SOME REMARKS ON DIOPHANTINE APPROXIMATIONS

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1. The present note contains some disconnected remarks on diophatine approximations.

First we collect a few well-known results about continued fractions, which we shall use later[1]. Let α be an irrational number, $q_1 < q_2 < \dots$ be the sequence of the denominators of its convergents. For almost all α we have for $k > k_0(\alpha)$, $q_{k+1} < q_k (\log q_k)^{1+\epsilon}$. Thus if n is large and $q_r \leqslant n < q_{r+1}$ we have $q_r > \frac{n}{(\log n)^{1+\epsilon}}$. Further for almost all α

$$\frac{1}{q_k^2 (\log q_k)^{1+\epsilon}} < \left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}, \tag{1}$$

the second inequality is true for all α.

Also if $|\alpha - a/b| < \frac{1}{2} b^2$ and $q_k \leqslant b < q_{k+1}$, then $b \equiv 0 \pmod{q_k}$. Hence if

$$\frac{1}{\{ m\alpha \}} > 2n, m < n \text{ then } m \equiv 0 \pmod{q_r}, \tag{2}$$

where $q_r \leqslant n < q_{r+1}$, and we denote by $\{u\}$ the distance of u from the nearest integer. It is easy to obtain from (1) that for almost all α and $m \geqslant m_0(\alpha)$

$$\frac{1}{\{ m\alpha \}} < m (\log m)^{1+\epsilon}. \tag{3}$$

A theorem of Behnke[2] states that for almost all α $(q_r \leqslant n < q_{r+1})$

1. The results in question can all be found in Koksma, Diophantische Approximation, *Ergebnisse der Math.* 4 (4).

2. *Hamburgische Abhandlengen*, 3 (1924), p. 289.

$$\sum_{\substack{m=1 \\ q_r \nmid m}}^{n} \frac{1}{\{m\alpha\}} < c_1 n \log n. \tag{4}$$

Denote by $N_n(a, b)$ the number of integers $m \leqslant n$ for which $a \leqslant n\alpha - [n\alpha] \leqslant b$. A theorem of Khintchine-Ostrowsky[1] states that

$$(b-a)n - c_2 (\log n)^{1+\epsilon} < N_n (a, b) < (b-a)n + c_3 (\log n)^{1+\epsilon}, \tag{5}$$

where c_2 and c_3 are independent of a, b and n and depend only on α and ϵ.

2. Denote by $d(n)$ the number of divisors of n, by $r_2(n)$ the number of representations of n as the sum of two squares and by $r_4(n)$ the number of representations of n as the sum of four squares. Walfisz[2] proved, sharpening previous results of Chowla[3], that for almost all α

$$\sum_{m=1}^{n} d(m)\, e^{2\pi i m\alpha} = O(n^{1/2} (\log n)^{1+\epsilon}) \tag{6}$$

$$\sum_{m=1}^{n} r_2(m)\, e^{2\pi i m\alpha} = O(n^{1/2} (\log n)^{1+\epsilon}) \tag{7}$$

$$\sum_{m=1}^{n} r_4(m)\, e^{2\pi i m\alpha} = O(n^{\frac{1}{2}} (\log n)^{2+\epsilon}). \tag{8}$$

By a slight modification of their argument we obtain that for almost all α

$$\sum_{m=1}^{n} d(m)\, e^{2\pi i m\alpha} = O(n^{\frac{1}{2}} \log n) \tag{9}$$

1. Khintchine, *Math. Zeitschrift*, 18 (1923), p. 297-300. See also Ostrowsky, *Hamburgische Abhandlungen*, 1 (1922), p. 95.

2. *Math. Zeitschrift*, 35 (1935), p. 774-778.

3. *Ibid.*, 33 (1935), p. 544-563.

$$\sum_{m=1}^{n} r_2(m)\, e^{2\pi i m\alpha} = O(n^{\frac{1}{2}} \log n) \tag{10}$$

$$\sum_{m=1}^{n} r_4(m)\, e^{2\pi i m\alpha} = O(n^{\frac{1}{2}} (\log n)^2). \tag{11}$$

(9), (10) and (11) were proved by Chowla[1] in case α has bounded partial fractions in its continued fraction development. But it is well known that these α's have measure 0.

It will suffice to prove (9), the proof of (10) and (11) follows the same pattern.

$$\sum_{m=1}^{n} d(m)\, e^{2\pi i m\alpha} = \sum_{ab \leqslant n} e^{2\pi i ab\alpha}$$

$$= 2 \sum_{a=1}^{n^{\frac{1}{2}}} \sum_{a \leqslant b \leqslant n/a} e^{2\pi i ab\alpha} - \sum_{a=1}^{n^{\frac{1}{2}}} e^{2\pi i a^2 \alpha}. \tag{12}$$

Now clearly for every irrational number α

$$\left| \sum_{a \leqslant b \leqslant m/a} e^{2\pi i ab\alpha} \right| < \frac{c_4}{\sin a\pi\alpha} < \frac{c_5}{\{a\alpha\}}. \tag{13}$$

Also trivially

$$\left| \sum_{a \leqslant b \leqslant n/a} e^{2\pi i ab\alpha} \right| < \frac{n}{a}. \tag{14}$$

Put $q_r \leqslant n^{1/2} < q_{r+1}$. We have from (12), (13), (14) and (3)

$$\left| \sum_{m=1}^{n} d(m)\, e^{2\pi i m\alpha} \right| < \sum_{\substack{a=1 \\ q_r \nmid a}}^{n^{\frac{1}{2}}} \frac{1}{\{a\alpha\}} + \sum{}' \min\left(\frac{r_5}{\{a\alpha\}}, \frac{n}{a} \right) + O(n^{\frac{1}{2}})$$

$$< c_6 n^{\frac{1}{2}} \log n + \sum{}'. \tag{15}$$

The dash indicates that the summation is extended over the $a \equiv 0 \pmod{q_r}$.

1. *Ibid.*, 33 (1935), p. 544-563.

Now we estimate Σ'. As stated in the introduction $q_r > n^{\frac{1}{2}}/\log n)^{1+\varepsilon}$. We distinguish two cases. In case I we have

$$n^{\frac{1}{2}}/(\log n)^{1+\varepsilon} < q_r < (n/\log n)^{\frac{1}{2}}. \tag{16}$$

From (1) we evidently have that for $k < (\log n)^2$, $\{k\, q_r\alpha\} = k\{q_r\alpha.\}$ Thus from (15), (16) and (2)

$$\sum{}' < \sum_{k<(\log n)^2} \frac{1}{\{kq_r\alpha\}} = \sum_{k<(\log n)^2} \frac{1}{k\{q_r\alpha\}} < q_r(\log q_r)^{1+\varepsilon}$$
$$\times \sum_{k<(\log n)^2} \frac{1}{k} < n^{\frac{1}{2}} (\log n)^{\frac{1}{2}+\varepsilon} \sum_{k<(\log n)^2} \frac{1}{k} = o(n^{\frac{1}{2}}\log n). \tag{17}$$

In case II, $q_r > \left(\frac{n}{\log n}\right)^{\frac{1}{2}}$. We evidently have from (14)

$$\sum{}' < \sum_{k \leq (\log n)^{\frac{1}{2}}} \frac{n}{kq_r} < (n \log n)^{1/2} \sum_{k<(\log n)} \frac{1}{k} = o(n^{1/2} \log n). \tag{18}$$

(9) clearly follows from (15), (17) and (18).

3. Spencer[1] proved that for almost all α

$$\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}} = O\left((\log n)^2\right). \tag{19}$$

He remarks that (19) is in a sense best possible since by a theorem of Hardy-Littlewood[2] we have for all irrational α

1. *Proc. Cambridge Phil. Soc.*, 35 (1939), p. 521-547. In fact Spencer considers $\sum_{m=1}^{n} \frac{\operatorname{cosec} m\pi\alpha}{m}$ but it is easy to see that asymptotically this is the same as $\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}}$.

2. *Bull. Calcutta Math. Soc.*, 20 (1930), p. 251-266.

$$\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}} > c_7 (\log n)^2.$$

Spencer conjectured[1] that for almost all α

$$\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}} = (1+o(1)) (\log n)^2. \tag{20}$$

We shall prove (20) and a few related results.

First we prove the following

LEMMA. *For almost all α we have*

$$\sum{}' \frac{1}{\{m\alpha\}} = (1+o(1))\, 2\, n \log n, \tag{21}$$

where in Σ' the summation is extended over the m for which $m \leqslant n$ and $\frac{1}{\{m\alpha\}} \leqslant 2n$.

We write

$$\sum{}' \frac{1}{\{m\alpha\}} = \sum\nolimits_1 + \sum\nolimits_2 \tag{22}$$

where in $\sum_1$, the summation is over all such m for which

$$\frac{1}{\{m\alpha\}} \leqslant \frac{n}{(\log n)^{10/9}}$$

and in $\sum_2$,

$$2n \geqslant \frac{1}{\{m\alpha\}} > \frac{n}{(\log n)^{10/9}}.$$

We obtain by (5) by a simple argument (re-ordering the terms in the summation) that

$$\sum\nolimits_1 = (1+o(1)) \sum_{k < n/(\log n)^{10/9}} \left(\mathcal{N}_n\left(0, \frac{1}{k}\right) + \mathcal{N}_n\left(1-\frac{1}{k}, 1\right)\right) =$$

$$(1+o(1)) \quad 2 \sum_{k < n/(\log n)^{10/9}} \frac{n}{k} = (1+o(1))\, n \log n. \tag{23}$$

1. Oral communication.

Next we estimate $\sum_2$. Put $A = \frac{(\log n)^{1/8}}{n}$. We evidently have from (5) and the fact that each summand in Σ_2 is less than $2n$

$$\sum_2 < 2n\left(\mathcal{N}_n(0, A) + \mathcal{N}_n(1-A, 1)\right) + 3(\log n)^{10/9} \frac{n}{(\log n)^{1/8}} \tag{24}$$

(by (5) the number of terms in Σ_2 is less then $3(\log n)^{10/9}$).

Now we have to estimate $\mathcal{N}_n(0, A) + \mathcal{N}_n(1-A, 1)$. Let $0 < x < 1$ be arbitrary. Denote by $v_1 < v_2 < \ldots < v_k$ the integers $\leqslant n$ for which $x \leqslant v_i\alpha - [v_i\alpha] \leqslant x + 1/2.n$. Clearly the numbers $(v_i - v_1)\alpha - [(v_i - v_1)\alpha]$ all are either in $(0\ 1/2.n)$ or in $(1 - 1/2.n, 1)$. Thus

$$\mathcal{N}_n(x, x+1/2.n) < \mathcal{N}_n(0, 1/2.n) + \mathcal{N}_n(1-1/2.n, 1) + 1,$$

or splitting $(0, A)$ and $(1-A, 1)$ into intervals of length $\frac{1}{2n}$ we have $\mathcal{N}_n(0, A) + \mathcal{N}_n(1-A, 1) <$

$$2(\log n)^{1/8}[\mathcal{N}_n(0, 1/2.n) + \mathcal{N}_n(1-1/2.n, 1)] + 2(\log n)^{1/8}. \tag{25}$$

By what has been said in the introduction all the integers m, for which $\frac{1}{\{m\alpha\}} \geqslant 2n$ satisfy $m \equiv 0 (\mathrm{mod}\ q_r)$, where $q_r \leqslant n < q_{r+1}$. We distinguish two cases.

Case I. $q_r \geqslant n/(\log n)^{1/2}$.

Then clearly

$$\mathcal{N}_n(0, 1/2.n) + \mathcal{N}_n(1-1/2.n, 1) < (\log n)^{1/2}. \tag{26}$$

Case II. $q_r < n/(\log n)^{1/2}$.

But then by (3)

$$\frac{1}{\{q_r\alpha\}} < q_r(\log q_r)^{1+\varepsilon} < n(\log n)^{1/2+\varepsilon}.$$

Thus if $k.q_r.\alpha - [k.q_r.\alpha]$ is in $(0, 1/2.n)$ or in $(1-1/2.n, 1)$ we have $k < (\log n)^{1/2+\varepsilon}$. Thus in case II

$$\mathcal{N}_n(0,\ 1/2.n)+\mathcal{N}_n(1-1/2.n,\ 1) < (\log n)^{1/2+\varepsilon}. \quad (27)$$

Hence from (26), (27) and (24) we obtain

$$\Sigma_2 = o(n \log n). \quad (28)$$

The lemma now follows from (23) and (28).

Now we prove (20). We have

$$\sum_{m=1}^{n} \frac{1}{m\{m\alpha\}} = \sum_3 + \sum_4, \quad (29)$$

where in $\sum_3$, $\frac{1}{(m\alpha)} \leqslant 2.n$

and in $\sum_4$, $\frac{1}{(m\alpha)} > 2.n.$

We obtain from (21) by partial summation that

$$\sum_3 = (1+o(1)) \sum_{m \leqslant n} \frac{2 \log m}{m} = (1+o(1)) (\log n)^2. \quad (30)$$

For the m in Σ_4 we have as before that $m \equiv 0 (\mathrm{mod}\ q_r)$, hence from $q_r > n/(\log n)^{1+\varepsilon}$ we have

$$\sum_4 \leqslant \sum_{k \leqslant n/q_r} \frac{1}{kq_r \{kq_r\alpha\}} \leqslant \sum_{k < (\log n)^2} \frac{1}{k^2 q_r \{q_r\alpha\}} <$$

$$(\log n)^{1+\varepsilon} \sum_{k=1}^{\infty} \frac{1}{k^2}\, o(\log n)^2. \quad (31)$$

(20) follows from (30) and (31).

Similarly we can prove that for almost all α and $0 < a < 1$

$$\sum_{m=1}^{n} \frac{1}{m^a \{m\alpha\}} = (1+o(1)) \frac{2n^{1-a} \log n}{a}.$$

Before concluding the paper we state a few results without proof:

I. For almost all α

$$\sum_{n=1}^{x} \frac{1}{\sum\limits_{m=1}^{n} \{m\alpha\}^{-1}} = (1+o(1)) \frac{\log \log x}{2}. \quad (32)$$

Thus in particular for almost all α,

$$\sum_{n=1}^{\infty} \frac{1}{\sum_{n=1}^{n} \{m\alpha\}^{-1}}$$

diverges.

The proof of (32) is not difficult, it follows from (21) without much difficulty.

II. Let $f(n)$ be an increasing function of n for which $f(n) > (2+c).n.\log n$ and $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ converges. Then for almost all α and $n > n_0(\alpha)$

$$\sum_{m=1}^{\infty} \frac{1}{\{m\alpha\}} < f(n).$$

The proof of (II) is not quite simple and is not given here. (I) and (II) were suggested to me by the beautiful work of Khintchine[1] and Paul Levy[2] on continued fractions.

1. *Compositio Math.*, 1 (1935), p. 381.
2. *Ibid.*, 3 (1936), p. 302.