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ON THE CONVERGENCE OF TRIGONOMETRIC SERIES

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A well known theorem of Kolmogoroff[1] states that if $\Sigma(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$ is a lacunary trigonometric series $\left(\text{i.e.} \frac{n_{k+1}}{n_k} > c > 1\right)$ and if $\Sigma(a_{n_k}^2 + b_{n_k}^2)$ converges, then $\Sigma(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$ converges almost everywhere. It has been observed by S. Sidon[2] that in many theorems about lacunary trigonometric series the condition of lacunarity can be replaced by the following condition (which Sidon called B_2): The number of solutions of $a = n_i + n_j$ is uniformly bounded for all integers a. This condition is trivially satisfied in the case of lacunary series and Sidon constructed B_2 sequences for which $n_k < ck$.[3] In the present note we are going to prove the following

Theorem. Let $\Sigma(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$ be a trigonometric series where the n_k satisfy property B_2 and $\Sigma(a_{n_k}^2 + b_{n_k}^2)$ converges. Then $\Sigma(a_{n_k} \cos n_k x + b_{n_k} \sin n_k x)$ converges almost everywhere.

In most cases where condition B_2 can replace lacunarity the proof of this fact is very simple, but in this case the proof will not be quite trivial. First we need the following

Lemma. Let $P(x) = a_m \cos mx + \cdots + a_n \sin nx$ and let $\sum_m^n a_k^2 = \epsilon$. We can always split $P(x)$ into two parts

$$P(x) = (a_m \cos mx + \cdots + a_{l-1} \cos (l-1)x + a_l' \cos lx) + (a_l'' \cos lx + \cdots + a_n \cos nx),\ a_l = a_l' + a_l'' \tag{1}$$

such that

$$a_m^2 + \cdots + a_l'^2 \leqslant \frac{\epsilon}{2}, \qquad a_l''^2 + \cdots + a_m^2 \leqslant \frac{\epsilon}{2} \tag{2}$$

and

$$a_m^2 + \cdots + a_l'^2 = a_l''^2 + \cdots + a_n^2 . \tag{3}$$

Proof. If there is a q such that $\sum_m^q a_k^2 = \sum_{q+1}^n a_k^2 = \frac{\epsilon}{2}$ then there is nothing to prove. Hence suppose

$$a_m^2 + \cdots + a_{l-1}^2 < \frac{\epsilon}{2} < a_m^2 + \cdots + a_l^2$$

Let

$$\delta = \frac{\epsilon}{2} - (a_m^2 + \cdots + a_{l-1}^2), \qquad 0 < \delta < a_l^2.$$

[1] Fund. Math. 5. (1924) p. 96–97.

[2] Acta Szeged VII. (1934–35) p. 85–94.

[3] Sidon, ibid.

We have

$$a_{l+1}^2 + \cdots + a_n^2 = \epsilon - a_l^2 - \left(\frac{\epsilon}{2} - \delta\right) = \frac{\epsilon}{2} + \delta - a_l^2$$

Take

$$a_l' = \frac{\delta}{a_l}, \qquad a_l'' = a_l - \frac{\delta}{a_l}.$$

(1) is clearly satisfied. Now

$$a_m^2 + \cdots + a_l'^2 = \frac{\epsilon}{2} - \delta + \left(\frac{\delta}{a_l}\right)^2,$$

$$a_l''^2 + \cdots + a_n^2 = \left(a_l - \frac{\delta}{a_l}\right)^2 + \frac{\epsilon}{2} + \delta - a_l^2 = \frac{\epsilon}{2} - \delta + \left(\frac{\delta}{a_l}\right)^2.$$

Thus (3) is satisfied. (2) is also satisfied, since

$$\frac{\epsilon}{2} - \delta + \left(\frac{\delta}{a_l}\right)^2 < \frac{\epsilon}{2}$$

is an immediate consequence of $0 < \delta < a_l^2$.

COROLLARY. By repeating this process it is clear that we can divide $P(x)$ into 2^r consecutive parts (with splitting of terms) such that the sum Σa_k^2 for each part is inferior to $\frac{\epsilon}{2^r}$, and that all the sums are equal. Also clearly for r sufficiently large $\frac{\epsilon}{2^r} < \min (a_k^2)$ ($\min (a_k^2)$ meaning the minimum of those a_k^2 which are $\neq 0$). Thus each block of terms will ultimately contain no more than two terms $\neq 0$.

PROOF OF THE THEOREM. Without loss of generality we can assume that our series is a pure cosine series $\Sigma a_{n_k} \cos n_k x$. Put

$$\sum_{n_k=2^m}^{2^{m+1}-1} a_{n_k} \cos_{n_k} x = P_m(x) = P_m, \qquad \sum_{2^m}^{2^{m+1}-1} a_{n_k}^2 = \epsilon_m$$

we have as a consequence of property B_2 [3]

$$\int_0^{2\pi} P_m^4\, dx < c\left(\int_0^{2\pi} P_m^2\, dx\right)^2. \tag{4}$$

We know that $\sum_m P_m$ converges almost everywhere.[4] Now let us split $P_m(x)$ into 2^s parts as indicated in the lemma and the corollary, this operation being successively performed for $s = 1, 2, \cdots, v$. (v sufficiently large.)

Let $P_{m,s}$ be one of the 2^s parts obtained in the s-th operation. Let $E_{m,s}$ be the set of x for which

$$P_{m,s} > \frac{\epsilon_m^{1/4}}{2^{s/4}} \sqrt{s} \tag{5}$$

[4] Kolmogoroff, ibid.

We have by (4)

$$\int_0^{2\pi} P_{m,s}^4\, dx < c\, \frac{\epsilon_m^2}{2^{2s}}$$

and thus

$$| E_{m,s} | < c\, \frac{\epsilon_m}{2^s}\, \frac{1}{s^2}, \qquad \text{where } | E | \text{ stands for the measure of } E.$$

Thus the sum of the 2^s, $| E_{m,s} |$ is less than $c\, \frac{\epsilon_m}{s^2}$ and the sum of all the $| E_{m,s} |$ $(s = 1, 2, \cdots v)$ is less than $B\epsilon_m$, B being a constant.

It follows that the x for which (5) holds for infinitely many m is of measure 0.

Now if x does not belong to this set and if $2^m \leqslant q < 2^{m+1}$ it is easy to see that (by writing q in the binary scale)

$$\sum_{n_k=2^m}^{q} a_{n_k} \cos n_k x \leq \frac{\epsilon_m^{1/4}\sqrt{1}}{2^{1/4}} + \frac{\epsilon_m^{1/4}\sqrt{2}}{2^{2/4}} + \cdots \frac{\epsilon_m^{1/4}\sqrt{v}}{2^{v/4}}\,\frac{\bar{1}}{4} + \max_{2^m \leq n_k < 2^{m+1}} | a_{n_k} | < \delta (\delta \to 0)$$

which completes the proof of our theorem.

It is easy to see that instead of

$$\int_0^{2\pi} P_m^4\, dx < c\left(\int_0^{2\pi} P_m^2\, dx\right)^2$$

it would have been sufficient to assume that

$$\int_0^{2\pi} | P_m |^{2+\alpha}\, dx < c\left(\int_0^{2\pi} P_m^2\, dx\right)^{\frac{2+\alpha}{2}}$$

holds for any positive α.

It is clear from our proof that it suffices to show that our n_k are such that the number of solutions of $a = n_i + n_j$, $2^m < n_i$, $n_j < 2^{m+1}$ is uniformly bounded. Thus we can construct a sequence n_k which satisfies our condition and for which $n_k < k^2$.[5]

It might be of some number-theoretic interest to investigate whether our theorem remains true for $n_k = k^3$. Mordell[6] proved that $n_k = k^3$ does not satisfy condition B_2.

Another theorem of Kolmogoroff[7] states that if $\Sigma(a_n \cos nx + b_n \sin nx)$ is a trigonometric series such that $\Sigma(a_n^2 + b_n^2)$ converges, and if we put $\zeta_n = \sum_{k=0}^{n} (a_k \cos kx + b \sin kx)$. Then if $\frac{n_{k+1}}{n_k} > c > 1$, S_{n_k} converges almost everywhere. This result we can not prove if the n_k satisfy condition B_2.

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[5] P. Erdös and P. Turán, London Math. Soc. Journal 16, (1941) p. 212–215.

[6] Oral communication. See also K. Mahler, London Math. Soc. Proc. 39, (1935) p. 431–436.

[7] Kolmogoroff, ibid.