lemma 3.6 we have, for $\epsilon = 1/\alpha^2$,

$$\Pr\,\{K(\alpha) \leq N(\alpha) \leq K(\alpha + \epsilon)\} \to 1,\ \alpha \to \infty$$

and by (3.6)

$$\Pr\left\{1 + \delta_1 e^{-2T(\alpha)} \leq \frac{N(\alpha)}{e^{2T(\alpha)}} \leq e^{-2(T(\alpha+\epsilon) - T(\alpha))} + \delta_2 e^{-2T(\alpha)}\right\} \to 1, \qquad \alpha \to \infty$$

where $0 \leq \delta_i \leq 1$, $i = 1, 2$, and by Lemma 3.7

$$\frac{N(\alpha)}{e^{2T(\alpha)}} \to 1$$

in probability. Hence $\log N(\alpha) - 2T(\alpha) \to 0$ in probability as asserted.

LEMMA 3.9. $\lim_{\alpha\to\infty} \Pr\{\log N(\alpha) > 2\mu(\alpha)y\} = e^{-y}$.

Proof. This follows directly from Lemma 3.4 on using Lemma 3.8.

We are now ready to prove Theorem 1 for Gaussian random variables. Since by Lemma 3.2 we want an expression for $\Pr\{N(\alpha) > n\}$, by Lemma 3.9 we need to solve $2\mu(\alpha)y = \log n$ with respect to α for large n.

LEMMA 3.10. *If* $2\mu(\alpha)y = \log n$, $y > 0$, *then for* $n \to \infty$

$$\alpha = (2 \log \log n)^{\frac{1}{2}} + \frac{\log \log \log n}{2(2 \log \log n)^{1/2}} - \frac{\log ((4\pi)^{\frac{1}{2}} y))}{(2 \log \log n)^{1/2}} + o\left(\frac{1}{(\log \log n)^{1/2}}\right). \tag{3.10}$$

Proof. From (3.9) we have to solve

$$\frac{2(2\pi)^{\frac{1}{2}}}{\alpha} e^{\alpha^2/2} y = \log n$$

asymptotically for $n \to \infty$ with respect to α, and this yields after some calculation (3.10).

Now from Lemmas 3.2, 3.9 $\Pr\{U_n < \alpha\} = \Pr\{N(\alpha) > n\} = \Pr\{\log N(\alpha) > \log n\} = \Pr\{\log N(\alpha) > 2\mu(\alpha)y\} \to e^{-y}$ so that

$$\Pr\left\{U_n < (2 \log \log n)^{\frac{1}{2}} + \frac{\log \log \log n}{2(2 \log \log n)^{1/2}} - \frac{\log ((4\pi)^{\frac{1}{2}} y))}{(2 \log \log n)^{1/2}}\right\} \to e^{-y}$$

and Theorem 1 follows by putting $t = -\log (4\pi)^{\frac{1}{2}} y)$.

4. **Use of the invariance principle.** In the preceding section we have proved Theorem 1 in the special case of Gaussian variables. Now if in the case of general X_i satisfying the hypotheses of Theorem 1 we prove that the limiting distribution of U_n *exists* and is *independent* of the parent distribution of the X_i we have proved the theorem in the general case. This is the so-called invariance principle of Erdös-Kac [3].

In the following, $\epsilon_1, \epsilon_2, \cdots$ will denote positive numbers which can be chosen independently and arbitrarily small, and $c_1, c_2, \cdots$ will be positive quantities depending on various parameters, but independent of the distribution of the X_i and of n.

We suppose that $X_1, X_2, \cdots$ are independent, with mean 0, variance 1, $E(|X_i|^3) < c_1$. Let

$$S_n = X_1 + X_2 + \cdots + X_n, \qquad F_n(x) = \Pr\{S_n/n^{\frac{1}{2}} \leq x\},$$

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

We need the following two known theorems:

(a) the Berry-Esseen estimate [1], [4],

$$|F_k(x) - \Phi(x)| < \frac{c_2}{k^{1/2}},$$

and its multidimensional analogue; c_2 is independent of k, x and the distribution of the X_i;

(b) the "law of the iterated logarithm"; Feller [5],

$$\Pr\left\{\overline{\lim}\, \frac{S_n}{(2n \log\log n)^{1/2}} = 1\right\} = 1.$$

We define the function $f_n(y)$

$$f_n(y) = (2\log\log n)^{\frac{1}{2}} + \frac{\log\log\log n}{2(2\log\log n)^{1/2}} + \frac{y}{(2\log\log n)^{1/2}} \tag{4.1}$$

and the events $A_k = A_{k,n}(y)$, $k = 1, 2, \cdots, n$,

$$A_k = A_{k,n}(y) = \text{event that } S_k/k^{\frac{1}{2}} > f_n(y) \tag{4.2}$$

and the function $g_n(y)$

$$g_n(y) = \Pr\left\{\bigcup_{k\le n} A_k\right\}. \tag{4.3}$$

Thus $g_n(y) = \Pr\{U_n > y\}$ where U_n is given by (1.1) and we need to show that $\lim g_n(y)$ exists and is independent of the distributions of the X_i.

LEMMA 4.1. *Given $\epsilon_1 > 0$ there exists a $c_3 > 0$ such that $n > c_3$ implies $\Pr\{\bigcup_{k\le(\log n)^3} A_k\} < \epsilon_1$ for all $y \ge 0$.*

Proof. Given any c_4 there exists a c_5 such that $n > c_5$ implies $\Pr\{\bigcup_{k\le c_4} A_k\} < \epsilon_1/2$ and there exists c_6 such that $\Pr\{S_k < 3(k\log\log k)^{\frac{1}{2}}, k > c_6\} > 1 - \epsilon_1/2$ by the law of the iterated logarithm, (b) above, for $n > c_6$. Then

$$\begin{aligned} 1 - \frac{\epsilon_1}{2} &\le \Pr\{S_k/k^{\frac{1}{2}} < 3(\log\log k)^{\frac{1}{2}}, (\log n)^3 \ge k > c_6\} \\ &\le \Pr\{S_k/k^{\frac{1}{2}} < c_7(\log\log\log n)^{\frac{1}{2}}, (\log n)^3 \ge k > c_6\} \\ &\le \Pr\{\bigcap_{c_6<k\le(\log n)^3} A_k'\} = 1 - \Pr\{\bigcup_{c_6<k\le(\log n)^3} A_k\}. \end{aligned}$$

Hence given $\epsilon_1 > 0$ choose c_6 such that $\Pr\{\bigcup_{c_6<k\le(\log n)^3} A_k\} \le \epsilon_1/2$ then c_5 such that $n > c_5$ implies $\Pr\{\bigcup_{k\le c_6} A_k\} \le \epsilon_1/2$ so that finally $n > c_5$ implies

$$\Pr\left\{\bigcup_{k\le(\log n)^3} A_k\right\} \le \Pr\left\{\bigcup_{k\le c_6} A_k\right\} + \Pr\left\{\bigcup_{c_6<k\le(\log n)^3} A_k\right\} < \epsilon_1$$

which proves the lemma.

LEMMA 4.2. *For $k > (\log n)^3$*

$$\Pr\{A_k\} = \frac{e^{-y}}{2\sqrt{\pi}\log n \log\log n}(1 + o(1))$$

where $o(1) \to 0$ uniformly in y and $n \ge k > (\log n)^3$, $n \to \infty$.

Proof. Since $\Pr\{A_k\} = 1 - F_k(f_n(y))$ the Berry-Esseen estimate, (a) gives

$$|\Pr(A_k) - (1 - \Phi(f_n(y)))| < \frac{c_2}{k^{1/2}}$$

and from the elementary formula

$$1 - \Phi(u) = \frac{1}{(2\pi)^{1/2}u} e^{-u^2/2}(1 + o(1)), \qquad u \to \infty$$

we obtain on using (4.1)

$$1 - \Phi(f_n(y)) = \frac{e^{-y}}{2(\pi)^{1/2} \log n \log \log n}(1 + o(1))$$

so that for $k > (\log n)^3$ we obtain the lemma.

Let $\xi > 0$ be given. We define an integer $N = N_n(\xi)$ and a sequence of integers $n_j = n_{j,n}(\xi)$ as follows

$$N = N_n(\xi) = \left[\frac{\log n \log \log n}{\xi}\right], \tag{4.4}$$

$$n_j = n_{j,n}(\xi) = \left[\exp\left(j \frac{\xi}{\log \log n}\right)\right], \qquad j = 1, 2, \cdots, N. \tag{4.5}$$

LEMMA 4.3. *Given $\epsilon_2 > 0$ there exist c_8, $c_9 > 0$ such that $n > c_8$, $\xi < c_9$ imply*

$$\Pr\left\{\bigcup_{k \le n} A_k - \bigcup_{j<N} A_{n_j}\right\} < \epsilon_2 .$$

Proof: Let

$$Z = \bigcup_{(\log n)^3 < k \le n} A_k - \bigcup_{(\log n)^3 < n_j \le n} A_{n_j}$$

$$H_\nu = A_\nu \bigcap_{(\log n)^3 < r < \nu} A'_r .$$

Then

$$P(Z) = \sum_{(\log n)^3 < \nu \le n} P(Z \mid H_\nu)P(H_\nu)$$

$$\le \max_{(\log n)^3 < \nu \le n} P(Z \mid H_\nu) \le \max_{(\log n)^3 < n_j \le n} P(A'_{n_{j+1}} \mid A_{n_j}).$$

Now

$$P(A_{n_j,n}(y + \delta) \mid A_{n_j,n}(y)) = \frac{P(A_{n_j,n}(y + \delta))}{P(A_{n_j,n}(y))} = e^{-\delta} + h_n$$

where $h_n \to 0$ uniformly in j, y, δ (but not ξ) from Lemma 4.2. It follows then that

$$P(Z) \le \max_{(\log n)^3 < n_j \le n} P(A'_{n_{j+1},n}(y) \mid A_{n_j,n}(y + \delta)) + 1 - e^{-\delta} + h_n .$$

Now given $\epsilon_2 > 0$ choose δ so that $1 - e^{-\delta} = \epsilon_2/3$. We have clearly

$$P(A'_{n_{j+1},n}(y) \mid A_{n_j}(y + \delta)) \le P(S_{n_{j+1}} - S_{n_j} \le (n_{j+1})^{\frac{1}{2}} f_n(y) - (n_j)^{\frac{1}{2}} f_n(y + \delta))$$

and from (4.1) and (4.5) it follows that, for any $\xi > 0$,

$$\frac{(n_{j+1})^{1/2} f_n(y) - (n_j)^{1/2} f_n(y + \delta)}{(n_{j+1} - n_j)^{1/2}} = \tfrac{1}{2}\xi^{1/2} - \frac{\delta}{\xi^{1/2}} + k_n ,$$

where $k_n \to 0$ uniformly in j, δ. We notice that $S_{n_{j+1}} - S_{n_j}$ has mean 0 and variance $n_{j+1} - n_j$, and that for any random variable Y with mean 0 and variance 1, and any $M > 0$,

$$P(Y \leq -M) \leq \frac{1}{1 + M^2} \cdot$$

Choose then $M > 0$ such that

$$\frac{1}{1 + M^2} = \frac{\epsilon_2}{3},$$

then c_9, c_{10} such that $\xi < c_9$, $n > c_{10}$ imply

$$\tfrac{1}{2}\,\xi^{1/2} - \frac{\delta}{\xi^{1/2}} + k_n \leq -M$$

for all $n_j > (\log n)^3$. Finally choose c_8 ($\geq c_{10}$) such that $n > c_8$ implies $h_n < \epsilon_2/3$ and the lemma is established.

LEMMA 4.4. *Given* $\epsilon_3 > 0$, $\xi > 0$, *there exist* c_{11} *and* c_{12} *such that* $n > c_{11}$, $a > c_{12}$ *imply* $\Pr\{\bigcup_{n_j \leq n} A_{n_j,n}(y + a)\} \leq \epsilon_3$.

Proof.
$$\Pr\left\{\bigcup_{n_j \leq n} A_{n_j,n}(y + a)\right\} \leq \Pr\left\{\bigcup_{n_j \leq (\log n)^3} A_{n_j,n}(y + a)\right\} + \sum_{n \geq n_j > (\log n)^3} \Pr\{A_{n_j,n}(y + a)\}.$$

For any $a > 0$ the first term can be made less than $\epsilon_3/2$ by making $n > c_{13}$ (Lemma 4.1) and since

$$\Pr\{A_{n_j,n}(y + a)\} = \frac{e^{-(y+a)}}{2(\pi)^{1/2}N\xi}(1 + o(1)), \qquad n_j > (\log n)^3$$

by Lemma 4.2 and (4.4), the second term is less than $c_{14}\, e^{-a}/\xi$ for $n > c_{15}$. Thus choose $n > c_{11} = \max(c_{13}, c_{15})$ and c_{12} such that $c_{14}\, e^{-c_{12}}/\xi < \epsilon_3/2$ and the lemma is proved.

Let us next define events B_j for any $a > 0$, $\xi > 0$, $j = 1, 2, \cdots$.

$$B_j = A_{n_j,n}(y) - A_{n_j,n}(y + a) \qquad \text{if } n_j \leq n \tag{4.6}$$
$$= \text{null event} \qquad \text{if } n_j > n.$$

The preceding four lemmas assert that given $\epsilon_4 > 0$ there exist n, a sufficiently large and $\xi > 0$ sufficiently small so that

$$\left| g_n(y) - \Pr\left\{\bigcup_{n_j > (\log n)^3} B_j\right\}\right| < \epsilon_4 .$$

LEMMA 4.5. *Given* $\epsilon_5 > 0$, $a > 0$, $\xi > 0$ *there exist* r, c_{16} *such that* $n > c_{16}$ *implies*

$$\max_j \sum_\nu \Pr\left\{B_{j+\nu} \,\middle|\, \frac{S_{n_j}}{(n_j)^{1/2}} = f_n(y + a)\right\} < \epsilon_5,$$

where $r < \nu \leq (\log\log n)^2$ *and* $(\log n)^3 < n_j$.

Proof. Let

$$b_\nu(j) = \Pr\left\{B_{j+\nu} \,\middle|\, \frac{S_{n_j}}{(n_j)^{1/2}} = f_n(y+a)\right\}. \tag{4.7}$$

Then

$$b_\nu(j) \le \Pr\left\{\frac{S_{n_{j+\nu}} - S_{n_j}}{(n_{j+\nu} - n_j)^{1/2}} > p_\nu\right\},$$

where

$$p_\nu = \frac{(n_{j+\nu})^{1/2} f_n(y) - (n_j)^{1/2} f_n(y+a)}{(n_{j+\nu} - n_j)^{1/2}}$$

$$= \frac{(n_{j+\nu})^{1/2} - (n_j)^{1/2}}{(n_{j+\nu} - n_j)^{1/2}} f_n(y) - \left(\frac{n_j}{n_{j+\nu} - n_j}\right)^{1/2} \frac{a}{(2 \log \log n)^{1/2}} .$$

We consider $n_j > (\log n)^3$ and treat two cases separately.

1. $r < \nu < \log \log n$. Now when $\nu < \log \log n$ using (4.4) a simple estimate shows

$$\frac{(n_{j+\nu})^{1/2} - (n_j)^{1/2}}{(n_{j+\nu} - n_j)^{1/2}} \ge \frac{c_{17}\nu^{1/2}}{(\log \log n)^{1/2}}$$

$$\left(\frac{n_j}{n_{j+\nu} - n_j}\right)^{1/2} \le \frac{c_{18}}{\nu^{1/2}} (\log \log n)^{1/2}$$

and hence

$$p_\nu > c_{19}\nu^{1/2} - \frac{c_{20}}{\nu^{1/2}} > c_{19}\nu^{1/2}$$

for all j, c_{19} independent of j, when $c_{22} < \nu < \log \log n$, c_{22} independent of j. Thus

$$b_\nu(j) < c_{23} e^{-\nu c_{20}}, \qquad c_{22} > \nu > \log \log n, \qquad c_{20} > 0$$

where $c_{20} c_{23}$ depend on ξ, y, a but not on j, n. Consequently there exists an r independent of j, n such that for any $\epsilon_5 > 0$

$$\sum_{r < \nu < \log \log n} b_\nu(j) < \frac{\epsilon_5}{2} \cdot$$

2. $\log \log n \le \nu \le (\log \log n)^2$. In this case we obtain

$$p_\nu \ge \left(1 - \left(\frac{n_j}{n_{j+\nu}}\right)^{1/2}\right) f_n(y) - c_{24}\left(\frac{n_j}{n_{j+\nu}}\right)^{1/2} \ge c_{25} f_n(y),$$

where c_{25} is independent of n, j, so that

$$b_\nu(j) < c_{26} \exp(-c_{27} \log \log n) \le \frac{c_{26}}{(\log n)^{c_{28}}} ,$$

and hence for $n > c_{29}$, c_{29} independent of j,

$$\sum_{\log \log n \le \nu \le (\log \log n)^2} b_\nu(j) < \frac{\epsilon_5}{2} , \qquad n > c_{29} .$$

Combining these two results the lemma is proved.

Also, when $(\log\log n)^2 < \nu$ we have $p_\nu = f_n(y)(1 + o(1))$ and

$$b_\nu(j) \leq \frac{e^{-y}}{2(\pi)^{1/2} N\xi}(1 + o(1))$$

and hence

$$\sum_{\substack{1 \\ nj > (\log n)^3}}^{N} b_\nu(j) < c_{30}$$

where c_{30} is independent of j.

An easy extension of this last result, using the multidimensional central limit theorem, shows that the events B_i are asymptotically mutually independent if the indices are sufficiently separated. We have, namely, from the preceding lemmas

$$\Pr\{B_i\} = \frac{c_{31}}{N}(1 + o(1)) \qquad \text{and} \qquad \Pr\{B_{i_1} B_{i_2} \cdots B_{i_k}\} = \left(\frac{c_{31}}{N}\right)^k (1 + o(1))$$

if $\min |j_\mu - j_\nu| > (\log\log n)^2$. c_{31} depends on y, a, ξ but not the distribution of the X_i nor on n. The function $o(1)$ may depend on the distribution of the X_i and not approach zero uniformly in the parameters.

Now introduce events D_k as follows, for any $r \geq 1$.

$$D_k = B_k B'_{k-1} B'_{k-2} \cdots B'_{k-r} \qquad k > r \tag{4.8}$$

$$D_k = B_k B'_{k-1} \cdots B'_1 \qquad k \leq r.$$

Clearly $\Pr\{\bigcup_{k\leq N} B_k\} = \Pr\{\bigcup_{i \leq N} D_i\}$ and also $\Pr\{D_{i_1} D_{i_2} \cdots D_{i_k}\} = 0$ unless $\min_{\mu,\nu} |j_\mu - j_\nu| > r$; but in any case $\Pr\{D_{i_1} \cdots D_{i_k}\} \leq \Pr\{B_{i_1} \cdots B_{i_k}\}$.

Given $\epsilon_6 > 0$ there exist ξ, a, r, c_{32} such that

$$\left| g_n(y) - \Pr\left\{\bigcup_{n \geq nj > (\log n)^3} D_j\right\}\right| < \epsilon_6 \qquad \text{for } n > c_{32}$$

and

$$\Pr\left\{\bigcup_{n \geq nj > (\log n)^3} D_j\right\} = \sigma_1 - \sigma_2 + \sigma_3 - \cdots$$

where

$$\sigma_k = \sum_{\substack{i_1 < i_2 < \cdots < i_k \\ n \geq nj_k > (\log n)^3}} \Pr\{D_{i_1} D_{i_2} \cdots D_{i_k}\}.$$

Now, as with the events B_i, the D_i are asymptotically independent if the indices have a mutual separation at least $(\log\log n)^2$, and $\sigma_1 = c_{33}(1 + o(1))$ where c_{33} depends on r, a, ξ but is independent of the distribution of the X_i. It is well known that, in addition

$$\left| \Pr\left\{\bigcup_{n \geq nj > (\log n)^3} D_j\right\} - \sigma_1 + \sigma_2 - \sigma_3 + \cdots + (-1)^k \sigma_{k-1} \right| \leq \sigma_k .$$

We now suppose $k \geq 2$ and write $\sigma_k = \sigma_k' + \sigma_k''$. The following estimates result directly from the preceding lemmas:

$$\sigma_k' = \sum_{\substack{j_1 < j_2 < \cdots < j_k \\ \min |j_\mu - j_\nu| \leq (\log \log n)^2 \\ n \geq n_{j_i} > (\log n)^3}} \Pr \{D_{j_1} D_{j_2} \cdots D_{j_k}\}$$

$$\leq \sum_{\substack{j_1 < j_2 < \cdots < j_k \\ r \leq \min |j_\mu - j_\nu| \leq (\log \log n)^2 \\ n \geq n_{j_i} > (\log n)^3}} \Pr \{B_{j_1} B_{j_2} \cdots B_{j_k}\}$$

$$\leq \Big(\sum_{n \geq n_j > (\log n)^3} \Pr \{B_j\} \Big) \max_j \Big(\sum_{r < \nu \leq (\log \log n)^2} b_\nu(j) \Big) \max_j \Big(\sum_{\nu=1}^{N} b_\nu(j) \Big)^{k-2}$$

$$\sigma_k'' = \sum_{\substack{j_1 < j_2 < \cdots < j_k \\ \min |j_\mu - j_\nu| > (\log \log n)^2 \\ n \geq n_{j_i} > (\log n)^3}} \Pr \{D_{j_1} D_{j_2} \cdots D_{j_k}\}$$

$$= \frac{c_{34}^k}{k!} (1 + o(1))$$

where c_{34} is independent of the distribution of the X_i, and of n. It follows that $\lim_{n \to \infty} \sigma_k = \tau_k$ exists, but the τ_k may be unbounded functions of ξ, a, r.

The final argument proceeds as follows: choose $\epsilon_7 > 0$ arbitrary, then choose ξ, n, a so that $| g_n(y) - \Pr \{\bigcup_{n \geq n_j > (\log) n^3} D_j\} | < \epsilon_7/2$ which is possible by Lemmas 4.1-4.4. Then choose k, n so large that $\sigma_k'' < \epsilon_7/4$, and finally by Lemma 4.5 and the above inequality on σ_k', choose r so large that $\sigma_k' < \epsilon_7/4$. We have then $| g_n(y) - \sigma_1 + \sigma_2 - \cdots + (-1)^k \sigma_{k-1} | < \epsilon_7$ and hence $\lim g_n(y) = g(y)$ exists, and since $\lim \sigma_j = \tau_j$ is independent of the distribution of the X_i so is $g(y)$, and the proof of Theorem 1 is complete.

5. **Additional remarks.** It is possible to get a theorem for $V_n = \max_{1 \leq k \leq n} | S_k |/k^{\frac{1}{2}}$ similar to Theorem 1 following an identical pattern as above. We get, in fact

THEOREM 2. *Let the X_i be as in Theorem 1 and set*

$$V_n = \max_{1 \leq k \leq n} | S_k |/k^{1/2}.$$

Then

$$\lim_{n \to \infty} \Pr \left\{ V_n < (2 \log \log n)^{1/2} + \frac{\log \log \log n}{2(2 \log \log n)^{1/2}} + \frac{t}{(2 \log \log n)^{1/2}} \right\} = \exp\left(- \frac{1}{\pi^{1/2}} e^{-t} \right).$$

We indicate also the two following strong analogues to Theorem 1, which we state here as conjectures. Define U_n as in (1.1) and let the X_i be as in Theorem 1.

1. There exists a $c_1 > 0$ such that the inequality

$$U_n > (2 \log \log n)^{1/2} + \frac{\log \log \log n}{2(2 \log \log n)^{1/2}} + \frac{(c_1 - h) \log \log \log \log n}{(2 \log \log n)^{1/2}}$$

holds for infinitely many n, or only finitely many n, according as $h > 0$ or $h < 0$, with probability 1.

2. There exists a $c_2 > 0$ such that the inequality

$$U_n > (2 \log \log n)^{1/2} + \frac{\log \log \log n}{2(2 \log \log n)^{1/2}} - \frac{(c_2 + h) \log \log \log \log n}{(2 \log \log n)^{1/2}}$$

holds for all except finitely many n, or fails for infinitely many n, with probability 1, according as $h > 0$ or $h < 0$.

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