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ON THE UNIFORM BUT NOT ABSOLUTE CONVERGENCE OF POWER SERIES WITH GAPS

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Hardy [1]) was the first to give an example of a power series $\sum_{k=1}^{\infty} a_k z^{n_k}$ which converges uniformly in $|z| \leqq 1$ but for which $\sum_{k=1}^{\infty} |a_k| = \infty$. Piranian asked me (in a letter) for what sequences of integers $n_1 < n_2 < \dots$ does there exist a power series $\sum_{k=1}^{\infty} a_k z^{n_k}$ which converges uniformly in $|z| \leqq 1$ but for which $\sum_{k=1}^{\infty} |a_k|$ diverges. In the present paper I shall prove the following

Theorem 1. *Let $n_1 < n_2 < \dots$ be a sequence of integers satisfying* $\liminf n_n^{1/k} = 1$. *Then there exists a power series $\sum_{k=1}^{\infty} a_k z^{n_k}$ which converges uniformly in $|z| \leqq 1$ but for which $\sum_{k=1}^{\infty} |a_k| = \infty$.*

The condition $\liminf n_k^{1/k} = 1$ can certainly not be weakened a great deal. In fact Zygmund [2]) proved that if $n_{k+1}/n_k > c > 1$, and if $\sum_{k=1}^{\infty} a_k z^{n_k}$ converges for all $|z| = 1$ then $\sum_{k=1}^{\infty} |a_k| < \infty$. On the other hand both Piranian and I observed that $\liminf n_k^{1/k} = 1$ is not a necessary condition. In fact I shall prove that the following somewhat stronger result holds:

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[1]) E. Landau, *Neuere Ergebnisse der Funktionentheorie*, p. 68.

[2]) Studia Math. 3 (1931), p. 77-91.

Theorem 2. *Let* $n_1 < n_2 < \ldots$ *satisfy*

$$(1) \qquad \liminf (n_j - n_i)^{1/j-i} = 1, \text{ where } j - i \to \infty.$$

Then there exists a power series $\sum_{k=1}^{\infty} a_k z^{n_k}$ *which converges uniformly for* $|z| = 1$ *but* $\sum_{k=1}^{\infty} |a_k| = \infty$.

It is clear that Theorem 2 is stronger than Theorem 1, since if $\liminf n_k^{1/k} = 1$ holds then $\liminf (n_j - n_i)^{1/j-i} = 1$ also holds (put $n_i = n_1$ and let $j \to \infty$).

It is not impossible that (1) is the necessary and sufficient condition for the existence of a power series $\sum_{k=1}^{\infty} a_k z^{n_k}$ which converges uniformly for $|z| \leqslant 1$ but $\sum_{k=1}^{\infty} |a_k| = \infty$.

We will prove Theorem 2 since the proof of Theorem 1 is not easier. The proof will use methods of probability theory and for this reason I am glad that the paper appears in a volume dedicated to Professor Steinhaus who has contributed so much to this subject.

It easily follows from (1) that there exists a sequence of indices satisfying

$$(2) \quad i_1 < j_1 < i_2 < j_2 < \ldots, \; 3 < n_{j_l} - n_{i_l} < \exp(j_l - i_l/2^l), \; j_l - i_l = 2A_l - 1,$$

$\exp z$ denoting e^z. The condition that $j_l - i_l$ is odd is assumed only to make some of the later computations simpler. From (2) we have

$$(3) \qquad 2A_l = j_l - i_l + 1 \leqslant n_{j_l} - n_{i_l} + 1 < 2(n_{j_l} - n_{i_l}).$$

Further by (2)

$$(4) \qquad j_l - i_l > 2^l \quad \text{or} \quad A_l > 2^{l-1}.$$

Define $a_k = 0$ if k is not in any of the intervals (i_l, j_l), $l = 1, 2, \ldots$, and $a_k = (l A_{l/2})^{-1}$ if $i_l \leqslant k \leqslant j_l$. Denote by $r_k(t)$ the k-th Rademacher function.

We have $\sum_{k=i_l}^{j_l} a_k = 1/l$ hence $\sum_{k=1}^{\infty} a_k = \sum_{l=1}^{\infty} 1/l = \infty$. Therefore Theorem 2 follows from

Theorem 3. *For almost all t*

$$f_t(z)=\sum_{k=1}^{\infty} r_k(t) a_k z^{n_k}$$

converges uniformly for $|z|\leqq 1$.

In other words if $\varepsilon_k=\pm 1$ and the ε-s are independent of each other, then $\sum_{k=1}^{\infty} \varepsilon_k a_k z^{n_k}$ converges uniformly for almost all choices of the ε-s (since $\sum_{k=1}^{\infty} a_k=\infty$ Theorem 3 includes Theorem 2).

To prove Theorem 3 we need a few lemmas. First of all put

$$\theta_t^{(l)}(z)=\sum_{k=i_l}^{j_l} r_k(t)\, z^{n_k-n_{i_l}}=\sum_{k=i_l}^{j_l} \varepsilon_k\, z^{n_k-n_{i_l}}. \tag{5}$$

By (5)

$$f_t(z)=\sum_{l=1}^{\infty} z^{n_{i_l}}\, \theta_t^{(l)}(z)/2lA_l.$$

The degree of $Q_t^{(l)}(z)$ is $n_{j_l}-n_{i_l}$ and it has $2A_l$ terms.

Lemma 1. *Suppose* $0\leqq b_i\leqq 1$, *and let* $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m$ *take on the values* ± 1 *in all possible ways (thus there are* 2^m *possible choices for the* ε-*s). Denote by* $g(m,r)$ *the number of choices of the* ε-s *for which*

$$\Big|\sum_{i=1}^{m} \varepsilon_i b_i\Big|<m-2r+1 \quad (if\ r>m/2,\ g(m,r)=0) \tag{6}$$

($\min g(m,r)$ *denotes the smallest possible value of* $g(m,r)$). *Then*

$$\min g(m,r)=\binom{m}{r}+\binom{m}{r+1}+\dots+\binom{m}{m-r}. \tag{7}$$

The Lemma means that $g(m,r)$ is minimal if $b_i=1$, $i=1,2,\dots,m$.

The lemma and its proof are due to Szekeres[3]). We use induction for m and r. We can assume without loss of generality that $b_1\leqq b_2\leqq\dots\leqq b_m=1$ (for if $b_m<1$, we replace b_i by b_i/b_m, $i=1,2,\dots,m$, and $g(m,r)$ is clearly not increased). If

$$\Big|\sum_{i=1}^{m-1} \varepsilon_i b_i\Big|<m-2r+1$$

[3]) Written communication.

and $\varepsilon_m = +1$ we obtain

$$(8) \qquad -(m-2r+2) < |\sum_{i=1}^{m-1} \varepsilon_i b_i| < m-2r.$$

If $\varepsilon_m = -1$ then

$$(9) \qquad -(m-2r) < |\sum_{i=1}^{m-1} \varepsilon_i b_i| < m-2r+2.$$

From (8) and (9) we obtain that

$$(10) \quad \min g(m,r) \geqslant \min [g(m-1, r)] + \min [g(m-1, r-1)].$$

Lemma 1 clearly holds for $r=0$ and any m, also it holds for $m=1$ and any n. Thus by (10) and a simple induction argument it holds for all m and r, which proves Lemma 1.

Lemma 2. *Let $z_1, z_2, \ldots, z_{2m}$ be $2m$ complex numbers, $|z_i|=1$, $i=1,2,\ldots,m$. Then the nuuber of choices of the ε-s for which*

$$|\sum_{i=1}^{2m} \varepsilon_i z_i| \geqslant (2s+1)\sqrt{2}$$

is less than

$$8m \cdot 2^{2m} \exp(-s^2/2m).$$

Put $z_j = a_j + i b_j$. If

$$|\sum_{j=1}^{2m} \varepsilon_j z_j| \geqslant (2s+1)\sqrt{2}$$

then either

$$|\sum_{j=1}^{2m} \varepsilon_j a_j| \geqslant 2s+1 \quad \text{or} \quad |\sum_{j=1}^{2m} \varepsilon_j b_j| \geqslant 2s+1.$$

By lemma 1 the number of solutions of

$$|\sum_{j=1}^{2m} \varepsilon_j a_j| \geqslant 2s+1$$

is less than or equal to

$$(11) \qquad \begin{aligned} 2^{2m} - \min [g(2m, m-s)] &= 2^{2m} - \binom{2m}{m-s} - \ldots - \binom{2m}{m+s} = \\ &= 2 \sum_{i=1}^{m-s-1} \binom{2m}{i} < 4m \binom{2m}{m-s} = \\ &= 4m \binom{2m}{m} \frac{m(m-1)\ldots(m-s+1)}{(m+1)(m+2)\ldots(m+s)} < \\ &< 4m \cdot 2^{2m} \prod_{i=1}^{s} \left(1 - \frac{2i-1}{2m}\right) < 4m \cdot 2^{2m} \exp(-s^2/2m). \end{aligned}$$

Similarly the number of solutions in the ε-s of

$$\left|\sum_{j=1}^{2m} \varepsilon_j b_j\right| \geqq 2s+1$$

satisfies (11), which proves Lemma 2.

Lemma 3. *For* $l > l_0$

$$\text{Prob}\,[\max_{|z|=1} |Q_t^{(l)}(z)| > ((2A_l+1)\sqrt{2}/l+\pi)] < 1/2^l.$$

In other words the measure in t of the set for which

$$\max_{|z|=1} |Q_t^{(l)}(z)| > (2A_l+1)\sqrt{2}/l+\pi$$

is less than $1/2^l$. Or in still other words: The number of choices of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2A_l}$ for which

$$(12) \qquad \max_{|z|=1} |Q_t^{(l)}(z)| = \max_{|z|=1} \left|\sum_{k=i_l}^{j_l} \varepsilon_k z^{n_k - n_{i_l}}\right| > (2A_l+1)\sqrt{2}/l+\pi$$

is less than $2^{2A_l - l}$.

Put

$$\xi_r = \exp\,[2\pi r i/(n_{j_l} - n_{i_l})^2, \quad 1 \leqq r \leqq (n_{j_l} - n_{i_l})^2.$$

Thus ξ_r runs through the $(n_{j_l} - n_{i_l})^2$-th roots of unity. $Q_t^{(l)}(\xi_r)$ is of the form $\sum_{l=1}^{2A_l} \varepsilon_l z_l$, $|z_l| = 1$. Thus lemma 2 can be applied and we obtain from Lemma 2 on putting $A_l = m$, $s = A_l/l$ that the number of choices of the ε-s for which

$$|Q_t^{(l)}(\xi_r)| > (2A_l+1)\sqrt{2}/l$$

is less than

$$(13) \qquad 4A_l \cdot 2^{2A_l} \exp\,(-A_l/2l^2).$$

Therefore for $l > l_0$ by (12), (2), (3) and (4) the number of ε-s for which

$$(14) \qquad \max_{1 \leqq r \leqq (n_{j_l} \cdot n_{i_l})^2} |Q_t^{(l)}(\xi_r)| > (2A_l+1)\sqrt{2}/l$$

is less than

$$4A_l\cdot 2^{2A_l}(n_{j_l}-n_{i_l})^2\exp(-A_l/2l^2)<$$

$$(15)\qquad <2^{2A_l+2}(n_{j_l}-n_{i_l})^3\exp(-A_l/2l^2)<$$

$$<2^{2A_l+2}\exp(3A_l/2^{l-1})\exp(-A_l/2l^2)<2^{2A_l+2}\exp(-A_l/4l^2)<2^{2A_l-l}.$$

In the last two inequalities of (15) we used the facts that for $l>l_0$ $3/2^{l-1}<1/4l^2$, and that for $l>l_0$ by (4)

$$A_l>2^{l-1}>4l^2(l+2).$$

Now let z_0 be any point on the circumference of the unit circle. Clearly

$$(16)\qquad \min_{1\leqslant r\leqslant(n_{j_l}-n_{i_l})^2}|z_0-\xi_r|<\pi/(n_{j_l}-n_{i_l})^2.$$

Further

$$(17)\qquad \max_{|z|=1}|(Q_t^{(l)}(z))'|\leqslant\sum_{k=i_l}^{j_l}(n_k-n_{i_l})<(n_{j_l}-n_{i_l})^2.$$

From (16) and (17) we see that if

$$\max_{1\leqslant r\leqslant(n_{j_l}-n_{i_l})^2}|Q_t^{(l)}(\xi_r)|\leqslant(2A_l+1)\sqrt{2}/l$$

then

$$(18)\qquad \max_{|z|=1}|Q_t^{(l)}(z)|<(2A_l+1)\sqrt{2}/l+\pi.$$

The formulae (15) and (18) imply that the number of choices of the ε-s for which (12) is not satisfied is less than 2^{2A_l-l}, which proves Lemma 3.

Now we can prove Theorem 3. Since $\sum 1/2^l<\infty$ it follows from Lemma 3 and the Borel-Cantelli lemma that for almost all t

$$(19)\qquad \max_{|z|=1}|Q_t^l(z)|\leqslant(2A_l+1)\sqrt{2}/l+\pi$$

except possibly for a finite number of l-s (these l-s of course may depend on t). Let t_0 be any real number for which (19) is false for only a finite number of l-s, and let $l_1=l_1(t)$ be such that (19) holds for $l>l_1(t)$. We shall prove that $f_{t_0}(z)$ converges uniformly for $|z|\leqslant 1$.

Let $k_1 > n_{j_{l_1}}$, assume that $n_{i_\lambda} < k_1 < n_{j_\lambda}$. We shall show that for $|z| \leqslant 1$

$$\left|\sum_{k=k_1}^{\infty} r_k(t_0)\, a_k z^{n_k}\right| < \frac{8}{\lambda}. \tag{20}$$

(20) clearly implies the uniform convergence of $f_{t_0}(z)$.

We evidently have for $|z| \leqslant 1$

$$\left|\sum_{k=k_1}^{\infty} r_k(t_0)\, a_k z^{n_k}\right| \leqslant \sum_{k=k_1}^{n_{j_\lambda}} |r_k(t_0)|\, a_k + \sum_{l=\lambda+1}^{\infty} |Q_{t_0}^{(l)}(z)|. \tag{21}$$

But (19) holds for $l > l_1$; hence from (21) and the definition of a_k and $f_{t_0}(z)$

$$\left|\sum_{k=k_1}^{\infty} r_k(t_0)\, a_k z^{n_k}\right| \leqslant \frac{1}{\lambda} + \sum_{l=\lambda+1}^{\infty} \left(\frac{(2A_l+1)\sqrt{2}}{2A_l l^2} + \frac{\pi}{2l A_l}\right).$$

But by (4) $A_l > 2^{l-1}$. Thus

$$\left|\sum_{k=k_1}^{k_2} r_k(t_0)\, a_k z^{n_k}\right| < \frac{1}{\lambda} + \sqrt{2}\sum_{l>\lambda}\frac{1}{l^2} + \sqrt{2}\sum_{l>\lambda}\frac{1}{2^l} + \pi\sum_{l>\lambda}\frac{1}{2^l} < \frac{8}{\lambda}.$$

Thus Theorem 3 and therefore Theorems 1 and 2 are proved.

This method can also be applied to entire functions. We can prove the following

Theorem 4. *Put*

$$f_t(z) = \sum_{k=0}^{\infty} r_k(t)\, z^k/k!$$

Then for $r > r_0(t)$ and almost all t

$$M_r(f_t) < \frac{e^r}{r^{1/4}} (\log r)^{c_1} \quad \left[M_r(f_t) = \max_{|z|<r} |f_t(z)|\right]$$

and for a sequence $r_n \to \infty$ and almost all t

$$M_{r_n}(f_t(z)) > \frac{e^r}{r^{1/4}} (\log r)^{c_2},$$

where $0 < c_2 < c_1$ are suitable constants.

We do not give the details of the proof.

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