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ON CONSECUTIVE INTEGERS

BY

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A theorem of SYLVESTER and SCHUR [1]) states that for every k and $n > k$ the product $n(n+1)\dots(n+k-1)$ is divisible by a prime $p > k$, or in other words the product of k consecutive integers each greater than k always contains a prime greater than k. Define now $f(k)$ as the least integer so that the product of $f(k)$ consecutive integers, each greater than k always contains a prime greater than k. The theorem of SYLVESTER and SCHUR states that $f(k) \leq k$. In the present note we shall prove

Theorem 1. There is a constant $c_1 > 1$ so that

$$f(k) \leq c_1 \frac{k}{\log k}. \tag{1}$$

In other words the sequence $u+1, u+2, \dots, u+t$, $t = \left[c_1 \frac{k}{\log k}\right]$, $u \geq k$ has at least one prime $> k$.

The exact determination of the order of $f(k)$ is an extremely difficult problem. It follows from a theorem of RANKIN [2]) that there exists a constant $c_2 > 0$ so that for every k we have consecutive primes p_r and p_{r+1} satisfying

$$k < p_r < p_{r+1} < 2k,\ p_{r+1} - p_r > c_2 \frac{\log k . \text{loglog}\, k . \text{logloglog log}\, k}{(\text{logloglog}\, k)^2} \tag{2}$$

Clearly all prime factors of the product $(p_r + 1)\dots(p_{r+1} - 1)$ are less than k. Thus

$$f(k) > c_3 \frac{\log k . \text{loglog}\, k . \text{logloglog log}\, k}{(\text{logloglog}\, k)^2} \tag{3}$$

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[1]) P. ERDÖS, A theorem of SYLVESTER AND SCHUR, Journal London Math. Soc. *9* (1934) 282—288.

[2]) R. A. RANKIN, The difference between consecutive prime numbers, ibid. *13* (1938) 242—247.

The gap between (1) and (3) is extremely large. It seems likely that $f(k)$ is not substantially larger than the greatest difference $p_{r+1} - p_r$, $k < p_r < p_{r+1} < 2k$. Thus by a conjecture of CRAMER [3]) one might guess

$$f(k) = (1 + o(1)) (\log k)^2. \tag{4}$$

The proof or disproof of (4) seems hopeless, there is of course no real evidence that (4) is true.

It would be interesting, but not entirely easy to determine $f(k)$ say for all $k < 100$. It is not even obvious that $f(k)$ is a non decreasing function of k (in fact I can not prove this). A theorem of PÓLYA and STÖRMER states that for $u > u_0(k)$, the product $u(u+1)$ always contains a prime factor greater than k, thus $f(k)$ can be determined in a finite number of steps, but as far as I know no explicite estimates are available for $u_0(k)$, which makes the determination of $f(k)$ difficult. In general it will be troublesome to prove that $f(k) < \pi(k)$ ($\pi(k)$ is the number of primes $\leq k$). It is easy to see that

$$f(2) = 2,\ f(3) = f(4) = 3,\ f(5) = f(6) = 4.$$

It seems likely that $f(7) = f(8) = f(9) = f(10) = 4$, but $f(13) \geq 6$.

In the proofs of theorems 1 and 2 we will make use of the following consequences of a result of HOHEISEL-INGHAM [4]):

$$\pi(x + x^\theta) - \pi(x) \sim \frac{x^\theta}{\log x} \qquad \tfrac{5}{8} \leq \theta \leq 1 \tag{*}$$

from which it follows for each pair of consecutive primes p_n, p_{n+1}:

$$p_{n+1} - p_n = O(p_n^{5/8}) \tag{**}$$

To prove Theorem 1 we first of all make use of (**): there exists a constant c_4, so that

$$p_{k+1} - p_k < c_4\, p_k^{5/8}. \tag{5}$$

It immediately follows from (5) that for $u \leq k^{3/2}$ at least one of the integers

$$u + 1,\ u + 2,\ \ldots,\ u + t,\ t = \left[c_1 \frac{k}{\log k}\right]$$

is a prime, for sufficiently large c_1.

[3]) H. CRAMER, On the order of magnitude of the difference between consecutive prime numbers, Acta Arithmetica *2* (1936) 23—46.

[4]) A. E. INGHAM, On the difference between consecutive primes. Quart. J. Math. *8* (1937) 255—266.

Thus in the proof of Theorem 1 we can assume $u > k^{3/2}$. If Theorem 1 would not be true then for each $c_1 > 0$ we could find a $u > k^{3/2}$ so that all prime factors of

$$\binom{u+t}{t},\ u > k^{3/2},\ t = \left[\frac{c_1 k}{\log k}\right]$$

would be less than or equal to k.

L e m m a. If $p^a \,//\, \binom{u+t}{t}$ then $p^a \leq u + t$. [5]

The Lemma is well known and follows easily from Legendre's formula for the decomposition of $n!$ into prime factors.

Clearly

$$\binom{u+t}{t} = \frac{(u+1)\,(u+2)\,\ldots\,(u+t)}{t!} \geq \left(\frac{u+t}{t}\right)^t > \left(\frac{u}{t}\right)^t \quad (6)$$

Now if all prime factors of $\binom{u+t}{t}$ would be less than or equal to k, we would have from (6) and from the above Lemma

$$\left(\frac{u}{t}\right)^t < \binom{u+t}{t} \leq (u+t)^{\pi(k)}. \quad (7)$$

Now by $u > k^{3/2}$ and $t < k$ ($t < k$ can be assumed by the theorem of SYLVESTER and SCHUR) we obtain from (7) and from

$$\pi(k) < \frac{3k}{2 \log k}$$

$$u^{t/3} < (u+t)^{\pi(k)} < u^{2k/\log k} \quad (8)$$

Thus (8) leads to a contradiction for $c_1 > 6$, which completes the proof of Theorem 1.

Define $g(k)$ as the smallest integer so that among k consecutive integers each greater than k there are at least $g(k)$ of them having prime factors greater than k. The theorem of SYLVESTER and SCHUR asserts that $g(k) \geq 1$. We prove

T h e o r e m 2.

$$g(k) = (1 + o(1))\,\frac{k}{\log k}.$$

The sequence $k + 1,\ \ldots,\ 2k$ clearly contains $\pi(2k) - \pi(k) = = (1 + o(1))\,\frac{k}{\log k}$ primes, or $g(k) \leq (1 + o(1))\,\frac{k}{\log k}$. Thus to

[5]) $p^a \,//\, u$ means that p^a / u and $p^{a+1} \times u$.

prove theorem 2 it will suffice to show that if $n \geq k$ the sequence

$$n+1,\ n+2,\ \ldots,\ n+k \tag{9}$$

contains at least $(1+o(1))\,\dfrac{k}{\log k}$ integers having prime factors greater than k.

a) If $k \leq n \leq 2k$ the integers (9) contain by the prime number theorem

$$\pi(n+k) - \pi(n) = (1+o(1))\,\frac{k}{\log k}$$

prime numbers. Thus we can assume $n > 2k$.

b) Assume first $2k < n \leq k^{3/2}$. By (*) there are least $(1+o(1))\,\dfrac{k}{\frac{3}{2}\log k}$ primes amongst the integers (9), but since $n > 2k$ there are also at least $(1+o(1))\,\dfrac{k}{2.\frac{3}{2}\log k}$ integers of the form $2p$, $p > k$, since among the integers

$$\left[\frac{n}{2}\right]+1,\ \ldots,\ \left[\frac{n+k}{2}\right]$$

there are at least $(1+o(1))\,\dfrac{k}{2.\frac{3}{2}\log k}$ primes.

Since

$$(1+o(1))\,\frac{k}{\frac{3}{2}\log k} + (1+o(1))\,\frac{k}{2.\frac{3}{2}\log k} = (1+o(1))\,\frac{k}{\log k}$$

we can assume $n > k^{3/2}$.

c) Next we show that there is a constant $k_0 > 0$ such that if $k > k_0$ and $n > k^{3/2}$ there are at least $k/6$ integers of (9) having prime factors greater than k. For if not, we have (as in the proof of theorem 1) by the Lemma and by (6) for an arbitrary large k a $n > k^{3/2}$ such that

$$\left(\frac{n+k}{k}\right)^k \leq \binom{n+k}{k} < (n+k)^{k/6+\pi(k)}$$

or

$$(n+k) < k(n+k)^{\frac{1}{6}+\frac{\pi(k)}{k}} < (n+k)^{\frac{2}{3}+\frac{1}{6}+\frac{\pi(k)}{k}}$$

which is clearly false if k is sufficiently large.

Remark: $k = 10$, $n = 12$ shows that $g(k)$ can be less than $\pi(2k) - \pi(k)$.

Theorem 3. Amongst the integers (9) there are at least $(\frac{1}{2} + o(1)) \dfrac{k}{\log k}$ which do not divide the product of the others.

Here we only assume $n \geq 0$ (and not $n \geq k$). If $n \geq k$ this follows immediately from Theorem 2 (since a prime greater than k can divide at most one of the integers (9)). If $n < k$ the primes $n + \dfrac{k}{2} < p < n + k$ divide only one of the integers (9) and their number is $\frac{1}{2} \dfrac{k}{\log k} + o\left(\dfrac{k}{\log k}\right)$. For $n = 0$ and $k > 5$ the sequence (9) contains exactly $\pi(k) - \pi\left(\dfrac{k}{2}\right) = \frac{1}{2} \dfrac{k}{\log k} + o\left(\dfrac{k}{\log k}\right)$ integers which do not divide the product of the others, thus Theorem 3 is best possible.