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ON THE LOWER LIMIT OF SUMS OF INDEPENDENT RANDOM VARIABLES

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1. Let $X_1, X_2, \cdots, X_n \cdots$ be independent random variables and let $S_n = \sum_{\nu=1}^{n} X_\nu$. In the so-called law of the iterated logarithm, completely solved by Feller recently, the upper limit of S_n as $n \to \infty$ is considered and its true order of magnitude is found with probability one. A counterpart to that problem is to consider the lower limit of S_n as $n \to \infty$ and to make a statement about its order of magnitude with probability one.

THEOREM 1. *Let $X_1, \cdots, X_n, \cdots$ be independent random variables with the common distribution*: $\Pr(X_n = 1) = p, \Pr(X_n = 0) = 1 - p = q$. *Let* $\psi(n) \downarrow \infty$ and

$$\sum_{n=1}^{\infty} \frac{1}{n\psi(n)} = \infty. \tag{1.1}$$

Then we have

$$\Pr\left(\varliminf_{n\to\infty} n^{1/2}\psi(n) \mid S_n - np \mid = 0\right) = 1. \tag{1.2}$$

Theorem 1 is a best possible theorem. In fact we shall prove the following

THEOREM 2. *Let X_n be as in Theorem 1 but let p be a quadratic irrational. Let $\phi(n) \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} \frac{1}{n\phi(n)} < \infty. \tag{1.3}$$

Then we have

$$\Pr\left(\varliminf_{n\to\infty} n^{1/2}\phi(n) \mid S_n - np \mid = 0\right) = 0. \tag{1.4}$$

By making use of results on uniform distribution mod 1 we can prove (1.4) for almost all p, however the proof is omitted here.

In order to extend the theorem to more general sequences of random variables, we need a theorem about the limiting distribution of S_n with an estimate of the accuracy of approximation. Cramér's asymptotic expansion is suitable for this purpose. The conditions on $F(x)$ in the following Theorem 3 are those under which the desired expansion holds.

THEOREM 3. *Let $X_1, \cdots, X_n, \cdots$ be independent random variables having the same distribution function $F(x)$. Suppose that the absolutely continuous part of $F(x)$ does not vanish identically and that its first moment is zero, the second is one, and the absolute fifth is finite. Let $\psi(n)$ be as in Theorem 1, then*

$$\Pr\left(\varliminf_{n\to\infty} n^{1/2}\psi(n) \mid S_n \mid = 0\right) = 1. \tag{1.5}$$

On the other hand, let $\phi(n)$ be as in Theorem 2; then

$$\Pr\left(\varliminf_{n\to\infty} n^{1/2}\phi(n)\,|S_n| = 0\right) = 0. \tag{1.6}$$

It seems clear that the result can be extended to other cases, however we shall at present content ourselves with this statement.

2. For $x > 0$ let $I(x)$ denote the integer nearest to x if x is not equal to $[x] + \frac{1}{2}$; in the latter case, let $I(x) = [x]$; let $\{x\} = x - I(x)$. We have then for any $x > 0$, $y > 0$, the inequality

$$|\{x - y\}| \leqq |\{x\} - \{y\}|.$$

We are now going to state and prove some lemmas. The first two lemmas are number-theoretic in nature; the third one supplies the main probability argument; and the fourth one is a form of zero-or-one law.

LEMMA 1. *Let $p > 0$ be a real number. Let $\psi(n) \uparrow \infty$. Arrange all the positive integers n for which we have,*

$$|\{np\}| < cn^{-1/2}\psi(n)^{-1} \tag{2.1}$$

in an increasing sequence n_i, $i = 1, 2, \cdots$. Then for any pair of positive integers i and k we have

$$n_{i+2k} \geqq n_i + n_k .$$

PROOF. Suppose the contrary:

$$n_{i+2k} < n_i + n_k .$$

CASE (i): $k \leqq i$. Consider the $2k + 1$ numbers

$$n_i, n_{i+1}, \cdots, n_{i+2k}$$

and the corresponding

$$\{n_ip\}, \{n_{i+1}p\}, \cdots, \{n_{i+2k}p\}. \tag{2.2}$$

There are at least $k + 1$ numbers among (2.2) which are of the same sign; without loss of generality we may assume that they are non-negative. Let the corresponding n_i be

$$n_{i_1} < n_{i_2} < \cdots < n_{i_{k+1}}.$$

Then we have

$$0 \leqq \{n_{i_j}p\} < cn_{i_j}^{-1/2}\psi(n_j)^{-1} \leqq cn_k^{-1/2}\psi(n_k)^{-1}, \qquad j = 1, \cdots, k+1;$$

since $i_j \geqq i \geqq k$; and

$$|\{n_{i_{k+1}}p - n_{i_j}p\}| < cn_k^{-1/2}\psi(n_k)^{-1}, \qquad j = 1, \cdots, k;$$

$$0 < n_{i_{k+1}} - n_{i_j} \leqq n_{i+2k} - n_i < n_k .$$

Thus there would be k different positive integers $n_{i_{k+1}} - n_{i_j}$, $j = 1, \cdots, k$ all $< n_k$, for which

$$|\{np\}| < cn_k^{-1/2}\psi(n_k)^{-1}.$$

This is a contradiction to the definition of n_k.

CASE (ii) $k > i$. Consider the $i + k + 1$ numbers

$$n_k, n_{k+1}, \cdots, n_{i+2k}$$

and the corresponding

$$\{n_k p\}, \{n_{k+1} p\}, \cdots, \{n_{i+2k} p\}.$$

Since $i + k + 1 > 2i + 1$, there are at least $i + 1$ of the numbers above which are of the same sign, say non-negative. Let the corresponding n_i be

$$n_{k_1} < n_{k_2} < \cdots < n_{k_{i+1}}.$$

By an argument similar to that in Case (i) we should have i numbers $n_{k_{i+1}} - n_{k_j}$, $j = 1, \cdots, i$, all $< n_i$ for which

$$|\{np\}| < cn_i^{-1/2}\psi(n_i)^{-1}.$$

This leads to a contradiction as before.

LEMMA 2. *Let n_i be defined as in Lemma 1. Then if*

$$\sum_{n=1}^{\infty} \frac{1}{n\psi(n)} = \infty, \tag{2.3}$$

we have

$$\sum_{i=1}^{\infty} n_i^{-1/2} = \infty. \tag{2.4}$$

PROOF. Consider the points

$$hcn^{-1/2}\psi(n)^{-1} \qquad h = \pm 1, \cdots, \pm[2^{-1}c^{-1}n^{1/2}\psi(n)].$$

They divide the interval $(-\frac{1}{2}, \frac{1}{2})$ into at most $[c^{-1}n^{1/2}\psi(n)] + 2$ parts. Hence at least one subinterval contains

$$l \geqq \frac{n}{[c^{-1}n^{1/2}\psi(n)] + 2}$$

members of the n numbers $\{mp\}$, $m = 1, 2, \cdots, n$. Let the corresponding n_i be

$$n_1 < n_2 < \cdots < n_l.$$

Then

$$0 < |\{n_l p - n_i p\}| < cn^{-1/2}\psi(n)^{-1} < c(n_l - n_i)^{-1/2}\psi(n_l - n_i)^{-1},$$
$$i = 1, \cdots, l - 1.$$

Hence if $g(n)$ denote the number of numbers among $1, \cdots, n$ for which

$$|\{np\}| < cn^{-1/2}\psi(n)^{-1},$$

we have, for n sufficiently large

$$g(n) > 2^{-1}cn^{1/2}\psi(n)^{-1}.$$

Now

$$\sum_{2^{k-1}<n\leq 2^k} n_i^{-1/2} \geq \frac{g(2^k) - g(2^{k-1})}{\sqrt{2^k}}.$$

Hence

$$\sum_{k=1}^{\infty} \sum_{2^{k-1}<r_i\leq 2^k} n_i^{-1/2} \geqq \sum_{k=1}^{\infty} \frac{g(2^k) - g(2^{k-1})}{\sqrt{2^k}}$$

$$= -\frac{g(1)}{\sqrt{2}} + \sum_{k=1}^{\infty} g(2^k)\left(\frac{1}{\sqrt{2^k}} - \frac{1}{\sqrt{2^{k+1}}}\right)$$

$$\geqq -\frac{g(1)}{\sqrt{2}} + \sum_{k=1}^{\infty} \frac{c}{2}\frac{\sqrt{2^k}}{\psi(2^k)}\left(\frac{1}{\sqrt{2^k}} - \frac{1}{\sqrt{2^{k+1}}}\right)$$

$$\geqq -1 + \frac{c}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\sum_{k=1}^{\infty} \frac{1}{\psi(2^k)}.$$

It is well-known[1] that if (2.3) holds then

$$\sum_{k=1}^{\infty} \frac{1}{\psi(2^k)} = \infty.$$

Thus (2.4) is proved.

LEMMA 3. *Let n_i, $i = 1, 2, \cdots$ be a monotone increasing sequence such that for any pair of positive integers i and k we have*

(2.5) $$n_{i+2k} \geqq n_i + n_k$$

and

(2.6) $$\sum_{i=1}^{\infty} n_i^{-1/2} = \infty.$$

Then if α and β are two integers, we have for any integer $h > 0$,

(2.7) $$\Pr(S_{n_i} = I(pn_i + p\alpha) + \beta \text{ at least once for } i \geqq h) \geqq \tfrac{1}{2}.$$

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[1] See e. g. Theory and Application of Infinite Series, London-Glasgow, Blackie and Son, 1928, p. 120.

PROOF. Denoting the joint probability of $E_1, E_2, \cdots$ by $\Pr(E_1; E_2; \cdots)$ we have

$$\Pr(S_{n_i} = I(pn_i + p\alpha) + \beta) = \Pr(S_{n_h} = I(pn_h + p\alpha) + \beta; \; S_{n_i - n_h} = I(pn_i + p\alpha) - I(pn_h + p\alpha))$$
$$+ \Pr(S_{n_h} \neq I(pn_h + p\alpha) + \beta; S_{n_{h+1}} = I(pn_{h+1} + p\alpha) + \beta; \; S_{n_i - n_{h+1}} = I(pn_i + p\alpha) - I(pn_{h+1} + p\alpha))$$
$$+ \cdots$$
$$+ \Pr(S_{n_h} \neq I(pn_h + p\alpha) + \beta; \cdots ; S_{n_{i-1}} \neq I(pn_{i-1} + p\alpha) + \beta; \; S_{n_i} = I(pn_i + p\alpha) + \beta).$$

Writing

$$p_i = \Pr(S_{n_i} = I(pn_i + p\alpha) + \beta),$$
$$w_k = \Pr(S_{n_j} \neq I(pn_j + p\alpha) + \beta \quad \text{for} \quad h \leqq j < k; S_{n_k} = I(pn_k + p\alpha) + \beta),$$
$$p_{k,i} = \Pr(S_{n_i - n_k} = I(pn_i + p\alpha) - I(pn_k + p\alpha)), \qquad p_{k,k} = 1;$$

and using the assumption of independence, we have

$$p_i = \sum_{k=h}^{i} w_k p_{k,i} .$$

Summing from h to m we get

$$\sum_{i=h}^{m} p_i = \sum_{i=h}^{m} \sum_{k=1}^{i} w_k p_{k,i} \leqq \sum_{k=1}^{m} w_k \sum_{i=k}^{m} p_{k,i} . \tag{2.8}$$

Now for any positive x and y, $I(x) - I(y) = I(x - y)$ or $I(x - y) \pm 1$; and it is well-known that for the random variables we have, given any $\epsilon > 0$, if $n > n_0(\epsilon)$, and $\theta = \pm 1$,

$$\Pr(S_n = I(np) + \theta) \leqq (1 + \epsilon)\Pr(S_n = I(np))$$

hence we have, if $i - k \geqq m_1(\epsilon)$,

$$p_{k,i} \leqq (1 + \epsilon/4)\Pr(S_{n_i - n_k} = I(pn_i - pn_k)). \tag{2.9}$$

From (2.5) if $i > k$, we have

$$n_i \geqq n_k + n_{[(i-k)/2]} . \tag{2.10}$$

Also it is well-known that as $i \to \infty$,

$$p_i \sim \frac{1}{\sqrt{2\pi pq n_i}} . \tag{2.11}$$

Hence from (2.9), (2.10) and (2.11) we have if $i - k \geqq m_2(\epsilon)$ where m_2 is a positive constant,

$$p_{k,i} \leqq (1 + \epsilon/2)\Pr(S_{n_{[(i-k)/2]}} = I(pn_{[(i-k)/2]})).$$

Since α and β are fixed, to any $\epsilon > 0$ there exists an integer $m_0 = m_0(\epsilon) > m_2$ such that if $i - k \geqq m_0(\epsilon)$,

$$\Pr(S_{n_{[(i-k)/2]}} = I(pn_{[(i-k)/2]})) \leqq (1 + \epsilon)p_{[(i-k)/2]}. \tag{2.12}$$

Thus for $i - k \geqq m_0(\epsilon)$,

$$p_{k,i} \leqq (1 + \epsilon)p_{[(i-k)/2]}.$$

Using (2.12) in (2.9), we obtain

$$\begin{aligned} \sum_{i=h}^{m} p_i &\leqq \sum_{k=1}^{m} w_j \left(\sum_{i=k}^{k+m_0-1} p_{j,i} + (1 + \epsilon) \sum_{i=k+m_0}^{m} p_{[(i-k)/2]} \right) \\ &\leqq \sum_{k=1}^{m} w_j \left(m_0 + 2(1 + \epsilon) \sum_{i=m_0}^{[m/2]} p_i \right). \end{aligned}$$

Therefore

$$\sum_{j=1}^{m} w_j \geqq \frac{\sum_{i=h}^{m} p_i}{m_0 + 2(1 + \epsilon) \sum_{i=m_0}^{[m/2]} p_i}.$$

Since by (2.11) and (2.6) the series $\sum_{i=1}^{\infty} p_i$ is divergent, we get, letting $n \to \infty$,

$$\sum_{j=1}^{\infty} w_j \geqq \frac{1}{2(1 + \epsilon)}.$$

Since ϵ is arbitrary and the left-hand side does not depend on ϵ this proves (2.7).

Lemma 4. *If for any integers α, β and $k > 0$, there exists a number $\eta > 0$ not depending on α, β and an integer $l = l(k, \eta)$ such that, n_i being any sequence $\uparrow \infty$,*

$$\Pr(S_{n_i} = I(pn_i + p\alpha) + \beta \textit{ at least once for } k \leqq i \leqq l) \geqq \eta; \tag{2.13}$$

then

$$\Pr(S_{n_i} = I(pn_i + p\alpha) + \beta \textit{ infinitely often}) = 1. \tag{2.14}$$

Proof. Take a sequence $k_1, k_2, \cdots$ and the corresponding $l_1, l_2, \cdots$ such that

$$k_1 < l_1 < k_2 < l_2 < \cdots.$$

Consider the event

E_r: $\quad S_{n_i} = I(pn_i + p\alpha) + \beta$ at least once for $k_r \leqq i \leqq l_r$,

and let the probability that E_r occurs under the hypothesis that none of $E_1, \cdots, E_{r-1}$ occurs, be denoted by $\Pr(E_r \mid E_1' \cdots E_{r-1}')$. Then the latter is a probability mean of the conditional probabilities of E_r under the various hypotheses:

H: $\quad S_{n_i} = \sigma_{n_i}, \quad k_t \leqq i \leqq l_t, \quad 1 \leqq t \leqq r - 1;$

where the σ_{n_i}'s are such that for all i, $\sigma_{n_i} \neq I(pn_i + p\alpha) + \beta$ but are otherwise arbitrary. Now under H, E_r will occur if the following event F occurs:

F: $S_{n_i - n_{l_{r-1}}} = I(pn_i + p\alpha) + \beta - \sigma_{n_{l_{r-1}}}$ at least once for $k_r \leq i \leq l_r$.

Hence

$$\Pr(E_r \mid E_1' \cdots E_{r-1}') \geq \min_H \Pr(E_r \mid H) \geq \Pr(F \mid H) = \Pr(F).$$

Writing the equality in F as

$$\begin{aligned} S_{n_i - n_{l_{r-1}}} &= I(p(n_i - n_{l_{r-1}}) + p(n_{l_{r-1}} + \alpha)) + \beta - \sigma_{n_{l_{r-1}}} \\ &= I(p(n_i - n_{l_{r-1}}) + p\alpha') + \beta' \end{aligned}$$

and consider the random variables $X_{n_{l_{r+1}}}, X_{n_{l_{r+1}}+1}, \cdots$ as $X_1', X_2', \cdots$ we see from (2.13) that

$$\Pr(E_r \mid E_1' \cdots E_{r-1}') \geq \Pr(F) \geq \eta.$$

Therefore the probability that none of the events E_r, $r = 1, \cdots, s$ occurs is $\Pr(E_1' \cdots E_s') = \Pr(E_1')\Pr(E_2' \mid E_1') \cdots \Pr(E_s' \mid E_1' \cdots E_{s-1}') \leq (1 - \eta)^s$. Hence

$$\Pr(S_{n_i} \neq I(pn_i + p\alpha) + \beta \text{ for all } l_r \leq i \leq k_r, r = 1, 2, \cdots) = 0$$

Since l_1 can be taken arbitrarily large, (2.14) is proved.

REMARK. Lemma 3 and 4 imply an interesting improvement of the well-known fact that $\Pr(S_n - np = \text{infinitely often}) = 1$ for a rational p. Let n_i be any monotone increasing sequence such that (2.6) holds; in addition if for a certain integer $m > 0$ and any pair of integers i and k we have

$$n_{i+mk} \geq n_i + n_k \tag{2.15}$$

then

$$\Pr(S_{n_i} - n_i p = 0 \text{ for infinitely many } i) = 1.$$

That the condition (2.6) alone is not sufficient can be shown by a counter-example. On the other hand, it is trivial that (2.6) is a necessary condition. The condition (2.15) can be replaced e.g. by the following condition:

$$n_{i+1} - n_i \geq An_i^{1/2}, \qquad A > 0.$$

The proof is different and will be omitted here.

PROOF OF THEOREM 1. Let the sequence n_i be defined as in Lemma 1. Then by Lemma 1 and 2 this sequence satisfies the conditions (2.5) and (2.6) in Lemma 3. Hence by Lemma 3 the condition (2.13) in Lemma 4 is satisfied with any $\eta < \frac{1}{2}$. Thus byLemma 4 we have (2.14). Taking $\alpha = \beta = 0$ therein we obtain

$$\Pr(S_{n_i} - n_i p = \{n_i p\} \text{ infinitely often}) = 1.$$

Hence by the definition (2.2)

$$\Pr(|S_n - np| < cn^{-1/2}\psi(n)^{-1} \text{ infinitely often}) = 1.$$

Since c is arbitrarily small (1.2) is proved.

REMARK. It is clear that (2.14) yields more than Theorem 1 since α and β are arbitrary. It is easily seen that we may even make α and β vary with n_i in a certain way, but we shall omit these considerations here.

PROOF OF THEOREM 2. Arrange all the positive integers n for which we have

$$|\{np\}| \leqq An^{-1/2}\phi(n)^{-1}, \qquad A > 0.$$

in an ascending sequence n_i, $i = 1, 2, \cdots$. Since

$$|\{n_i p\}| \leqq An_i^{-1/2}\phi(n_i)^{-1}$$

we have

$$|\{n_{i+1}p - n_i p\}| \leqq 2An_i^{-1/2}\phi(n_i)^{-1}. \tag{2.16}$$

On the other hand, since p is a quadratic irrational, it is well-known[2] that there exists a number $M > 0$ such that

$$|\{n_{i+1}p - n_i p\}| > \frac{M}{n_{i+1} - n_i}. \tag{2.17}$$

From (2.16) and (2.17) we get with $A_1 = M/2A$,

$$n_{i+1} - n_i > A_1 n_i^{1/2}\phi(n_i) \tag{2.18}$$

Without loss of generality we may assume that $\phi(n_i) \leqq n_i^{1/2}$. For we may replace $\phi(n)$ by $\phi_1(n)$ defined as follows:

$$\phi_1(n) = \begin{cases} \phi(n) & \text{if } \phi(n) \leqq n^{1/2}; \\ n^{1/2} & \text{if } \phi(n) > n^{1/2}. \end{cases}$$

After this replacement (1.3) remains convergent, while if (1.4) holds for $\phi_1(n)$, it holds *a fortiori* for $\phi(n)$.

Now if $\phi(n_i) \leqq n_i^{1/2}$, and the constant A_2 is such that $2A_2 + A_2^2 < A_1$, we have from (2.18)

$$n_{i+1}^{1/2} > n_i^{1/2} + A_2\phi(n_i).$$

Hence by iterating,

$$n_{i+1}^{1/2} > A_2 \sum_{k=1}^{i} \phi(n_k) > A_2 \sum_{k=[i/2]}^{i} \phi(n_k) > A_2 \frac{i}{2}\, \phi\left(\left[\frac{i}{2}\right]\right).$$

Therefore by (1.3)

$$\sum_{i=1}^{\infty} n_i^{-1/2} < \infty. \tag{2.19}$$

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[2] See e. g. HARDY AND WRIGHT, Introduction to the Theory of Numbers, Oxford 1938, p. 157.

Define

$$p_i = \Pr(S_{n_i} = I(pn_i)).$$

As in (2.11) we have

$$p_i \sim \frac{1}{\sqrt{2\pi pqn_i}}.$$

Hence from (2.18)

$$\sum_{i=1}^{\infty} p_i < \infty.$$

By the classical Borel-Cantelli lemma it follows that

$$\Pr(S_{n_i} = I(pn_i) \text{ infinitely often}) = 0.$$

By the definition of n_i this is equivalent to (1.4).

3. LEMMA 5. *Let $X_1, \cdots, X_n, \cdots$ be independent random variables having the same distribution function $F(x)$ which satisfies the conditions in Theorem 3. Then if $x_1 < x_2$ and $x_1 = o(1)$, $x_2 = o(1)$ as $n \to \infty$, we have*

$$\Pr(x_1 \leqq n^{-1/2} S_n \leqq x_2) = (2\pi)^{-1/2}(x_2 - x_1) + o(x_2 - x_1) + 0(n^{-3/2}) \tag{3.1}$$

PROOF. By Cramér's asymptotic expansion[3] we have, if we denote the r^th^ moment of $F(x)$ by α_r,

$$\Pr\left(\frac{S_n}{\sqrt{n}} \leqq x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy - \frac{\alpha_3}{6\sqrt{2\pi}\sqrt{n}} (x^2 - 1)\, e^{-x^2/2}$$

$$+ \frac{\alpha_4 - 3\alpha_2^2}{24\sqrt{2\pi}\, n}(-x^3 + 3x)e^{-x^2/2} + \frac{\alpha_3^2}{72\sqrt{2\pi}\, n}(-x^5 + 10x^3 - 15x)e^{-x^2/2} + R(x)$$

where

$$| R(x) | \leqq Qn^{-3/2},$$

and Q is a constant depending only on $F(x)$,

It follows, using elementary estimates, that

$$\Pr\left(x_1 \leqq \frac{S_n}{\sqrt{n}} \leqq x_2\right) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{-y^2/2}\, dy$$

$$+ 0\left((x_2 - x_1)\left(\frac{|x_1| + |x_2|}{\sqrt{n}} + \frac{1}{n}\right)\right) + 0\left(\frac{1}{\sqrt{n^3}}\right)$$

Since $x_1 = o(1)$, $x_2 = o(1)$ this reduces immediately to (3.1).

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[3] CRAMÉR, Random Variables and Probability Distributions, Cambridge 1937, Ch. 7. For a simplified proof see P. L. HSU, *The Approximate Distribution of the Mean and Variance of a Sample of Independent Variables*, Ann. Math. Statistics, 16 (1945), pp. 1–29.

LEMMA 6. *Let z_n be any real number such that $z_n = 0(n^{1/2})$, c any positive number, and h any positive integer. Let $\psi(n) \uparrow \infty$ and*

$$\sum_{n=1}^{\infty} \frac{1}{n\psi(n)} = \infty. \tag{3.2}$$

Then if the random variables X_n satisfy the conditions of Theorem 3, we have

$$\Pr(|S_n - z_n| \leqq cn^{-1/2}\psi(n)^{-1} \text{ at least once for } n \geqq h) = 1. \tag{3.3}$$

PROOF. Write

$$P_n = \Pr(|S_n - z_n| \leqq cn^{-1/2}\psi(n)^{-1});$$

$$W_k = \Pr(|S_j - z_j| > cj^{-1/2}\psi(j)^{-1} \text{ for } h \leqq j < k; \quad |S_k - z_k| \leqq ck^{-1/2}\psi(k)^{-1})$$

$$P_{k,n} = \Pr(|S_n - z_n| \leqq cn^{-1/2}\psi(n)^{-1} \mid |S_j - z_j| > cj^{-1/2}\psi(j)^{-1} \text{ for } h \leqq j < k; \; |S_k - z_k| \leqq ck^{-1/2}\psi(k)^{-1}).$$

Then by a similar argument as in Lemma 3, we have

$$\sum_{n=h}^{m} P_n \leqq \sum_{k=h}^{m} W_k \sum_{n=k}^{m} P_{k,n}. \tag{3.4}$$

Our next step is to show that to any $\epsilon > 0$ there exists a constant $A(\epsilon)$ such that for $n - k > A$, we have

$$P_{k,n} \leqq (1 + \epsilon)P_{n-k}. \tag{3.5}$$

To prove this we divide the x-interval $|x - z_k| \leqq ck^{-1/2}\psi(k)^{-1}$ into disjoint subintervals I_j; of lengths $\leqq \epsilon' cn^{-1/2}\psi(n)^{-1}$ where $\epsilon' > 0$ is arbitrary. If we write

$$P_{k,n}^{(j)} = \Pr(|S_n - z_n| \leqq cn^{-1/2}\psi(n)^{-1} \mid S_k - z_k \subset I_j)$$

we have

$$P_{k,n}^{(j)} \leqq \Pr(S_n - S_k \subset I_j')$$

where I_j' is an interval of lengths $\leqq (2 + \epsilon')cn^{-1/2}\psi(n)^{-1} \leqq (2 + \epsilon')c(n - k)^{-1/2}$ $\psi(n - k)^{-1}$ lying within the interval $|x - z_n + z_k| \leqq cn^{-1/2}\psi(n)^{-1} + ck^{-1/2}\psi(k)^{-1}$. From Lemma 5 it is seen that if $n - k \geqq A_1(\epsilon')$,

$$P_{k,n}^{(j)} \leqq \frac{2(1 + \epsilon')c}{\sqrt{2\pi}\,(n - k)\psi(n - k)};$$

since $P_{k,n}$ is a probability mean of $P_{k,n}^{(j)}$, we have

$$P_{k,n} \leqq \max_j P_{k,n}^{(j)} \leqq \frac{2(1 + \epsilon')c}{\sqrt{2\pi}\,(n - k)\psi(n - k)}. \tag{3.6}$$

On the other hand, we have again from Lemma 5, if $n - k \geqq A_2(\epsilon')$,

$$P_{n-k} \geqq \frac{2(1 - \epsilon')}{\sqrt{2\pi}\,(n - k)\psi(n - k)}. \tag{3.7}$$

From (3.6) and (3.7) follows (3.5).

Using (3.5) in (3.4) we get

$$\sum_{n=h}^{m} P_n \leqq \sum_{k=h}^{m} W_k \left(\sum_{n=k}^{k+A-1} P_{k,n} + (1+\epsilon) \sum_{n=k+A}^{m} P_{n-k} \right)$$

$$\leqq \sum_{k=h}^{m} W_k \left(A + (1+\epsilon) \sum_{n=A}^{m} P_n\right)$$

$$\sum_{k=h}^{m} W_k \geqq \frac{\sum_{n=h}^{m} P_n}{A + (1+\epsilon) \sum_{n=A}^{m} P_n} \tag{3.8}$$

Now $\sum_{n=h}^{\infty} P_n = \infty$ by (3.7) and (3.1). It follows from (3.8) by letting $n \to \infty$ that

$$\sum_{k=h}^{\infty} W_k \geqq \frac{1}{1+\epsilon}.$$

Since ϵ is arbitrary and the left-hand side does not depend on ϵ we have

$$\sum_{k=h}^{\infty} W_k \geqq 1. \tag{3.9}$$

Thus (3.3) follows.

PROOF OF THEOREM 3. Taking $z = 0$ in (3.9) and denoting by E_k the event

$$| S_n | \leqq cn^{-1/2} \psi(n)^{-1},$$

we can write (3.9) as follows:

$$\Pr \left(\sum_{n=h}^{\infty} E_n \right) = 1,$$

where the sign $\sum$ denotes disjunction of events. Now the event which consists in the realization of an infinite number of the E_n's can be written as

$$\prod_{h=1}^{\infty} \left(\sum_{n=h}^{\infty} E_n \right)$$

where the sign $\prod$ denotes conjunction of events. Hence

$$\Pr \left(\prod_{h=1}^{\infty} \left(\sum_{n=h}^{\infty} E_n \right) \right) = \lim_{h \to \infty} \left(\sum_{n=h}^{\infty} E_n \right) = 1.$$

Thus (1.5) is proved. The proof of (1.6) follows immediately from Lemma 5 and Borel-Cantelli lemma.

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