OVER-CONVERGENCE ON THE CIRCLE OF CONVERGENCE

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1. **Introduction and summary.** M. Riesz has proved [7] that *if* $\lim_{n\to\infty} a_n = 0$ *and the function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is regular at each point of the arc* $C(z = e^{i\alpha}, \alpha_1 \leq \alpha \leq \alpha_2)$, *the series* $\sum a_n z^n$ *converges to* $f(z)$ *uniformly on the arc* C. If the sequence a_n fails to converge to zero, the series $\sum a_n z^n$ cannot possibly converge at any point on the unit circle; but it is still possible that a subsequence of the partial sums $s_n(z) = \sum_{k=0}^{n} a_k z^k$ converges at some points on the unit circle. In fact, Ostrowski's gap theorem [5] and [4; 204–207] asserts that *if the function* $f(z) = \sum a_n z^n$ *is regular at all points of the arc* C *(including the end points) and* m_i *and* n_i *are two sequences such that* $\liminf_{i\to\infty} n_i/m_i > 1$ *and* $a_n = 0$ *when* $m_i < n < n_i$, *the partial sums* $s_{m_i}(z)$ *converge to* $f(z)$ *uniformly in some domain that contains the arc* C.

If the requirement $\liminf_{i\to\infty} n_i/m_i > 1$ in Ostrowski's theorem is relaxed, convergence of the partial sums $s_{m_i}(z)$ to $f(z)$ at points on the unit circle may still occur provided that the function $f(z)$ or its Taylor coefficients satisfy certain conditions [1], [6] — the theorem by M. Riesz may be regarded as providing a special example. The present paper establishes certain general conditions sufficient for uniform convergence of the sequence $s_{m_i}(z)$ to the function $f(z)$ on an arc C of the unit circle.

2. **The general principle.** Riesz's proof of his theorem has been greatly simplified by Landau [3; 73]: Let the function $f(z) = \sum a_n z^n$ be regular on the arc C of the unit circle; then there exists a circular segment Γ (vertex at the origin) containing the arc C and its end points, and such that $f(z)$ is regular at all points of Γ and its boundary. The functions

$$g_n(z) = [f(z) - s_n(z)]z^{-n-1}(z - z_1)(z - z_2)$$

(where z_1 and z_2 are the intersections of the unit circle with the boundary of Γ), are regular in Γ; each therefore takes its maximum modulus on the boundary of Γ. Landau shows that the sequence $g_n(z)$ tends to zero uniformly on the boundary of Γ, and it follows that the sequence $f(z) - s_n(z)$ tends to zero uniformly on the arc C.

Landau's proof in turn invites certain modifications. The general principle stated below gives the results that can be obtained by these modifications. The results become interesting when the modifications are made specific.

PRINCIPLE. (i) *Let* Γ *be a region such that the function* $f(z) = \sum a_n z^n$ *and its immediate analytic extension are regular in the interior of* Γ *and continuous on* Γ, *and let the boundary of* Γ *meet the unit circle at* z_1 *and* z_2.

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(ii) *Let the arc C of the unit circle (together with its end points) be interior to* Γ, *and let* $L = \min | (z - z_1)(z - z_2) |$ (z *on* C).

(iii) *Let m_i and n_i be increasing sequences of integers* ($i = 0, 1, 2, \cdots ; m_i < n_i$) *such that $a_n = 0$ when $m_i \leq n \leq n_i$. Then a sufficient condition that $s_{m_i}(z) \to f(z)$ uniformly on C is the existence of a positive sequence p_i such that the sequence of functions*

$$g_i(z) = [f(z) - s_{m_i}(z)]\,[(z - z_1)(z - z_2)/L]^{p_i} z^{-(m_i+n_i)/2}$$

tends to zero uniformly on the boundary of Γ.

The principle is virtually obvious: each function $g_i(z)$ is regular in Γ; the principle of the maximum therefore applies; and since on C

$$| f(z) - s_{m_i}(z) | = | g_i(z) |\, [| (z - z_1)(z - z_2) |/L]^{-p_i}$$

and the quantity in brackets is not less than unity, the proof is complete.

3. **Application to Taylor series with bounded coefficients.** It seems reasonable to expect that sufficient conditions for over-convergence will generally involve the size of the Taylor coefficients and the lengths and positions of the gaps. Remarkably enough, the following theorem assures over-convergence at an astonishingly rapid rate which is independent of the position of the gaps.

THEOREM 1. *If*

(i) $| a_n | < A$ ($n = 0, 1, 2, \cdots$);

(ii) $a_n = 0$ *when* $m_i \leq n \leq n_i$ ($\lim_{i\to\infty} (n_i - m_i) = \infty$);

(iii) *the function $f(z) = \sum a_n z^n$ is regular on the arc C of the unit circle (including the end points)*;

then the sequence $s_{m_i}(z)$ converges to $f(z)$ uniformly on C; moreover, if k is any constant less than unity, there exists an integer i_0 such that on C

$$| f(z) - s_{m_i}(z) | < (n_i - m_i)^{-(n_i - m_i)^k}$$

when $i > i_0$.

To prove that the sequence $s_{m_i}(z)$ converges to $f(z)$ uniformly on C, it would be sufficient to apply the general principle with the region Γ chosen as for Landau's proof of Riesz's theorem, and with $p_i = 2$ ($i = 0, 1, 2, \cdots$). The proof of the complete theorem is carried out with the same region Γ, but it calls for a more delicate choice of the sequence p_i.

For simplicity, the following notation is used:

$$(n_i + m_i)/2 = N_i\,,$$

$$(n_i - m_i)/2 = M_i = N_i - m_i = n_i - N_i\,.$$

A constant h is chosen so that $k < h < 1$, and the sequence p_i is defined by the relation

$$p_i = M_i^h + 1.$$

It is to be shown that on the boundary of Γ the relation

$$\left| [f(z) - s_{m_i}(z)]\,[(z - z_1)(z - z_2)/L]^{1+M_i^h} z^{-N_i} \right| < (2M_i)^{-(2M_i)^k}$$

holds uniformly provided the index i is sufficiently large.

The proof is carried out in three parts: for the segment $z = rz_1$ $(0 \leq r \leq 1)$; for the segment $z = rz_1$ $(1 \leq r \leq R)$; and for the arc $|z| = R$. Symmetry permits the suppression of all discussion regarding the segment $z = rz_2$ $(0 \leq r \leq R)$.

On the first part the relation

$$| f(z) - s_{m_i}(z) | \leq \sum_{n=n_i}^{\infty} | a_n | r^n < Ar^{n_i}/(1 - r)$$

gives the estimate

$$| g_i(z) | < AcL^{-1}r^{M_i}[(1 - r)c/L]^{M_i^h},$$

where c denotes the diameter of the region Γ. Since

$$\max r^s(1 - r)^t = \left(\frac{s}{s + t}\right)^s\left(\frac{t}{s + t}\right)^t \qquad (0 < r < 1),$$

suppression of the subscript i gives the inequalities

$$\begin{aligned} | g(z) | &< AcL^{-1}\left(\frac{M}{M + M^h}\right)^M \left(\frac{M^h c/L}{M + M^h}\right)^{M^h} \\ &< AcL^{-1}(1 + 1/M^{1-h})^{-M^{1-h}M^h}\,(M^{1-h}L/c)^{-M^h} \\ &< AcL^{-1}(2eM^{1-h}L/c)^{-M^h}, \end{aligned}$$

and therefore

$$| g(z) | < (KM^{1-h})^{-M^h},$$

where K is a certain positive constant; this implies that if M_i is sufficiently large,

$$\begin{aligned} \log | g(z) | &< -M^h[\log K + (1 - h) \log M] \\ &< -(2M)^k \log (2M), \end{aligned}$$

as was to be shown.

On the segment $z = rz_1$ $(1 \leq r \leq R)$,

$$| g_i(z) | < \left[F + \sum_{n=0}^{m_i-1} | a_n | r^n \right][(r - 1)c/L]^{1+M_i^h} r^{-N_i}$$

where $F = \max | f(z) |$ (z on Γ). This inequality implies that (subscripts suppressed)

$$| g(z) | < \left[F + A\,\frac{r^m - 1}{r - 1} \right][(r - 1)c/L]^{1+M^h} r^{-N}$$

and, for sufficiently large values of i,

$$| g(z) \quad < 2AcL^{-1}r^{-M}[(r-1)c/L]^{M^h}.$$

Because

$$\max r^{-s}(r-1)^t = \left(\frac{s}{s-t}\right)^{-s}\left(\frac{t}{s-t}\right)^t \qquad (r > 1),$$

$$\begin{aligned} | g(z) | &< 2AcL^{-1}\left(\frac{M}{M-M^h}\right)^{-M}\left(\frac{M^h c/L}{M-M^h}\right)^{M^h} \\ &= 2AcL^{-1}(1-M^{h-1})^{-M^{1-h}M^h}[(M^{1-h}-1)L/c]^{-M^h} \\ &< (KM^{1-h})^{-M^h} < (2M_i)^{-(2M_i)^k} \end{aligned}$$

provided M_i is sufficiently large.

On the arc $| z | = R$,

$$\begin{aligned} | g_i(z) | &< \left[F + A\sum_{n=0}^{m_i-1} R^n\right](c/L)^{1+M_i^h}R^{-N_i} \\ &< \frac{2A}{R-1}R^{-M_i}(c/L)^{1+M_i^h} \\ &= [(1-\epsilon)R]^{-M_i} \end{aligned}$$

($\lim_{i\to\infty} \epsilon = 0$). A simple comparison shows that the last member is less than

$$(2M_i)^{-(2M_i)^k}$$

when M_i is sufficiently large, and the proof is complete.

4. **Functions of finite order.** A function $f(z) = \sum a_n z^n$ is said to be of finite order on the unit circle provided

$$\limsup_{n\to\infty} \log | a_n |/\log n < \infty$$

[2; 171]. For functions of this type, two closely related theorems will be proved simultaneously.

THEOREM 2. *If a_n is a sequence such that*

(i) *there exists a constant t such that $| a_n | < n^t$ ($n = 0, 1, 2, \cdots$);*

(ii) *there exist two sequences of integers m_i, n_i such that $\lim_{i\to\infty} (n_i - m_i)/\log n_i = \infty$ and such that $a_n = 0$ when $m_i \leq n \leq n_i$;*

(iii) *the function $f(z) = \sum a_n z^n$ is regular on the arc C of the unit circle (including its end points);*

then the sequence $s_{m_i}(z)$ converges to $f(z)$ uniformly on C; moreover, if k is any constant, there exists an integer i_0 such that on C

$$|f(z) - s_{m_i}(z)| < n_i^{-k}$$

when $i > i_0$

THEOREM 3. *Let Γ be any region and C an arc on the unit circle interior (together with its end points) to Γ. Corresponding to every constant t there exists a second constant $k(t, C, \Gamma)$ such that for every function $f(z) = \sum a_n z^n$ which is regular in the interior of Γ and continuous on Γ, the sequence $s_{m_i}(z)$ converges to $f(z)$ uniformly on C provided*

(i) $|a_n| < n^t$ $(n = 0, 1, 2, \cdots)$,

(ii) $a_n = 0$ *when* $m_i \leq n \leq n_i$ $(i = 0, 1, 2, \cdots)$, *where n_i is a sequence such that*

$$\liminf_{i \to \infty} (n_i - m_i)/\log n_i > k(t, C, \Gamma).$$

The following version of the proof, wasteful in its treatment of inequalities, is presented because of its simplicity. Without loss in generality it may be assumed that the region Γ in Theorem 3 is a sector of an annulus: $z = re^{i\alpha}$; $R_1 \leq r \leq R_2$, $R_1 < 1 < R_2$; $\alpha_1 \leq \alpha \leq \alpha_2$. The symbols c, M_i, N_i, and F are used with the same meaning as in the previous section; the symbol θ_i denotes the quantity $M_i/\log n_i$. The sequence p_i is defined by the relation

$$p_i = t + 1 + \log n_i$$

Part 1. On the segment $z = rz_1$ $(R_1 \leq r \leq 1)$

$$|g_i(z)| < \sum_{n=n_i}^{\infty} n^t r^n [(1 - r)c/L]^{p_i} r^{-N_i}$$

$$< n_i^t r^{M_i} \sum_{n=0}^{\infty} n^t r^n [(1 - r)c/L]^{\log n_i} [(1 - r)c/L]^{t+1}$$

$$< C_1 [r^{\theta_i}(1 - r)e^t c/L]^{\log n_i},$$

where C_1 is a constant depending on c, L, and t.

Part 2. On the segment $z = rz_1$ $(1 \leq r \leq R_2)$

$$|g_i(z)| < \left| F + \sum_{n=0}^{m_i - 1} n^t r^n \right| [(r - 1)c/L]^{p_i} r^{-N_i}$$

$$< 2m_i^{t+1} r^{-M_i} [(r - 1)c/L]^{\log n_i} [(r - 1)c/L]^{t+1}$$

$$< C_2 [r^{-\theta_i}(r - 1)e^{t+1} c/L]^{\log n_i}.$$

Part 3. On the arc $z = R_2 e^{i\alpha}$ $(\alpha_1 \le \alpha \le \alpha_2)$

$$| g_i(z) | < \left[F + \sum_{n=0}^{m_i - 1} n^t R_2^n \right] (c^2/L)^{p_i} R_2^{-N_i}$$

$$< m_i^{t+1} [R_2^{-\theta_i} c^2/L]^{\log n_i} (c^2/L)^{t+1}$$

$$< C_3 (R_2^{-\theta_i} e^{t+1} c^2/L)^{\log n_i}$$

Part 4. On the arc $z = R_1 e^{i\alpha}$ $(\alpha_1 \le \alpha \le \alpha_2)$

$$| g_i(z) | < \sum_{n=n_i}^{\infty} n^t R_1^n (c^2/L)^{p_i} R_1^{-N_i}$$

$$< n_i^t R_1^{M_i} \sum_{n=0}^{\infty} n^t R_1^n (c^2/L)^{p_i}$$

$$< C_4 [R_1^{\theta_i} e^t c^2/L]^{\log n_i}.$$

In Part 1, the quantity $| g_i(z) |$ is dominated by $C_1[(1 - R_1)e^t c/L]^{\log n_i}$, in Part 2 by $C_2[(R_2 - 1)e^{t+1} c/L]^{\log n_i}$. If the contents of the two pairs of brackets are not less than unity, they may be made so by replacing the radii R_1 and R_2 by radii R_1' and R_2' nearer to unity. The quantities dominating $| g_i(z) |$ in Parts 3 and 4 are then replaced by $C_3[R_2'^{-\theta_i} e^{t+1} c^2/L]^{\log n_i}$, $C_4[R_1'^{\theta_i} e^t c^2/L]^{\log n_i}$. The contents of the last two pairs of brackets are less than unity and bounded away from unity provided *that*

$$\log R_2' \limsup_{i \to \infty} \theta_i > t + 1 + \log c^2/L,$$

$$-\log R_1' \limsup_{i \to \infty} \theta_i > t + \log c^2/L.$$

Theorem 3 now follows at once. The conclusion of Theorem 2 follows, for any fixed constant k, if R_1' and R_2' are chosen so that

$$R_1' > 1 - e^{-k-t} L/c,$$

$$R_2' < 1 + e^{-k-t-1} L/c.$$

Since nothing hinders this choice, the proof is complete.

The preceding proof of Theorem 3 passes from inequality to inequality with utter disregard of economy, and it finally accomplishes its purpose by heaping most of its burden on the quantity $\limsup_{i \to \infty} \theta_i$. In other words, the quantity $k(t, C, \Gamma)$ is made unnecessarily large. Whereas a proof constructed to obtain the best possible value for $k(t, C, \Gamma)$ would be so loaded with parameters that all its salient features would be obliterated, it is of interest to mention some of the possible refinements:

The diameter c of the region Γ may be replaced by quantities c_j $(j = 1, 2, 3, 4)$

where c_1 and c_2 represent the maxima of $| z - z_2 |$ on Parts 1 and 2 of the boundary of Γ, respectively, and c_3^2 and c_4^2 represent the maxima of $| (z - z_1)(z - z_2) |$ on Parts 3 and 4, respectively.

The factors $r^{\theta i}$ and $r^{-\theta i}$ in the majorants for $| g_i(z) |$ on Parts 1 and 2 of the boundary of Γ need not be discarded; they can perform useful work, as in the proof of Theorem 1. This approach makes it unnecessary to choose R_1' and R_2' near to unity. Further freedom (of value when R_2 can be chosen quite large) is then obtained if the sequence $p_i = t + 1 + \log n_i$ is replaced by the sequence $p_i = t + 1 + s \log n_i$, where s is a convenient constant between zero and unity.

5. **Two general theorems.** In the application of the general method of this paper an essential step consists of estimating the quantity $| f(z) - s_{m_i}(z) |$ by means of the Taylor series $\sum_{n=n_i}^{\infty} | a_n | r^n$. The previous two sections deal with cases in which the sequence $| a_n |$ is dominated by the special sequence n^t; the present section deals with a more general dominating sequence.

If a sequence a_n has the property $| a_n | \leq e^{\varphi(n)}$ $(n = 0, 1, 2, \cdots)$, where the function $\varphi(x)$ is continuous and has a right-hand derivative $\varphi'(x)$ tending to zero monotonically as $x \to \infty$, it follows at once (since $\log | a_n |^{1/n} \leq \varphi(n)/n$) that $\limsup_{n\to\infty} | a_n |^{1/n} \leq 1$. If on the other hand the sequence a_n has the property $\limsup_{n\to\infty} | a_n |^{1/n} \leq 1$, it is graphically obvious that there exists a continuous function $\varphi(x)$ with a right-hand derivative tending monotonically to zero as $x \to \infty$ and such that $\log | a_n | \leq \varphi(n)$ $(n = 0, 1, 2, \cdots)$; in fact, in any case in which the inequality $| a_m | \leq \limsup_{n\to\infty} | a_n |$ holds for all non-negative integers m, there exists a least function $\varphi(x)$ with these properties. The concave sequences $\varphi(n)$ dominating the sequences $\log | a_n |$ will be used in estimating the quantities $| f(z) - s_{m_i}(z) |$.

THEOREM 4. *If*

(i) *the function* $f(z) = \sum a_n z^n$ *is regular on the arc* C *of the unit circle (including its end points)*;

(ii) $\varphi(x)$ *is a function such that* $\varphi'(x) \searrow 0$ *as* $x \to \infty$ *and such that* $| a_n | \leq e^{\varphi(n)}$ $(n = 0, 1, 2, \cdots)$;

(iii) N_i *and* θ_i *are sequences such that*

$$\liminf_{i\to\infty} \frac{\varphi(N_i) \log \theta_i}{\log N_i} > 1,$$

$$\lim_{i\to\infty} \theta_i = \infty,$$

$$a_n = 0 \text{ when } | n - N_i | \leq \theta_i \varphi(N_i);$$

then the sequence $s_{N_i}(z)$ *converges to* $f(z)$ *uniformly on the arc* C.

The proof follows the pattern that produced Theorem 1. The same region is used; but the consideration of the function

$$g_i(z) = [f(z) - s_{m_i}(z)][(z - z_1)(z - z_2)/L]^{\varphi(N_i)} z^{-N_i}$$

on the segment from the origin to the point z_1 is carried out separately for the two segments $0 \leq |z| \leq 1 - 2\varphi(N_i)/N_i$, $1 - 2\varphi(N_i)/N_i \leq |z| \leq 1$. The symbols F and c shall have the same meaning as in §2. The symbols m_i and n_i represent the quantities $N_i - \theta_i\varphi(N_i)$, $N_i + \theta_i\varphi(N_i)$.

Part 1. When $z = rz_1$ $(0 \leq r \leq 1 - 2\varphi(N_i)/N_i)$,

$$\begin{aligned}|f(z) - s_{m_i}(z)| &< \sum_{n=n_i}^{\infty} r^n e^{\varphi(n)} \\ &< r^{n_i}e^{\varphi(n_i)} \sum_{n=0}^{\infty} r^n e^{n\varphi'(n_i)} \\ &< r^{n_i}e^{\varphi(n_i)}/[1 - re^{\varphi'(N_i)}].\end{aligned}$$

Since $(1 - 2x)e^x < 1 - x$ $(x > 0)$ and $\varphi'(N_i) < \varphi(N_i)/N_i$ (the assumption that $\varphi(0) \geq 0$ entails no loss in generality), the following relations exist:

$$\begin{aligned}|f(z) - s_{m_i}(z)| &< r^{n_i}e^{\varphi(n_i)}[1 - (1 - 2\varphi(N_i)/N_i)e^{\varphi(N_i)/N_i}] \\ &< r^{n_i}e^{\varphi(n_i)}N_i/\varphi(N_i), \\ |g_i(z)| &< \frac{N_i}{\varphi(N_i)} r^{n_i - N_i}e^{\varphi(n_i)}[(1 - r)c/L]^{\varphi(N_i)} \\ &< N_i[r^{\theta_i}(1 - r)(1 + \epsilon)ec/L]^{\varphi(N_i)}\end{aligned}$$

(the assumption that $e^{\varphi(n_i)} = [(1 + \epsilon)e]^{\varphi(N_i)}$) is justified if $\limsup n_i/N_i < \infty$; if this relation does not hold, the gaps can be divided into two classes so that the relation holds for one of the classes and Ostrowski's theorem fills the breach for the other class).

Since

$$\max_{0<r<1} r^{\theta_i}(1 - r) = \left(\frac{\theta_i}{\theta_i + 1}\right)^{\theta_i + 1} \frac{1}{\theta_i} \sim 1/(e\theta_i),$$

$$|g_i(z)| < N_i(k\theta_i)^{-\varphi(N_i)} = \exp\{\log N_i - \varphi(N_i) \log k\theta_i\},$$

where k is an appropriate constant. The hypotheses on $\varphi(N_i) \log \theta_i/\log N_i$ and on θ_i imply that $g_i(z) \to 0$ uniformly on Part 1.

Part 2. When $z = rz_1$ $(1 - 2\varphi(N_i)/N_i \leq r \leq 1)$,

$$|f(z) - s_{m_i}(z)| < F \sum_{n=0}^{m_i - 1} e^{\varphi(n)} < N_i e^{\varphi(N_i)},$$

and (subscripts suppressed)

$$\begin{aligned}|g(z)| &< Ne^{\varphi(N)}r^{-N}[(1 - r)c/L]^{\varphi(N)} \\ &< Ne^{\varphi(N)}[1 - 2\varphi(N)/N]^{-N}[2\varphi(N)c/(NL)]^{\varphi(N)} \\ &= N\{[1 - 2\varphi(N)/N]^{-N/\varphi(N)}(2ec/L)\varphi(N)/N\}^{\varphi(N)} \\ &= N[(1 + \epsilon)k\varphi(N)/N]^{\varphi(N)},\end{aligned}$$

where $k = 2e^3c/L$ and $\epsilon \to 0$ as $i \to \infty$. With the notation $\varphi(N_i) = e^{s_i}$, $N_i = e^{s_i+t_i}$, the last member becomes

$$\exp \{s_i + t_i - e^{s_i}[t_i - \log (1 + \epsilon)k]\},$$

and since $\lim s_i = \lim t_i = \infty$, this tends to zero as $i \to \infty$.

Part 3. When $z = rz_1$ $(1 \le r \le R)$

$$| f(z) - s_{m_i}(z) | < F + \sum_{n=0}^{m_i} r^n e^{\varphi(n)} < N_i e^{\varphi(m_i)} r^{m_i},$$

$$| g_i(z) | < N_i e^{\varphi(m_i)} [r^{-\theta_i}(r - 1)c/L]^{\varphi(N_i)}$$

$$< N_i[r^{-\theta_i}(r - 1)k]^{\varphi(N_i)},$$

where $k = ec/L$. Since

$$\max_{1<r} r^{-\theta_i}(r - 1) = \left(\frac{\theta_i}{\theta_i - 1}\right)^{\theta_i+1} \frac{1}{\theta_i} \sim e/\theta_i ,$$

the conclusion is again obvious.

Part 4. When $| z | = R$,

$$| g_i(z) | < N_i[R^{-\theta_i} ec^2/L]^{\varphi(N_i)},$$

and the proof of Theorem 4 is complete.

It should be pointed out that the condition $\lim \theta_i = \infty$ in Theorem 4 is strictly analogous to the condition $\lim (n_i - m_i)/\log n_i = \infty$ in Theorem 2. In both theorems it is required that the length of the gap beginning at $n = m_i$ be large compared with $\varphi(m_i)$, where the sequence $\varphi(n)$ is a concave majorant of the sequence $\log | a_n |$. If in Theorem 1 the constant majorant A of the bounded sequence $| a_n |$ is taken greater than unity (so that its logarithm is positive), Theorem 1 immediately falls into one pattern with Theorems 2 and 4.

Now Theorem 4 is of interest only in those cases in which $\limsup \log | a_n |/\log n > 0$ (all other cases are covered better by Theorems 1 and 2). There remains the case of functions whose Taylor coefficients a_n are unbounded but are ultimately less than n^α in absolute value, where α is an arbitrarily small positive constant. This case is covered by Theorem 2, but in a way that requires the lengths of the gaps to be very much greater than the least possible concave majorant of the sequence $\log | a_n |$. Theorem 5 abolishes this state of discrimination; more generally, it improves the status of all Taylor series of finite order with unbounded coefficients; it includes Theorem 2.

THEOREM 5. *If*

(i) *the function* $f(z) = \sum a_n z^n$ *is regular on the arc* C *of the unit circle (end points included)*;

(ii) $\varphi(x)$ *is a function such that*

$$\varphi(x) \to \infty, \qquad \varphi'(x) \searrow 0 \qquad\qquad (x \to \infty),$$

$$\limsup_{x\to\infty} \varphi(x)/\log x < \infty,$$

$$| a_n | < e^{\varphi(n)} \qquad\qquad (n = 0, 1, \cdots);$$

(iii) N_i *and* θ_i *are sequences such that*

$$\lim \theta_i = \infty,$$

$$a_n = 0 \text{ when } | n - N_i | \leq \theta_i \varphi(N_i);$$

then the sequence $s_{N_i}(z)$ *converges to* $f(z)$ *uniformly on the arc* C; *moreover, if* ϵ *is any positive constant, there exists an integer* i_0 *such that* $| f(z) - s_{N_i}(z) | < \epsilon^{\varphi(N_i)}$ *when* $i > i_0$.

The proof is carried out with the same region Γ (and with the same notation) as the proof of Theorem 4. The exponent p_i has the value $\varphi(N_i) + q$, where q is any constant greater than $1 + \max \varphi(n)/\log n$ ($n = 2, 3, \cdots$). It is to be shown that the function

$$g_i(z) = [f(z) - s_{m_i}(z)][(z - z_1)(z - z_2)/L]^{\varphi(N_i)+q} z^{-N_i}$$

tends to zero uniformly (and with a certain rapidity) on the boundary of Γ, as $i \to \infty$.

When $z = rz_1$ $(0 \leq r \leq 1)$,

$$\begin{aligned} | f(z) - s_{m_i}(z) | &\leq \sum_{n=n_i}^{\infty} r^n e^{\varphi(n)} \\ &\leq \sum_{n=0}^{\infty} r^{n_i+n} e^{\varphi(n_i)+\varphi(n)} \\ &< r^{n_i} e^{\varphi(n_i)} (| a_0 | + | a_1 | + \sum_{n=2}^{\infty} r^n n^{q-1}) \\ &< C_1 r^{n_i} [(1 + \eta)e]^{\varphi(N_i)} (1 - r)^{-q}, \end{aligned}$$

$$| g_i(z) | < [C_2 r^{\theta_i}(1 - r)]^{\varphi(N_i)}.$$

Since the maximum value of $r^{\theta_i}(1 - r)$ on the interval $0 \leq r \leq 1$ tends to zero as θ_i becomes large, the required result follows for the segment from the origin to the point z_1.

When $z = rz_1$ $(1 \leq r \leq R)$,

$$| f(z) - s_{m_i}(z) | \leq F + \sum_{n=0}^{m_i-1} r^n e^{\varphi(n)}$$

$$< F + e^{\varphi(m_i)} \sum_{n=0}^{m_i-1} r^n$$

$$< 2e^{\varphi(m_i)} r^{m_i}/(r - 1),$$

$$| g_i(z) | < [C_3 r^{-\theta_i}(r - 1)]^{\varphi(N_i)},$$

and again the required result follows immediately.

Finally, on the arc $| z | = R$,

$$| g_i(z) | < [C_4 R^{-\theta_i}]^{\varphi(N_i)},$$

and the proof is complete.

6. **A theorem on double gaps.** *A Taylor series $\sum a_n z^n$ shall be said to have double gaps with partitions of thickness k if there exist infinite sequences m_i, n_i, n_i' with the property that*

$$\lim_{i \to \infty} (n_i - m_i) = \lim_{i \to \infty} (n_i' - n_i) = \infty,$$

$a_n = 0$ when $m_i < n \leq n_i$ or $n_i + k < n \leq n_i'$,

$a_n \neq 0$ for at least one of the values $n = n_i + j$ $\qquad (j = 1, 2, \cdots, k)$.

THEOREM 6. *If*

(i) *the Taylor series $\sum a_n z^n$ has double gaps with partitions of thickness k;*

(ii) *the coefficients a_n are bounded;*

(iii) *one of the k sequences a_{n_i+j} $(j = 1, 2, \cdots, k)$ fails to converge to zero as $i \to \infty$;*

then the function $f(z) = \sum a_n z^n$ cannot be extended analytically beyond the unit circle.

It follows from Theorem 1 that if the function $f(z)$ is regular on any arc C of the unit circle, the sequences $s_{n_i}(z)$ and $s_{n_i+k}(z)$ must converge to $f(z)$ uniformly on the arc C. Now

$$s_{n_i+k}(z) - s_{n_i}(z) = z_{n_i} \sum_{j=1}^{k} a_{n_i+j} z^j,$$

and

$$\left| \sum_{j=1}^{k} a_{n_i+j} z^j \right| = A_i \left| \sum_{j=1}^{k} (a_{n_i+j}/A_i) z^j \right|$$

where $A_i = \max | a_{n_i+j} |$ $(j = 1, 2, \cdots, k)$. Since the polynomial in the last member has at least one coefficient of modulus unity, its maximum modulus on the arc C is bounded away from zero; and since $\limsup A_i > 0$, the overconvergence guaranteed by Theorem 1 cannot take place. Therefore the unit circle has no arcs of regularity for the function $f(z)$, and the theorem is proved.

It is clear that Theorems 2, 4 and 5 induce theorems analogous to Theorem 6. Moreover, it is easily seen that condition (iii) in Theorem 6 can be weakened: it is not necessary that the sequence $a_{n_i} + j$ does not tend to zero, merely that, for some constant k less than unity and for infinitely many values of the integer i, the two inequalities

$$| a_{n_i} + j | > (n_i - m_i)^{-(n_i - m_i)^k},$$

$$| a_{n_i} + j | > (n'_i - n_i)^{-(n'_i - n_i)^k}$$

are satisfied.

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