ON A PROBLEM OF SIDON IN ADDITIVE NUMBER THEORY AND ON SOME RELATED PROBLEMS

ADDENDUM

P. ERDÖS.

In a note in this *Journal* [16 (1941), 212–215], Turan and I proved, among other results, the following: Let $a_1 < a_2 < \dots < a_x \leqslant n$ be a sequence of positive integers such that the sums $a_i + a_j$ are all different. Then $x < n^{\frac{1}{2}} + O(n^{\frac{1}{4}})$. On the other hand, there exist such sequences with $x > n^{\frac{1}{2}}(2^{-\frac{1}{2}} - \epsilon)$, for any $\epsilon > 0$.

Recently I noticed that J. Singer, in his paper "A theorem in finite projective geometry and some applications to number theory" [*Trans. Amer. Math. Soc.*, 43 (1938), 377–385], proves, among other results, that, if m is a power of a prime, then there exist $m+1$ numbers $a_1 < a_2 < \dots < a_{m+1} < m^2 + m + 1$ such that the differences $a_i - a_j$ are congruent, mod (m^2+m+1), to the integers $1, 2, \dots, m^2+m$. Clearly the sums $a_i + a_j$ are all different, and since the quotient of two successive primes tends to 1, Singer's construction gives, for any large n, a set with $x > n^{\frac{1}{2}}(1-\epsilon)$, for any $\epsilon > 0$. Singer's method is quite different from ours. His result shows that the above upper bound for x is best possible, except perhaps for the error term $O(n^{\frac{1}{4}})$.

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NOTE ON H_2 SUMMABILITY OF FOURIER SERIES

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1. Let $s_n(t)$ denote a partial sum of the Fourier series of an integrable function $f(t)$, periodic with period 2π, and let $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2s\}$. Recently I proved the following result of Hardy and Littlewood†:

If $\int_0^t |\phi(u)| \{1 + \log^+ |\phi(u)|\}\, du = o(t)$ *as* $t \to 0$, *then the Fourier series of* $f(t)$ *is summable* H_2 *to sum* s *for* $t = x$, *i.e.*

$$\sum_{\nu=0}^{n} |s_\nu(x) - s|^2 = o(n).$$

* Received 10 February, 1945; read 1 March, 1945.

† Hardy and Littlewood (1), *Fund. Math.*, 25 (1935), 162–189; Wang (2), *Duke Math. Journal* (in the press).

ON A PROBLEM OF SIDON IN ADDITIVE NUMBER THEORY, AND ON SOME RELATED PROBLEMS

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Let $a_1 < a_2 < \ldots$ be a sequence of positive integers, and suppose that the sums $a_i + a_j$ (where $i \leqslant j$) are all different. Such sequences, called B_2 sequences by Sidon†, occur in the theory of Fourier series. Suppose that n is given, and that $a_x \leqslant n < a_{x+1}$; the question was raised by Sidon how large x can be; that is, how many terms not exceeding n a B_2 sequence can have. Put $x = \phi(n)$, and denote by $\Phi(n)$ the maximum of $\phi(n)$ for given n. Sidon observed that $\Phi(n) > cn^{\frac{1}{4}}$, where c is a positive constant. In the present note we prove that

$$\Phi(n) > \left(\frac{1}{\sqrt{2}} - \epsilon\right) \sqrt{n}$$

for any positive ϵ and all $n > n_0(\epsilon)$. In the opposite direction, it is clear that $\Phi(n) < \sqrt{(2n)} + 1$ [for the numbers $a_i - a_j$, where $1 \leqslant j < i \leqslant x$, must all be different, whence $\frac{1}{2}x(x-1) \leqslant n-1$]. We prove that

$$\Phi(n) < (1+\epsilon) \sqrt{n}$$

for any positive ϵ and all $n > n_0(\epsilon)$. Thus

$$\frac{1}{\sqrt{2}} \leqslant \varliminf \frac{\Phi(n)}{\sqrt{n}} \leqslant \varlimsup \frac{\Phi(n)}{\sqrt{n}} \leqslant 1.$$

It is very likely that $\lim \Phi(n)/\sqrt{n}$ exists, but this we have not been able to prove.

We also prove the following result: let $f(n)$ denote the number of representations of n as $a_i + a_j$, where the a's are an arbitrary sequence of positive integers; then it is impossible that $f(n)$ should be constant for all $n \geqslant n_0$.

I. Let p be a prime, and let

$$a_k = 2pk + (k^2) \quad \text{for} \quad k = 1, 2, \ldots, (p-1),$$

* Received 17 July, 1941; read 11 December, 1941.

† S. Sidon, *Math. Annalen*, 106 (1932), 539.

where (k^2) denotes the unique integer u satisfying $k^2 \equiv u \pmod{p}$, $1 \leqslant u \leqslant p-1$. Clearly the a's are all less than $2p^2$. We show that

$$a_i + a_j \neq a_k + a_l \tag{1}$$

if the pairs (i, j) and (k, l) are different. If (1) does not hold, we clearly have

$$i+j = k+l, \quad i^2+j^2 \equiv k^2+l^2 \pmod{p}, \tag{2}$$

and hence $i-k = l-j$, $i^2-k^2 \equiv l^2-j^2 \pmod{p}$. Thus either $i-k = l-j = 0$, or $i+k \equiv l+j \pmod{p}$. In the latter case, it follows from (2) that $i \equiv l \pmod{p}$ and $k \equiv j \pmod{p}$, whence $i = l$ and $k = j$, and the pairs (i, j) and (k, l) are not different.

Since the a's satisfy (1), we have $\Phi(2p^2) \geqslant p-1$; and, since the quotient of consecutive primes tends to 1, it follows that

$$\varliminf \frac{\Phi(n)}{\sqrt{n}} \geqslant \frac{1}{\sqrt{2}}.$$

II. Let $a_1 < a_2 < \ldots < a_x \leqslant n$ be positive integers such that the sums $a_i + a_j$ $(i \leqslant j)$ are all different. Let m be a positive integer less than n, and consider the intervals

$$(-m+1, 1), \quad (-m+2, 2), \quad \ldots, \quad (n, n+m).$$

Let A_u denote the number of a's in the interval $-m+u \leqslant a_i < u$. Since each a_i occurs in exactly m intervals, we have

$$\sum_{u=1}^{m+n} A_u = mx.$$

The number of pairs a_i, a_j $(j > i)$ which lie in the above interval is

$$\tfrac{1}{2} A_u (A_u - 1).$$

The total number of these is

$$\sum_{u=1}^{m+n} \tfrac{1}{2} A_u (A_u - 1),$$

and, by an elementary inequality, this is greater than or equal to

$$\tfrac{1}{2}(m+n)\left(\frac{mx}{m+n}\right)\left(\frac{mx}{m+n} - 1\right).$$

For any such pair, $a_j - a_i$ is an integer r satisfying $1 \leqslant r \leqslant m-1$, and to each value of r there corresponds at most one such pair, since the numbers $a_j - a_i$ are all different. The pair which corresponds to r occurs in exactly $m-r$ of the intervals. Hence the total number of pairs is less than or equal to

$$\sum_{r=1}^{m-1} (m-r) = \tfrac{1}{2}m(m-1).$$

Comparing these results, we have

$$\tfrac{1}{2}mx(mx-m-n) \leqslant \tfrac{1}{2}m(m-1)(m+n),$$

whence
$$x(mx-2n) < m(m+n),$$

and
$$x < \frac{n}{m} + \left(n+m+\frac{n^2}{m^2}\right)^{\frac{1}{2}}.$$

Taking $m=[n^{\frac{1}{4}}]$, we obtain $x < n^{\frac{1}{2}}+O(n^{\frac{1}{4}})$. This proves the second result.

It is easy to see that, for every *infinite* B_2 sequence, $\underline{\lim}\, \phi(n)/\sqrt{n} = 0$. On the other hand, it is not difficult to give an example of a B_2 sequence with $\overline{\lim}\, \phi(n)/\sqrt{n} > 0$.

III. Let $a_1, a_2, \ldots$ be an arbitrary sequence of positive integers, and suppose that $f(n)=k$ for $n \geqslant n_0$, where $f(n)$ denotes the number of representations of n as a_i+a_j. Clearly $\phi(n)=o(n)$. For, if not, there would be arbitrarily large values of n for which the number of pairs a_i, a_j both less than n would be greater than cn^2, and so there would be a number $m<2n$ for which $f(m)>cn^2/2n$, which is contrary to hypothesis.

Therefore, by Fabry's gap theorem, the power series $\sum_{i=1}^{\infty} z^{a_i}$ has the unit circle as its natural boundary. But

$$\left(\sum_{i=1}^{\infty} z^{a_i}\right)^2 = \sum_{n=1}^{\infty} f(n)\, z^n = \psi(z) + \frac{kz^{n_0}}{1-z}, \tag{4}$$

where $\psi(z)$ is a polynomial of degree not exceeding n_0-1. Clearly (4) gives a continuation of $\Sigma\, z^{a_i}$ over the whole plane as an algebraic function, which is an obvious contradiction. This proves the result.

It would be of interest to have an elementary proof of this result, but we have not succeeded in finding one. Perhaps the following conjectures on the behaviour of $f(n)$ may be of some interest.

(1) It is impossible that

$$\sum_{m=1}^{n} f(m) = cn + O(1),$$

where c is a constant. If, for example, $a_i = i^2$, the error term is known not to be even $O(n^{\frac{1}{4}})$.

(2) If $f(n) > 0$ for $n > n_0$, then $\overline{\lim} f(n) = \infty$. Here we may mention that the corresponding result for $g(n)$, the number of representations of n as $a_i a_j$, can be proved*.

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ON THE SUMMABILITY FACTORS OF FOURIER SERIES

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1. Suppose that the Fourier series of the L-integrable function $f(x)$ is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \tfrac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n(x). \tag{1}$$

It was proved by B. N. Prasad that‡, *if* $\{\lambda_n\}$ *is one of the sequences*

$$\left\{\frac{1}{(\log n)^{1+\delta}}\right\}, \quad \left\{\frac{1}{\log n(\log_2 n)^{1+\delta}}\right\}, \quad \left\{\frac{1}{\log n \log_2 n(\log_3 n)^{1+\delta}}\right\}, \quad \dots \quad (\delta > 0),$$

the series

$$\sum_{n=2}^{\infty} \lambda_n c_n(x) \tag{2}$$

is summable $|A|$ *for almost all values of* x. This result has been generalized by Izumi and Kawata, who proved that§, *if* $\{\lambda_n\}$ *is a convex sequence and the series* $\Sigma n^{-1}\lambda_n$ *converges, the series* (2) *is summable* $|A|$ *for almost all values of* x. Izumi and Kawata proved also that||, *if* $f(x)$ *belongs to the class* H, *i.e. if* $f(x)$ *and its conjugate function are both* L*-integrable, and if* $\{\lambda_n\}$ *is a bounded sequence such that the series*

$$\sum_{n=1}^{\infty} n(\Delta\lambda_n)^2, \quad \sum_{n=1}^{\infty} \frac{\lambda_n^2}{n}, \tag{3}$$

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* The proof is similar to that used by P. Erdös in *Mitt. Tomsk Univ.*, 2 (1938), 74–82, but is considerably more complicated.

† Received 2 September, 1941; read 11 December, 1941.

‡ B. N. Prasad, *Proc. London Math. Soc.* (2), 35 (1933), 407–424. A series Σc_n is said to be *summable* $|A|$ if $F(x) = \Sigma c_n x^n$ converges for $|x| < 1$ and $F(x)$ is of bounded variation in (0, 1).

§ S. Izumi and T. Kawata, *Proc. Imp. Acad. Japan*, 14 (1938), 32–35.

|| S. Izumi and T. Kawata, *Tôhoku Math. Journal*, 45 (1938), 194–196.