# ON A TAUBERIAN THEOREM FOR EULER SUMMABILITY

by

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Let $\Sigma a_n$ be an infinite series. Put

$$a_n' = \frac{1}{2^{n+1}} \left( \sum_{k=0}^{n} \binom{n}{k} a_k \right). \tag{1}$$

$\Sigma a_n'$ is said to be the Euler sum of $\Sigma a_n$.[1] It is easy to see that $\Sigma a_n'$ converges if $\Sigma a_n$ converges, but the converse is not true. Euler summability was first studied by Knopp.[2]

W. Meyer-König proved[3] that if $\Sigma a_n$ is Euler summable and $a_n = 0$ except if $n = n_i$, $n_{i+1}/n_i > c > 1$, then $\Sigma a_n$ is convergent. He also conjectured that the conclusion of the theorem would follow from the following weaker condition: $a_n = 0$ except if $n = n_i$, where $n_{i+1} - n_i > c n_i^{1/2}$, $c > 0$ any constant. In fact he proved[4] this conjecture under the further assumption that $|a_n| < n^{\alpha}$ where $\alpha$ is any constant. It is easy to see that this conjecture if true is best possible i. e. if $f(n)$ tends to infinity arbitrarily slowly there exists a series $\Sigma \omega_n$ which is Euler summable but not convergent and for which $a_n = 0$ except if $n = n_i$, $n_{i+1} - n_i > n_i^{1/2} / f(n_i)$.

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[1]) (1) gives a series to series transformation method. The corresponding sequence to sequence method would be

$$s_n' = \frac{1}{2^{n+1}} \left( \sum_{k=0}^{n} \binom{n}{k} s_k \right).$$

The two methods are equivalent, but for our present purpose the series to series transformation seems to be more suitable.

[2]) Math. Zeitschrift 15 (1922), p. 226—253 and 18 (1923) p. 125—156.

[3]) Math. Zeitschrift 49 (1943), p. 151—160.

[4]) Math. Zeitschrift 45 (1939), p. 479—494.

In the present note we are going to prove the following

**Theorem.** *There exists a constant* $A > 0$ *so that if* $\Sigma a_n$ *is a series which is Euler summable, and for which* $a_n = 0$ *except if* $n = n_i$

$$n_{i+1} - n_i > A n_i^{1/2}, \tag{2}$$

*then* $\Sigma a_n$ *is convergent.*

At present I am unable to decide whether $A$ can be any constant greater than 0, in other words I am unable to prove Meyer-König's conjecture.

Let $\binom{n}{m} a_m / 2^{n+1}$ be the summand of greatest absolute value in (1). If there are several such terms we consider the one with the greatest index $m$. Put $m = f(n)$, $F(n) = \left| \binom{n}{m} a_m / 2^{n+1} \right|$.

**Lemma 1.** *$f(n)$ is a non-decreasing function of $n$.*

To prove lemma 1 it will clearly suffice to show that if

$$\left| \binom{n}{m} a_m \right| \geqslant \left| \binom{n}{l} a_l \right| \text{ for } m > l, \text{ then } \left| \binom{n+1}{m} a_m \right| > \left| \binom{n+1}{l} a_l \right|.$$

This is true since

$$\left| \binom{n+1}{m} a_m \cdot \left( \binom{n}{m} a_m \right)^{-1} \right| > \left| \binom{n+1}{l} a_l \cdot \left( \binom{n}{l} a_l \right)^{-1} \right|$$

i. e.

$$\frac{n+1}{n-m+1} > \frac{n+1}{n-l+1} \text{ for } m > l.$$

**Lemma 2.** *Assume that* $f(n) > n/2$. *Then* $F(n+1) \geqslant F(n)$.

Put $f(n) = m > n/2$. We obtain from $F(n+1) \geqslant \frac{1}{2^{n+2}} \left| \binom{n+1}{m} a_m \right|$

$$F(n+1)/F(n) \geqslant \binom{n+1}{m} \Big/ 2 \binom{n}{m} = \frac{n+1}{2(n-m+1)} \geqslant 1,$$

which proves the lemma.

**Lemma 3.** Let $\alpha$ be arbitrary. Assume that $|a_n| < n^{\alpha}$ for all $n$, and $a_n = 0$ except if $n = n_i$, $n_{i+1} - n_i > c n_i^{1/2}$, where $c > 0$ is an arbitrary positive constant. Then if $\Sigma a_n$ is Euler summable it is convergent.

This is a theorem of Meyer-König.[4)]

Because of lemma 3 we can now assume that, for infinitely many $n$, $|a_n| > n$. We shall show that if an infinite series satisfies (2) and

[4)] Math. Zeitschrift 45 (1939), p. 479—494.

$|a_n| > n$ for infinitely many $n$ then it cannot be Euler summable. This together with lemma 3 will complete the proof of our theorem. First we prove

**Lemma 4.** *Let $c_1 > 0$ be suitable constant. Then there exist infinitely many integers $n$ satisfying*

$$n/2 \leqslant f(n) = f(n+1) = \cdots = f(n+t), \quad t \geqslant \frac{A}{3} n^{1/2} \tag{3}$$

*and*

$$F(n) > c_1 n^{1/2}, \; F(n+1) > c_1 n^{1/2}, \ldots, F(n+t) > c_1 n^{1/2}. \tag{4}$$

First of all it is easy to see that there exist infinitely many integers $n_i$ satisfying

$$|a_{n_i}| > n_i, \quad |a_k| < |a_{n_i}| \text{ for } 1 \leqslant k < n_i. \tag{5}$$

To prove (5) it suffices to choose $|a_n| > n$ and define $a_{n_i}$ as the $a_k$ of largest absolute value for $1 \leqslant k \leqslant n$.

Put

$$a'_{2n_i} = \frac{1}{2^{2n_i+1}} \sum_{k=0}^{2n_i} \binom{2n_i}{k} a_k .$$

By the second inequality of (5) we have $f(2n_i) \geqslant n_i$, and by the first inequality of (5) for sufficiently large $n_i$

$$F(2n_i) \geqslant \left| \binom{2n_i}{n_i} a_{n_i} / 2^{2n_i+1} \right| > n_i \binom{2n_i}{n_i} / 2^{2n_i+1} > c_2 n_i^{1/2}. \tag{6}$$

Assume first that for infinitely many $n$ satisfying (5) we have $f(2n_i) = n_i$. From lemma 1 we have for $x \geqslant 0$

$$f(2n_i - x) \leqslant f(2n_i) = n_i . \tag{7}$$

Further, a simple argument shows that for $t \leqslant n_i - n_{i-1}$ and $j \geqslant 1$

$$\binom{2n_i - t}{n_i} \geqslant \binom{2n_i - t}{n_{i-j}}. \tag{8}$$

Therefore by the second inequality of (5)

$$\left| \binom{2n_i - t}{n_i} a_{n_i} \right| > \left| \binom{2n_i - t}{n_{i-j}} a_{n_{i-j}} \right|. \tag{9}$$

(7) and (9) imply that for $0 \leqslant t \leqslant n_i - n_{i-1}$

$$f(2n_i - t) = f(2n_i) = n_i . \tag{10}$$

A simple computation gives that for $t < A n_i^{1/2}$

$$\binom{2n_i - t}{n_i} > c_3 \binom{2n_i}{n_i} / 2^t . \tag{11}$$

Therefore from (2)[5] (6) and (11) we have for $t < A n_i^{1/2}$

$$F(2n_i - t) \geqslant \frac{1}{2^{2n_i - t + 1}} \binom{2n_i - t}{n_i} | a_{n_i} | > c_3 / 2^{2n_i + 1} \binom{2n_i}{n_i} | a_{n_i} | > \\ > c_2 c_3 n_i^{1/2} > 1 \tag{12}$$

(10) and (12) prove our lemma.

Assume next that for all sufficiently large $n_i$ satisfying (5) we have $f(2n_i) > n_i$. Put $n_i = n_{i_0}$, $f(2n_{i_0}) = n_{i_1}$, $f(2n_{i_1}) = n_{i_2}$ . . . There thus exists an infinite sequence $n_{i_0}$, $n_{i_1}$, . . . satisfying

$$n_{i_0} < n_{i_1} < \ldots, \ 2n_{i_r} \geqslant n_{i_{r+1}}, \ f(2n_{i_r}) = n_{i_{r+1}} . \tag{13}$$

To simplify the notation we shall write $n_r$ instead of $n_{i_r}$ whenever there is no danger of confusion. First of all we show that all the $n_r$ satisfy (5). We use induction. By assumption $n_0$ satisfies (5). Assume that $n_r$ satisfied (5). A simple computation gives for sufficiently large $A$

$$\binom{2n_r}{n_{r+1}} < \binom{2n_r}{n_r + A[n_r]^{1/2}} < \frac{1}{2} \binom{2n_r}{n_r} .$$

Thus

$$\left| \binom{2n_r}{n_{r+1}} a_{n_{r+1}} \right| \geqslant \left| \binom{2n_r}{n_r} a_{n_r} \right|$$

implies

$$| a_{n_{r+1}} | > 2 | a_{n_r} | > 2n_r \geqslant n_{r+1}$$

which is the first inequality of (5). Further since the binomial coefficients $\binom{2n_r}{n_r + l}$ decrease as $l$ increases, it follows from $f(2n_r) = n_{r+1}$ that

$$| a_{n_{r+1}} | > | a_n | \quad \text{for} \quad n_r \leqslant n < n_{r+1} .$$

But then since $n_r$ satisfied the first inequality of (5) it clearly follows that $n_{r+1}$ also satisfied it, which completes our proof.

Next we prove that for all $n \geqslant 2n_0$

$$F(n) > c_4 n^{1/2} . \tag{14}$$

[5]) This is the only place where our assumption that $A$ is sufficiently large is essential.

From (13) and lemma 1 it follows that for $n \geqslant 2n_0$, $f(n) > n/2$. Hence we have from lemma 2 that for $n \geqslant 2n_0$ $F(n)$ is an increasing function of $n$. Let

$$2n_r \leqslant n < 2n_{r+1} \leqslant 4n_r.$$

Since $n_r$ satisfies (5) we have

$$F(n) \geqslant F(2n_r) \geqslant \left| \binom{2n_r}{n_r} a_{n_r} \right| > c_5 n_r^{1/2} > c_7 n^{1/2} \text{ q. e. d.}$$

Consider now the interval $2n_{i_0} \leqslant n \leqslant 4n_{i_0}$. Clearly $n_{i_0} < f(n) \leqslant 4n_{i_0}$. Also $f(n)$ must be one of the $n_j$'s. But by (2) the difference of two consecutive $n_j$'s is greater than $A n_{i_0}^{1/2}$, $(n_j > n_{i_0})$. Thus the number of $n_j$'s in the interval $(n_{i_0}, 4n_{i_0})$ is less than $3n_{i_0}^{1/2}/A$. Hence there must be at least

$$2n_{i_0} | (3n_{i_0}/A) | = \frac{2A}{3} n_{i_0}^{1/2}$$

integers in the interval $(n_{i_0}, 4n_{i_0})$ with the same $f(n)$ and by Lemma 1 they must be consecutive integers say $n, n+1, \ldots n+t$ $t > A/3\, n^{1/2}$. Thus (14) completes the proof of Lemma 4.

Now we can prove our theorem. Let $n$ satisfy lemma 4 and choose $t = \left[\frac{A}{3} n^{1/2}\right] + 1$. Put $\left[\frac{2n+t}{2}\right] = M$. We have $a'_M = \frac{1}{2^{M+1}} \sum_{k=0}^{M} \binom{M}{k} a_k$.

We shall show that $|a'_M| > c_8 M^{1/2}$ where $c_8$ is an absolute constant independent of $n$. This will of course show that $\Sigma a'_n$ can not converge, hence $\Sigma a_n$ was not Euler summable and the proof of our theorem will be complete.

Put $f(M) = n_j$. 'Ve have by (4)

$$F(M) = \frac{1}{2^{M+1}} \left| \binom{M}{n_j} a_{n_j} \right| > c_1 n^{1/2} > c_1/2\, M^{1/2}. \tag{15}$$

We have

$$\begin{aligned} |a'_M| &\geqslant \frac{1}{2^{M+1}} \left[ \binom{M}{n_j} |a_{n_j}| - \sum_{n_r > n_j} \binom{M}{n_r} |a_{n_r}| - \sum_{n_r < n_j} \binom{M}{n_r} |a_{n_r}| \right] = \\ &= \frac{1}{2^{M+1}} \left[ \binom{M}{n_j} |a_{n_j}| - \Sigma_1 - \Sigma_2 \right]. \end{aligned} \tag{16}$$

For an estimate of $\Sigma_1$ put $r - j = k$, then $n_r - n_j > A k n_j^{1/2}$. Put

$$n + t = M + x, \quad \frac{A}{6} n^{1/2} \leqslant x \leqslant \frac{A}{6} n^{1/2} + 1.$$

We have by

$$f(n) = f(n+t) = n_j \leqslant x > \frac{A}{12} M^{1/2}$$

$$\left| \binom{M+x}{n_r} a_{n_r} \right| \leqslant \left| \binom{M+x}{n_j} a_{n_j} \right|$$

Hence

$$\left| \binom{M}{n_r} a_{n_r} \right| \leqslant \left| \binom{M}{n_j} a_{n_j} \right| \frac{M-n_r+1}{M-n_j+1} \cdot \frac{M-n_r+2}{M-n_j+2} \cdot \, \cdot \, \cdot \frac{M-n_r+x}{M-n_j+x} <$$

$$< \left| \binom{M}{n_j} a_{n_j} \right| \left(1 - \frac{kA\, n_j^{1/2}}{M}\right)^x < \left| \binom{M}{n_j} a_{n_j} \right| \left(1 - \frac{kA\,(M/2)^{1/2}}{M}\right)^{\frac{A}{12} M^{1/2}} <$$

$$< \left| \binom{M}{n_j} a_{n_j} \right| \left(1 - \frac{kA}{2M^{1/2}}\right)^{\frac{A}{12} M^{1/2}} < \left| \binom{M}{n_j} a_{n_j} \right| e^{-k\frac{A^2}{24}}$$

since from $f(M) = n_j$, $n_j \geqslant M/2$ (lemma 4). Thus for sufficiently large $A$ [5]

$$\Sigma_1 < \left| \binom{M}{n_j} a_{n_j} \right| \sum_{k=1}^{\infty} e^{-k\frac{A^2}{24}} < \frac{1}{4} \left| \binom{M}{n_j} a_{n_j} \right|. \tag{17}$$

In the same way we can show

$$\Sigma_2 < \frac{1}{4} \left| \binom{M}{n_j} a_{n_j} \right|. \tag{18}$$

Thus by (15), (16), (17) and (18)

$$|a'_M| > \frac{1}{2} \left| \binom{M}{n_j} a_{n_j} \right| / 2^{M+1} = \frac{1}{2} F(M) > \frac{c_1}{2} M^{1/2}$$

which completes the proof of the theorem.

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