ON A THEOREM OF RÅDSTRÖM

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The purpose of this note is to give a new and simplified proof of the following theorem.

THEOREM. *Let* $f(z)=\sum_{\nu=0}^{\infty} a_\nu z^\nu$ *be an entire function. Denote* $M(r) = \max_{|z|\leq r} |f(z)|$. *Assume that* $\lim\sup \log M(r)/r=\infty$. *Then there exists* ω_ν, $\nu=0, 1, \cdots$, *with* $|\omega_\nu|=1$, *so that the origin is a limit point of the roots of the derivatives of* $k(z)=\sum_{\nu=0}^{\infty} \omega_\nu a_\nu z^\nu$.

In other words the theorem holds if the order ρ of $f(z)$ is greater than 1 or if $\rho=1$ and $f(z)$ is of maximal type.

This theorem is due to Rådström and was proved by him for the case $\rho>1$ in a recent note.[1] The result as announced here is best possible with respect both to order and to type, as is shown by the example e^{cz}, where c is a constant (cf. footnote 1, p. 400).

We need the following two lemmas.

LEMMA 1. *Let* $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ *be a power series with radius of convergence* $R\leq\infty$ *and such that* $|a_0/a_1|<R$. *Then it is possible to find* $\omega_0, \omega_1, \cdots$ *with* $|\omega_\nu|=1$ *so that* $\sum_{\nu=0}^{\infty}\omega_\nu a_\nu z^\nu$ *has a zero* z_0 *with* $|z_0|\leq|a_0/a_1|$.

PROOF. We put $\omega_0=\omega_1=1$ and $a_0+a_1z=P_1(z)$. Obviously $P_1(z)$ has a zero with the required property. We proceed by induction. Suppose that we have succeeded in determining $\omega_0, \omega_1, \cdots, \omega_{n-1}$ such that the polynomial $P_{n-1}(z)=\sum_{\nu=0}^{n-1}\omega_\nu a_\nu z^\nu$ has a zero z_0 with $|z_0| \leq|a_0/a_1|$. Consider $P_{n-1}(z)+\omega a_n z^n$, $|\omega|=1$. Three cases may occur:

1. The equation $|P_{n-1}(z)|=|a_nz^n|$ has a solution on $|z|=|a_0/a_1|$.
2. $|P_{n-1}(z)|>|a_nz^n|$ for all z with $|z|=|a_0/a_1|$.
3. $|P_{n-1}(z)|<|a_nz^n|$ for all z with $|z|=|a_0/a_1|$.

In case 1, it will obviously be possible to choose ω so that $P_{n-1}(z) +\omega a_nz^n=0$ on the circle $|z|=|a_0/a_1|$. In case 2, $P_{n-1}(z)+\omega a_nz^n$ has by Rouché's theorem as many zeros inside the circle $|z|=|a_0/a_1|$ as $P_{n-1}(z)$, that is, at least one, by the induction hypothesis. In case 3, again by Rouché's theorem, $P_{n-1}(z)+\omega a_nz^n$ has as many zeros in $|z|=|a_0/a_1|$ as a_nz^n, that is, n zeros. In all these cases we can therefore choose $\omega=\omega_n$, $|\omega_n|=1$ so that $P_{n-1}(z)+a_n\omega_nz^n$ has a zero in the circle $|z_0|=|a_0/a_1|$. Consider now the power series $\sum_{\nu=0}^{\infty}\omega_\nu a_\nu z^\nu$. We know that all its partial sums have zeros in or on the circle $|z|=|a_0/a_1|$. As this circle is strictly inside the circle of convergence

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[1] H. Rådström, Proc. Nat. Acad. Sci. U.S.A. vol. 35 (1949) pp. 399–404.

the same must hold for the infinite series, which proves the lemma.

LEMMA 2. *Let* $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ *satisfy the conditions of Lemma* 1, *and let* ϵ *be a positive number. Then there exists an integer* n *and numbers* $\omega_0, \omega_1, \cdots, \omega_n$ *with* $|\omega_\nu| = 1$ *such that the series* $\sum_{\nu=0}^{\infty} \omega_\nu a_\nu z^\nu$ *has a zero in the circle* $|z| \leq |a_0/a_1| + \epsilon$, *irrespective of the choice of the numbers* ω_ν *for* $\nu \geq n+1$.

PROOF. Let r be a number with $|a_0/a_1| < r < \min(R, |a_0/a_1| + \epsilon)$ and such that the series $f(z) = \sum_{\nu=0}^{\infty} \omega_\nu a_\nu z^\nu$ constructed in Lemma 1 has a positive minimum m on the circle $|z| = r$. Put $\delta_\mu = \sum_{\nu=\mu+1}^{\infty} |a_\nu| r^\nu$. We have $\delta_\mu \to 0$ monotonically. Choose n so large that $2 \cdot \delta_n < m$, and let $g(z)$ be any series which coincides with $f(z)$ in the first $n+1$ terms whereas in the rest of the terms arbitrary changes of the arguments are allowed. Obviously $|g(z) - f(z)| < 2 \cdot \delta_n$ for $|z| \leq r$. Therefore, by Rouché's theorem, $g(z)$ has as many roots in $|z| \leq r$ as $f(z)$, that is, at least one (since $r > |a_0/a_1|$). This proves the lemma.

In order to prove the theorem we first observe that if $\limsup \log M(r)/r = \infty$, it follows that $\liminf |a_n/(n+1)a_{n+1}| = 0$, for otherwise there would exist a $k > 0$ such that for all sufficiently large n, $a_{n+1} < ka_n/(n+1)$. Iterating this we would get, for sufficiently large n, $a_n < ck^n/n!$, which as is well known implies $\limsup \log M(r)/r \leq k$, an evident contradiction. Therefore there exists a sequence n_ν of integers such that $a_{n_\nu}/(n+1)(a_{n_\nu+1}) \to 0$. We also observe that $f^{(n)}(z)/n! = a_n + (n+1)a_{n+1}z + \cdots$. Now choose a sequence ϵ_ν of positive numbers with $\epsilon_\nu \to 0$. According to Lemma 2 we can find numbers $\omega_{n_1}, \omega_{n_1+1}, \cdots, \omega_{n_1+p_1}$ so that if in $f^{(n_1)}(z)/n_1!$ we multiply each coefficient with the corresponding ω, we shall get a function which has a zero in $|z| < |a_{n_1}/(n_1+1)a_{n_1+1}| + \epsilon_1$, and we shall still be able to choose ω_μ arbitrarily if $\mu > n_1 + p_1$ without destroying this property. Therefore we can repeat this process, now starting with the smallest $n_\nu > n_1 + p_1$. Call that number m_2 and put $m_1 = n_1$. Then we get a new set of ω's, $\omega_{m_1}, \cdots, \omega_{m_2+p_2}$, and if $\mu > m_2 + p_2$ we still have the free choice of the ω_μ. Iterating this process we shall obtain a sequence of nonoverlapping blocks of ω's and we complete it if necessary by choosing ω_ν arbitrarily for those ν which do not correspond to an ω in a block. In this way we get a sequence $\omega_0, \omega_1, \cdots$ and we construct the corresponding power series $k(z) = \sum_{\nu=0}^{\infty} \omega_\nu a_\nu z^\nu$. From the construction and Lemma 2 it is then obvious that $k(z)$ will have the property: $k^{(m_\nu)}(z)$ has a zero z_ν satisfying $z_\nu < |a_{m_\nu}/(m_\nu+1)a_{m_\nu+1}| + \epsilon_\nu$. As the sequence m_ν is a subsequence of n_ν and $\epsilon_\nu \to 0$, it is clear that $z_\nu \to 0$, which proves the theorem.

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