SEQUENCES OF PLUS AND MINUS

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SUPPOSE n one's and an equal number of minus one's are arranged in a series. In all there are ${}_{2n}C_n$ possible arrangements. For example, when $n = 2$ the following 6 $(= {}_4C_2)$ arrangements are possible:

$$\begin{array}{ll} 1+1-1-1 & \qquad -1+1+1-1 \\ 1-1+1-1 & \qquad -1+1-1+1 \\ 1-1-1+1 & \qquad -1-1+1+1 \end{array}$$

The sum of any of these series is, of course, 0. A partial sum, formed by breaking off a series at a point, can be either positive or negative; in any case it lies between n and $-n$. In connection with an investigation being made by one of the authors, the following question arose: in how many of the arrangements are all the partial sums non-negative?

Of the 6 arrangements above, two (the first two) are acceptable. Similarly, of the 20 arrangements for $n = 3$, the following 5 are acceptable:

$$\begin{array}{l} 1+1+1-1-1-1 \\ 1+1-1+1-1-1 \\ 1+1-1-1+1-1 \\ 1-1+1+1-1-1 \\ 1-1+1-1+1-1 \end{array}$$

and of the 70 arrangements for $n = 4$, one can verify that there are 14 good ones. It is now easy to guess the right formula: in general ${}_{2n}C_n/(n+1)$ of the ${}_{2n}C_n$ arrangements fulfill the condition.

It is a curious fact that, in order to prove this conjecture, it seems to be wise to generalize as follows: let there be m one's and n minus one's and let it be required that all partial sums are at least $m - n$. Let us denote by $f(m, n)$ the number of arrangements that fulfill this condition. If $m > n + 1$, it is evident that already the first partial sum cannot fulfill the condition, for it cannot be greater than 1. Thus

$$f(m, n) = 0 \qquad (m > n + 1). \tag{1}$$

If $m = n$ or $n + 1$, we shall have to begin the series with 1. Then we are left with $m-1$ one's and n minus one's, and the partial sums are now to be greater than $m-n-1$. Hence

$$f(m, n) = f(m-1, n) \qquad (m = n \text{ or } n+1). \tag{2}$$

Finally if $m < n$, we are entitled to begin with either 1 or -1 and we find similarly

$$f(m, n) = f(m-1, n) + f(m, n-1) \qquad (m < n). \tag{3}$$

One can now easily verify by induction that the solution of equations (1), (2), (3), with the boundary conditions $f(1, 0) = f(0, n) = 1$, is given by (1), (4), and (5):

$$f(m, n) = \frac{n-m+1}{n+1}\, {}_{m+n}C_m \qquad (m < n) \tag{4}$$

$$f(n+1, n) = {}_{2n}C_n/(n+1). \tag{5}$$

By taking $m = n$ in (4), we obtain in particular the result earlier conjectured.

The problem can be given in a chess-board setting. Take a one-dimensional board stretching to infinity to the right and bounded to the left, and place a king at the left-hand end. Then $f(n, n)$ is the number of ways for the king to make $2n$ moves which return it to its starting point. The corresponding problem for a two-dimensional board seems to be quite difficult if we permit the king its diagonal moves; however, if we restrict the king to horizontal and vertical moves the answer is just $[f(n, n)]^2$.

A problem that further suggests itself is to place the king in the middle of the board and ask for the number of ways for it to take a trip to another designated square. In the one-dimensional case this is conveniently formulated as follows: in how many ways can m one's and n minus one's be arranged so that all partial sums are at least $m-n-a$? If we let the desired number be $g(m, n, a)$ then $g(m, n, 0) = f(m, n)$, and for $a < 0$ we have $g(m, n, a) = 0$ since the final sum cannot be greater than $m-n$. We can get a recurrence formula by splitting the acceptable arrangements into two subsets: those which finish with 1 and those which finish with -1. In the former case we are left with $m-1$ one's and n minus one's to be arranged with partial sums of at least $m-n-a$, and there are $g(m-1, n, a-1)$ such arrangements. Similarly there are $g(m, n-1, a+1)$ in the latter group so that we have

$$g(m, n, a) = g(m-1, n, a-1) + g(m, n-1, a+1).$$

Setting $a = 0, 1, 2$ in succession we find

$$g(m, n, 1) = f(m, n + 1)$$
$$g(m, n, 2) = f(m, n + 2) - f(m - 1, n + 1)$$
$$g(m, n, 3) = f(m, n + 3) - 2f(m - 1, n + 2)$$

and the general formula is

$$g(m, n, a) = \sum_{i=0}^{[a/2]} (-1)^i \, {}_{a-i}C_i f(m - i, n + a - i)$$

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CURIOSA

118. Factorials and Sub-factorials. The number of permutations P_n of n distinct objects n at a time is $P_n = n! = 1 \cdot 2 \cdot 3 \ldots n$. Obviously, $P_{n+1} = (n + 1)P_n$. The number of permutations of n objects so that none of them occupies its original position is $P_n' = n![1 - 1/1! + 1/2! - 1/3! + 1/4! - \ldots + (-1)^n 1/n!]$, sometimes called *sub-factorial* of n. The first eight successive values of P_n' are: $P_0' = 1$, $P_1' = 0$, $P_2' = 1$, $P_3' = 2$, $P_4' = 9$, $P_5' = 44$, $P_6' = 265$, $P_7' = 1854$, $P_8' = 14{,}833$. The recurrences $p_{n+1}' = n(P_n' + P_{n-1}')$, and $P_{n+1}' = (n + 1)P_n' + (-1)^{n+1}$ are known in the literature.

The following relationship seems to be new:

$$P_n = (1 + P')^n = 1 + C_1P' + C_2P'^2 + C_3P'^3 + \ldots + C_nP'^n,$$

where $C_1, C_2 \ldots$ are the corresponding binomial coefficients and P'^k stands for P_k'.

For example, $4! = 1 + 4 \cdot 0 + 6 \cdot 1 + 4 \cdot 2 + 9$; $5! = 1 + 5 \cdot 0 + 10 \cdot 1 + 10 \cdot 2 + 5 \cdot 9 + 44 = 120$, etc.

S. GUTTMAN

119. The Dual of the above Formula. The formula

$$P_n = (P' + 1)^n$$

remains valid when P and P' are interchanged and the plus is changed to a minus, yielding

$$P_n' = (P - 1)^n.$$

For example, $P_4' = 9 = 4! - 4.3! + 6.2! - 4.1! + 1$,
$P_5' = 44 = 5! - 5.4! + 10.3! - 10.2! + 5.1! - 1$.

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