Annals of Mathematics
Vol. 44, No. 2, April, 1943

ON FAMILIES OF MUTUALLY EXCLUSIVE SETS

By P. Erdös and A. Tarski

(Received August 11, 1942)

In this paper we shall be concerned with a certain particular problem from the general theory of sets, namely with the problem of the existence of families of mutually exclusive sets with a maximal power. It will turn out—in a rather unexpected way—that the solution of these problems essentially involves the notion of the so-called "inaccessible numbers." In this connection we shall make some general remarks regarding inaccessible numbers in the last section of our paper.

§1. Formulation of the problem. Terminology[1]

The problem in which we are interested can be stated as follows: Is it true that every field $\mathfrak{F}$ of sets contains a family of mutually exclusive sets with a maximal power, i.e. a family $\mathfrak{G}$ whose cardinal number is not smaller than the cardinal number of any other family $\mathfrak{H}$ of mutually exclusive sets contained in $\mathfrak{F}$.

By a field of sets we understand here as usual a family $\mathfrak{F}$ of sets which together with every two sets X and Y contains also their union $X \cup Y$ and their difference $X - Y$ (i.e. the set of those elements of X which do not belong to Y) among its elements. A family $\mathfrak{G}$ is called a family of mutually exclusive sets if no set X of $\mathfrak{G}$ is empty and if any two different sets of $\mathfrak{G}$ have an empty intersection.

A similar problem can be formulated for other families e.g. for rings of sets, i.e. for families which together with any two sets X and Y also contain their union $X \cup Y$ and their intersection $X \cap Y$ among their elements. We obtain an especially interesting particular case of this problem by referring it to the ring of open sets of a topological space S with power $2^{\aleph_0}$.

It turns out that the solution of our problem is in general positive; however it is negative in certain exceptional cases. To examine the problem thoroughly we must first subject it to a certain transformation by using some notions from the arithmetic of cardinal numbers.

We shall denote the cardinal number (or power) of a set S by $\mathfrak{c}(S)$.

A cardinal number $\mathfrak{n}$ is called a limit number if $\mathfrak{n} \neq 0$ and if among the cardinal numbers $\mathfrak{x} < \mathfrak{n}$ there is no largest one. The number $\mathfrak{n}$ is called *singular* if it can be expressed as a sum of less than $\mathfrak{n}$ numbers $\mathfrak{m}$, each of which is smaller than $\mathfrak{n}$.

[1] For the concepts and results of the general theory of sets, which are applied in this paper, see Hausdorff, Mengenlehre; however as regards the concept of an inaccessible number cf. Tarski, Über unerreichbare Kardinalzahlen, Fund. Math. Vol. 30 (1938) p. 68–89. For the concepts and results from the theory of partially ordered sets, lattices, Boolean algebras, etc., see G. Birkhoff, Lattice theory. For topological concepts see Kuratowski, Topologie I.

If such a representation is impossible the number $\mathfrak{n}$ is called *regular*. Regular limit numbers are also referred to as "inaccessible" or "weakly inaccessible" numbers.

As is well known, every limit number is an infinite number, and every singular infinite number is a limit number. The problem of the existence of regular limit numbers $> \aleph_0$ is thus far unsolved, and presumably will never be solved on the basis of the axiom systems upon which the general theory of sets is constructed at present. At any rate the existence of the numbers in question cannot be derived from these axiom systems provided they are consistent; on the other hand it seems highly improbable that these systems cease to be consistent if we enrich them by adding new existential axioms which secure the existence of the inaccessible numbers.

$\mathfrak{F}$ being a family of sets let us denote by $\mathfrak{d}(\mathfrak{F})$ the smallest cardinal number which is $> \mathfrak{c}(\mathfrak{G})$ for every family $\mathfrak{G}$ of mutually exclusive sets contained in F. If $\mathfrak{d}(\mathfrak{F})$ is not a limit number, the family F obviously contains a subfamily $\mathfrak{G}$ of mutually exclusive sets with a maximal power. Thus our problem reduces now to the following one:

$\mathfrak{n}$ being a limit number is it true that for every field (or ring) of sets we have $\mathfrak{d}(\mathfrak{F}) \neq \mathfrak{n}$?

We shall show that the solution of this problem depends on the properties of the number $\mathfrak{n}$: the answer is affimative if $\mathfrak{n}$ either $= \aleph_0$ or is a regular number (Theorem 1), is negative only for the hypothetical regular limit numbers $> \aleph_0$ (Theorem 2). If in particular the problem is applied to the ring of all open sets of a topological space[2] with $\mathfrak{c}(S) = 2^{\aleph_0}$, then its positive solution proves to be equivalent with the statement that there is no inaccessible number $> \aleph_0$ and $\leqq 2^{\aleph_0}$ (Corollary 3).

In order to formulate the positive part of our result in as general form as possible, we shall use the terminology of partially ordered sets.

Let S be an arbitrary set which is partially ordered by the binary relation $\leqq$. If x is an element of S, we write $S(x)$ to denote the partially ordered set of all elements $y \epsilon S$ which are $\leqq x$. The symbol $\wedge$ will denote a null element of S, i.e. an element x such that $x \leqq y$ for every $y \epsilon S$. Two elements y and z of S are called disjoint if $y \neq \wedge$, $z \neq \wedge$ and if for every $x \epsilon S$ the formulas $x \leqq y$ and $x \leqq z$ imply $x = \wedge$. We do not here assume that the partially ordered set necessarily contains a null element. In fact without loss of generality we could confine ourselves to the consideration of sets which do not contain such elements; and in this case we could simply say that two elements y and z are called disjoint if there is no element x such that $x \leqq y$ and $x \leqq z$.

A subset T of a partially ordered set S such that every element of T is $\neq \wedge$ and every two different elements of T are disjoint is called a set of mutually

[2] By topological spaces we mean here the spaces with the closure operation satisfying the axioms I–III of Kuratowski (op. cit. p. 77). However the space which will be constructed in the proof of Corollary 2 will also satisfy axiom IV (normality) (pp. 95–101, ibid.).

exclusive elements. Again we denote by $\mathfrak{d}(\mathfrak{S})$ the smallest cardinal number $> \mathfrak{c}(T)$ for every $T \subseteq S$ of sets of mutually exclusive elements; moreover we write for every element $x \in T$

$$\mathfrak{d}(x) = \mathfrak{d}(\mathfrak{S}(x))$$

In view of this formula $\mathfrak{d}$ constitutes an example of a function f which correlates with every element of a partially ordered set a cardinal number $f(x)$. This function is obviously increasing, for we have

$$\mathfrak{d}(x) \leqq \mathfrak{d}(y)$$

for every two elements x and y such that $x \leqq y$. Many other examples of this kind of increasing functions are also known; e.g. the function c defined for every $x \in S$ by the formula

$$\mathfrak{c}(x) = \mathfrak{c}(S(x)).$$

Still another example is constituted by the function g defined in the following way: for every $x \in S$, $g(x)$ is the smallest cardinal number $\mathfrak{n}$ such that there is a basis B of the set $S(x)$ with power $c(B) = \mathfrak{n}$; by a basis we here understand a set $B \subseteq S(x)$ such that every element of $S(x)$ is the union (the least upper bound) of elements of B. To every increasing function f of the kind considered there corresponds a certain notion of *homogeneity* of partially ordered sets. We say generally that an element x of a partially ordered set S is homogeneous with respect to an increasing function f, which is defined over the set S and assumes cardinal numbers as values, or simply that x is f-homogeneous, if $x \neq \Lambda$ and if $f(x) = f(y)$ for every element $y \in S$ such that $y \neq \Lambda$ and $y \leqq x$. If the set S contains a unit element u i.e. an element x such that $y \leqq x$ for every $y \in S$), and if u is f-homogeneous, the whole set S is called f-homogeneous.

SOLUTION OF THE PROBLEM

We shall begin with two simple lemmas concerning f-homogeneous elements

LEMMA 1. *Let S be a partially ordered set, and f an increasing function which correlates with every element $x \in S$ a cardinal number $f(x)$. Then for every element $x \neq \Lambda$ there exists an f-homogeneous element $y \leqq x$.*

PROOF. Consider all the cardinal numbers $f(y)$ correlated with the elements $y \leqq x, y \neq \Lambda$. Among these cardinal numbers there certainly exists a smallest, say $\mathfrak{n}$ (by the well ordering theorem); and it is easily seen that every element such that

$$y \leqq x, \qquad f(y) = \mathfrak{n}$$

is f-homogeneous.

LEMMA 2. *Under the hypothesis of* LEMMA 1. *there exists a set $T \subseteq S$ of mutually exclusive f-homogeneous elements such that no element of S is disjoint with all elements of T.*

PROOF. It can be easily shown (e.g. with the help of well ordering) that there exists a maximal set T of mutually exclusive f-homogeneous elements of S; i.e., a set T of mutually exclusive f-homogeneous elements of S which is not a proper subset of any other set with the same property. Hence by LEMMA 1 it follows that no element of S—whether homogeneous or not—is disjoint with every element of T, q.e.d.

As an immediate consequence of LEMMA 2 we obtain the following theorem which, however, will not be applied in this paper.

Let B be a Boolean algebra, and f an increasing function which correlates with every element x of B a cardinal number $f(x)$. Then every element of B—and, in particular, the unit element—can be represented as the union of mutually exclusive f-homogeneous elements of B; and therefore B is isomorphic with a direct sum of f-homogeneous Boolean algebras.

The following three lemmas will lead us directly to THEOREM 1, which is one of the main results of this paper.

LEMMA 3. *If S is a partially ordered set and $\mathfrak{d}(S)$ is a limit number, then S contains a $\mathfrak{d}$-homogeneous element x with $\mathfrak{d}(x) = \mathfrak{d}(S)$.*

PROOF. Assume that, on the contrary, S does not contain a $\mathfrak{d}$-homogeneous element x with $\mathfrak{d}(x) = \mathfrak{d}(S)$. By applying LEMMA 2 to the function $f = \mathfrak{d}$ we obtain a set $T \subseteq S$ of mutually exclusive $\mathfrak{d}$-homogeneous elements, with the property that no element of S is disjoint with every element of T. According to our assumption we have:

$$\mathfrak{d}(t) < \mathfrak{d}(S) \quad \text{for every element} \quad t \in T; \tag{1}$$

moreover, the definition of $\mathfrak{d}$ implies:

$$\mathfrak{c}(T) < \mathfrak{d}(S) \tag{2}$$

Since $\mathfrak{d}(S)$ is an infinite cardinal number, we have

$$(\mathfrak{d}(S))^2 = \mathfrak{d}(S); \tag{3}$$

hence, by (1) and (2), we obtain:

$$\sum_{t \in T} \mathfrak{d}(t) \leqq \mathfrak{c}(T)\cdot\mathfrak{d}(S) \leqq (\mathfrak{d}(S))^2 = \mathfrak{d}(S). \tag{4}$$

We want now to show that in the latter formula '$\leqq$' may be replaced by '$=$'. In fact, consider an arbitrary set $U \subseteq S$ of mutually exclusive elements. As was mentioned before, no element of U can be disjoint with every element of T. Hence (by using the axiom of choice) we can correlate with every element $u \in U$ first an element $t_u \in T$, and then an element $v_u \in S$ such that $v_u \neq \Lambda$, $v_u \leqq u$, and $v_u \leqq t_u$. Let V be the set of all these elements v_u. It can easily be seen that the correspondence between the elements of U and those of V is one-to-one, and that therefore the sets U and V have the same power. If, on the other hand, we denote by V_t (where t is a given element of T) the set of all those elements v_u which are $\leqq t$, we see at once that V is the union of all these sets V_t;

$$V = \bigcup V_t$$

and that

$$\mathfrak{c}(V_t) < \mathfrak{d}(t) \quad \text{for every} \quad t \in T.$$

Hence

$$\mathfrak{c}(U) = \mathfrak{c}(V) \leqq \sum_{t \in T} \mathfrak{d}(t).$$

Since the latter formula holds for every set $U \subseteq S$ of mutually exclusive elements, we infer from the definition of $\mathfrak{d}$ that either

$$\mathfrak{d}(S) \leqq \sum_{t \in T} \mathfrak{d}(t) \tag{5}$$

or else $\mathfrak{d}(S)$ is the cardinal number which immediately follows $\sum_{t \in T} \mathfrak{d}(t)$. However, the second alternative is excluded, $\mathfrak{d}(S)$ being by hypothesis a limit number, and therefore (5) holds. The formulas (4) and (5) give at once:

$$\sum_{t \in T} \mathfrak{d}(t) = \mathfrak{d}(S). \tag{6}$$

From (1), (2), (3), and (6) it follows that for every cardinal number $\mathfrak{x} < \mathfrak{d}(S)$ there exists an element $t \in T$ such that $\mathfrak{x} < \mathfrak{d}(t)$. For if we had:

$$\mathfrak{d}(S) > \mathfrak{x} \geqq \mathfrak{d}(t) \quad \textit{for every} \quad t \in T$$

we should have, by (1), (2), and (3),

$$\sum_{t \in T} \mathfrak{d}(t) \leqq c(T)\mathfrak{x} < (\mathfrak{d}(S))^2 = \mathfrak{d}(S)$$

which obviously contradicts (6). Hence we can easily construct (with the help of the well ordering theorem) a well ordered transfinite sequence of elements $t_0, t_1, \cdots, t_\xi, \cdots \in T$ of an ordinal type τ which satisfy the following conditions:

(7) $\quad \mathfrak{d}(t_{\xi_1}) < \mathfrak{d}(t_{\xi_2}) \quad$ for $\quad \xi_1 < \xi_2 < \tau, \qquad \tau$ being a limit ordinal number,

and

$$\sum_{\xi < \tau} \mathfrak{d}(t_\xi) = \mathfrak{d}(S). \tag{8}$$

Consider an arbitrary ordinal number $\xi < \tau$. By (7) we have

$$\mathfrak{d}(t_\xi) < \mathfrak{d}(t_{\xi+1})$$

Hence by virtue of the definition of $\mathfrak{d}$, there exists a set $W_\xi \subseteq S(t_{\xi+1})$ of mutually exclusive elements with power:

$$\mathfrak{c}(W_\xi) = \mathfrak{d}(t_\xi) \quad \text{for every} \quad \xi < \tau. \tag{9}$$

Putting

$$W = \bigcup_{\xi < \tau} W_\xi \tag{10}$$

we easily see that W is a set of mutually exclusive elements of S, the sets $W_0, W_1, \cdots, W_\xi, \cdots$ being mutually exclusive. We have moreover, by (8), (9), and (10),

$$\mathfrak{c}(W) = \mathfrak{d}(S).$$

In this way we have arrived at a contradiction; for $\mathfrak{d}(S)$ is by definition $> \mathfrak{c}(X)$ for every set $X \subseteq S$ of mutually exclusive elements. Thus we must reject our original supposition, and assume that S has a d-homogeneous element x with $\mathfrak{d}(x) = \mathfrak{d}(S)$, q.e.d.

LEMMA 4. *If x is a $\mathfrak{d}$-homogeneous element of a partially ordered set S, then* $\mathfrak{d}(x) \neq \aleph_0$.

PROOF. Assume $\mathfrak{d}(x) = \aleph_0$. By the definition of $\mathfrak{d}$ there exist two disjoint elements x_1 and x_2 which are $\leqq x$. Since x is $\mathfrak{d}$-homogeneous, we have further $\mathfrak{d}(x_2) = \aleph_0$, and therefore there exist two disjoint elements, $x_{2,1}$ and $x_{2,2}$ which are $\leqq x_2 \leqq x$. By continuing this procedure indefinitely we obtain (with the help of the axiom of choice) an infinite sequence of mutually exclusive elements $x_1, x_{2,1}, x_{2,2,1}, \cdots$ which are all $\leqq x$; but this clearly contradicts our assumption. Hence $\mathfrak{d}(x) \neq \aleph_0$, q.e.d.

LEMMA 5. *If x is a $\mathfrak{d}$-homogeneous element of a partially ordered set, then $\mathfrak{d}(x)$ is not a singular limit number.*

PROOF. Assume, on the contrary, that $\mathfrak{d}(x)$ is a singular limit number. Thus $\mathfrak{d}(x)$ can be represented in the form:

$$\mathfrak{d}(x) = \sum_{i \in C} \mathfrak{m}_i \tag{11}$$

where C is a certain set of power $< \mathfrak{d}(x)$, and every number $\mathfrak{m}_i$ is also $< \mathfrak{d}(x)$. Since $\mathfrak{c}(C) < \mathfrak{d}(x)$, we can correlate with every element $i \in C$ an element $x_i \leqq x$ in such a way that any two elements x_{i_1} and x_{i_2} $(i_1 \neq i_2)$ are disjoint. The element x being $\mathfrak{d}$-homogeneous, we have for every element x_i:

$$\mathfrak{d}(x_i) = \mathfrak{d}(x) > \mathfrak{m}_i ;$$

consequently we can correlate with every element x_i a set $T_i \subseteq S(x_i)$ of mutually exclusive elements with power $\mathfrak{c}(T_i) = \mathfrak{m}_i$. Hence in view of (11) it is easily seen that the set T defined by means of the formula

$$T = \bigcup_{i \in C} T_i$$

is a set of mutually exclusive elements of $S(x)$ with power

$$\mathfrak{c}(T) = \sum_{i \in C} \mathfrak{M}_i = \mathfrak{d}(x).$$

But this is impossible, since $\mathfrak{d}(x)$ must be by definition $> \mathfrak{c}(T)$. Thus we must assume that $\mathfrak{d}(x)$ is not a singular limit number, q.e.d.

Lemmas 3, 4, and 5 imply directly

THEOREM 1. *If $\mathfrak{n}$ is either equal to $\aleph_0$ or is a singular limit number then there is no partially ordered set S such that $\mathfrak{d}(S) = \mathfrak{n}$.*

REMARK. Theorem 1 applies directly to various special partially ordered sets, e.g., to lattices or Boolean algebras. It can also be applied to an arbitrary family $\mathfrak{F}$ of sets, for every such family is partially ordered by the relation of inclusion $\subseteq$. It should be noticed, however, that two sets of a family $\mathfrak{F}$ which are disjoint from the point of view of the theory of partially ordered sets are not necessarily disjoint in the usual set-theoretic meaning. On the other hand, it is easily seen that the two meanings of the notion of disjointness coincide if the family $\mathfrak{F}$ contains the empty set among its elements and if, with any two sets X and Y belonging to $\mathfrak{F}$ their intersection $X \cap Y$ also belongs to $\mathfrak{F}$. Thus Theorem 1 applies literally to every field of sets, and even to every ring of sets which contains the empty set, e.g., to the ring of all open sets of a topological space; and it can be very easily shown that the theorem holds for an arbitrary ring of sets, even if it does not contain the empty set (for in this case the ring does not contain any two sets which are disjoint in the set-theoretic sense).

THEOREM 2. *If $\mathfrak{n}$ is a regular cardinal number $> \aleph_0$, then for every set S of power $\mathfrak{n}$ there exists a field $\mathfrak{F}$ of subsets of S such that $\mathfrak{c}(F) = \mathfrak{d}(\mathfrak{F}) = n$.*

PROOF. We could assume that $\mathfrak{n}$ is a limit number, for otherwise the proof presents no difficulty; however, no use will be made here of this assumption.

We shall first prove the theorem for a particular set N of power $\mathfrak{n}$, which will be defined as follows. Let us write for every ordinal number α:

(1) $\mathfrak{c}(\alpha)$ = the power of the set of all ordinal numbers $\xi < \alpha$. By the well-ordering theorem there exists an ordinal number ν such that

(2) $\mathfrak{c}(\nu) = \mathfrak{n}$, while $\mathfrak{c}(\xi) < \mathfrak{n}$ for every number $\xi < \nu$.

(3). N = the set of all transfinite sequences σ of ordinal numbers $\sigma_0, \sigma_1, \cdots$, which satisfy the following conditions:

(i) $\sigma_\xi \leqq \xi$ for every number $\xi < \nu$;

(ii) there are only finitely many numbers $\xi < \nu$ such that $\sigma_\xi \neq 0$.

Since

(4). $n = n^0 + n^1 + \cdots + n^k + \cdots \qquad (k < \aleph_0)$,

it is easily seen from (1), (2), and (3) that N has in fact power $\mathfrak{n}$.

We are now going to correlate to every number $\xi < \nu$ a family $\mathfrak{H}$ of subsets X of N so as to satisfy the following conditions:

(5). $\mathfrak{H}_\xi$ is a family of mutually exclusive sets, and $\bigcup_{X \in \mathfrak{H}_\xi} X = N$

(6). $\mathfrak{c}(\mathfrak{H}_\xi) = \mathfrak{c}(\xi + 1) < \mathfrak{n}$;

(7). if $\xi_1, \xi_2, \cdots \xi_n$ is any finite sequence of distinct ordinal numbers $< \nu$, and $X_1, X_2, \cdots, X_n$ any finite sequence of sets such that $X_1 \in \mathfrak{H}_{\xi_1}$, $X_2 \in \mathfrak{H}_{\xi_2}, \cdots, X_n \in \mathfrak{H}_{\xi_n}$, then the intersection $X_1 \cap \cdots \cap X_n$ is not empty. To obtain such families $\mathfrak{H}_\xi$ we put:

(8). $N_{\xi,\eta}$ = the set of all sequences $\sigma \in N$ such that $\sigma_\xi = \eta$, $\eta \leqq \xi < \nu$

(9). $\mathfrak{H}_\xi$ = the family of all sets $N_{\xi,0}, N_{\xi,1} \cdots N_{\xi,\eta} \cdots$ where $\eta \leqq \xi < \nu$.

The proof that the families $\mathfrak{H}_\xi$ thus defined satisfy the conditions (5), (6), and (7) does not present any difficulties.

Finally we construct the field $\mathfrak{F}$ by putting

(10). $\mathfrak{H} = \bigcup_{\xi < \nu} \mathfrak{H}_\xi$

(11). $\mathfrak{F}$ = the smallest field of sets which contains all the sets of $\mathfrak{H}$; or, in other words,: $\mathfrak{F}$ = the family of all sets which are finite unions of finite intersections of sets $X \in \mathfrak{H}$ and their complements $N - X$.

We shall prove that $\mathfrak{F}$ satisfies the conclusion of our theorem. If $\mathfrak{n}$ is an infinite number, it is easily seen from (2), (6), and (10) that the family $\mathfrak{H}$ has power $\mathfrak{n}$. Hence by (4) and (11) it follows that $\mathfrak{F}$ has also power $\mathfrak{n}$. Furthermore, (2), (5), (6), (10), and (11) imply that, for every number $\mathfrak{r} < \mathfrak{n}$, $\mathfrak{F}$ does contain $\mathfrak{r}$ mutually exclusive sets. Hence we have

(12). $\mathfrak{c}(\mathfrak{F}) = \mathfrak{n} \leqq \mathfrak{d}(\mathfrak{F})$.

It remains to show that every family $\mathfrak{G} \subseteq \mathfrak{F}$ of mutually exclusive sets has a power $<\mathfrak{n}$. We shall show it first for families of a rather special character.

Let us agree to say that a set $X \in \mathfrak{F}$ is of the l^{th} order (where l is any positive integer) if it is not empty and can be represented as an intersection of l different sets of the family $\mathfrak{H}$. We are going to establish certain simple properties of sets of the l^{th} order.

(13). Every set X of the l^{th} order can be represented uniquely in the form

$$X = X_1 \cap \cdots \cap X_l$$

where $X_1 \in \mathfrak{H}_{\xi_1}, \cdots, X_l \in \mathfrak{H}_{\xi_l}$, and $\xi_i < \xi_k < \nu$ for $1 \leqq i < k \leqq l$.

In fact, the possibility of such a representation follows directly from the definition of the sets of the l^{th} order; two different sets X_i and X_k cannot belong to the same family $\mathfrak{H}$, for by (5) the set $X \subseteq X_i \cap X_k$ would be then empty. Assume that the set X has two representations of this kind:

$$X = X_1 \cap \cdots \cap X_l = Y_1 \cap \cdots \cap Y_l$$

where $X_i \in \mathfrak{H}_{\xi_i}$, $Y_i \in H_{\eta_i}$, $\xi_i < \xi_k < \nu$, and $\eta_i < \eta_k < \nu$ for $1 \leqq i < k \leqq l$. If these representations are different, at least one of the sets $X_1, \cdots, X_l$, let us say X_i, cannot occur among the sets $Y_1, \cdots, Y_l$; and similarly a certain set Y_j cannot occur among the sets $X_1, \cdots, X_l$. Hence the number ξ_i must be different from each of the numbers $\eta_1, \cdots, \eta_l$. For, if we had $\xi_i = \eta_k$ $(1 \leqq k \leqq l)$, the sets X_i and Y_k $(X_i \neq Y_k)$ would belong to the same class $\mathfrak{H}_{\xi_i} = \mathfrak{H}_{\eta_k}$; and therefore by (5) the set $X \subseteq X_i \cap Y_k$ would be empty. For the same reason the number η_j must be different from each of the numbers $\xi_1, \cdots, \xi_l$. Thus, in particular, $\xi_i \neq \eta_j$, and therefore at least one of these two numbers is $\neq 0$. Assume, e.g., $\xi_i \neq 0$. By (6) there is a set $X_i' \in \mathfrak{H}_{\xi_i}$ which is different from X_i. By (5) the sets X_i and X_i' are disjoint; consequently the intersection

$$X_i' \cap X = X_i' \cap X_1 \cap \cdots \cap X_l$$

is empty. Hence the intersection

$$X_i' \cap X = X_i' \cap Y_1 \cap \cdots \cap Y_l$$

must also be empty; but this clearly contradicts (7), all the numbers $\xi_i, \eta_1, \eta_2, \cdots \eta_l$ being distinct. Thus the two representations of X cannot be different.

In what follows we shall refer to the sets $X_1, \cdots, X_l$ occurring in the representation (13) of a set X as the *factors* of X.

(14). In order that two sets X and Y of the l^{th} order be disjoint it is necessary and sufficient that they have two factors X_i and Y_i which are different, but belong to the same family $\mathfrak{H}$.

This follows directly from (5), (7), and (13).

(15). If two disjoint sets X and Y of the $(l+1)^{th}$ order have a common factor $X_i = Y_j$, and if X^* and Y^* are the intersections of the remaining factors of X and Y respectively, then X^* and Y^* are disjoint sets of the l^{th} order.

This can be easily obtained from (13) and (14).

Now we can prove by induction with respect to l:

(16). Every family $\mathfrak{G} \subseteq \mathfrak{F}$ of mutually exclusive sets of the l^{th} order has power $< \mathfrak{n}$.

(16) is clearly true for $l = 1$. In fact, in this case $\mathfrak{G}$ is contained in the family

$$\mathfrak{H} = \bigcup_{\xi < \nu} \mathfrak{H}_\xi .$$

If there were two sets X_1 and X_2 of $\mathfrak{G}$ which belonged to two different families $\mathfrak{H}_{\xi_1}$ and $\mathfrak{H}_{\xi_2}$, they would not be disjoint, on account of (7). Therefore there must be a $\xi < \nu$ such that $\mathfrak{G} \subseteq \mathfrak{H}_\xi$, and hence, by (6), $\mathfrak{c}(\mathfrak{G}) < \mathfrak{n}$.

Now assume that (16) has been proved for a given positive integer l, and consider a family $\mathfrak{G} \subseteq \mathfrak{F}$ of mutually exclusive sets of the $(l+1)$st order. Let X be any set of $\mathfrak{G}$, and let $\mathfrak{H}_{\xi_1}, \mathfrak{H}_{\xi_2}, \cdots, \mathfrak{H}_{\xi_{l+1}}$ $(\xi_1 < \xi_2 < \cdots < \xi_{l+1} < \nu)$ be those families $\mathfrak{H}_\xi$ which by (13) contain among their elements a factor of X. By (14) every set of $\mathfrak{G}$ must have a factor belonging to at least one of the families $\mathfrak{H}_{\xi_1}, \cdots, \mathfrak{H}_{\xi_{l+1}}$, and thus also to their union

$$\mathfrak{H}^* = \mathfrak{H}_{\xi_1} \cup \cdots \cup \mathfrak{H}_{\xi_{l+1}} . \tag{17}$$

For every set Z of H^* let us denote by $\mathfrak{G}(Z)$ the family of all those sets $Y \in \mathfrak{G}$ which have Z as a factor. We have thus a decomposition of $\mathfrak{G}$ in subfamilies $\mathfrak{G}(Z)$ (which are not unnecessarily mutually exclusive):

$$\mathfrak{G} = \bigcup_{Z \in \mathfrak{H}^*} \mathfrak{G}(Z).$$

Hence

$$\mathfrak{c}(\mathfrak{G}) \leq \sum_{Z \in \mathfrak{H}^*} \mathfrak{c}[\mathfrak{G}(Z)]. \tag{18}$$

Consider a particular family $\mathfrak{G}(Z)$ where Z is any set of $\mathfrak{H}^*$. If Y is a set of $\mathfrak{G}(Z)$, it has Z as a factor. Denote by Y^* the intersection of the remaining factors of Y, and by $\mathfrak{G}^*(Z)$ the family of all sets Z^* thus obtained. $\mathfrak{G}(Z)$ being a family of mutually exclusive sets, we easily infer from (13) and (15) that the family $\mathfrak{G}^*(Z)$ is also a family of mutually exclusive sets; furthermore that the correspondence $Y \dashrightarrow Y^*$ between the sets of $\mathfrak{G}(Z)$ and $\mathfrak{G}^*(Z)$ is one-to-one, and that therefore these two families have the same power. On the other hand, $\mathfrak{G}^*(Z)$ consists of sets of the l^{th} order; thus, by applying to $\mathfrak{G}^*(Z)$ our inductive premise, we obtain:

$$(19) \qquad \mathfrak{c}[\mathfrak{G}(Z)] = \mathfrak{c}[\mathfrak{G}^*(Z)] < \mathfrak{n} \quad \text{for every} \quad Z \in \mathfrak{H}^*.$$

$\mathfrak{n}$ being a regular, and thus an infinite, cardinal number, we also have, by (6) and (17),

$$\mathfrak{c}(\mathfrak{H}^*) \leqq \mathfrak{c}(\mathfrak{H}_{\xi_1}) + \cdots + \mathfrak{c}(\mathfrak{H}_{\xi_{l+1}}) < \mathfrak{n};$$

and hence, in view of (19),

$$(20) \qquad \sum_{Z \in \mathfrak{H}^*} \mathfrak{c}[\mathfrak{G}(Z)] < \mathfrak{n}.$$

From (18) and (20) it follows at once that $\mathfrak{G}$ has power $< \mathfrak{n}$. Thus (16) holds for every positive integer l.

We can now extend (16) in the following way:

(21) Every family $\mathfrak{G} \subseteq \mathfrak{F}$ of mutually exclusive sets of any finite orders has power $< \mathfrak{n}$.

In fact, denote by $\mathfrak{G}_l$ the family of those sets $X \in \mathfrak{G}$ which are of the l^{th} order. We obtain the decomposition

$$\mathfrak{G} = \mathfrak{G}_1 \cup \cdots \cup \mathfrak{G}_l \cup \cdots,$$

whence

$$\mathfrak{c}(\mathfrak{G}) \leqq \mathfrak{c}(\mathfrak{G}_1) + \cdots + \mathfrak{c}(\mathfrak{G}_l) + \cdots.$$

(The families $\mathfrak{G}_1, \cdots, \mathfrak{G}_l, \cdots$ are not necessarily mutually exclusive.) On the other hand, we have by (16):

$$C(\mathfrak{G}_l) < \mathfrak{n} \text{ for every positive integer } l.$$

Hence, $\mathfrak{n}$ being by hypothesis a regular number $> \aleph_0$, we easily obtain:

$$\mathfrak{c}(\mathfrak{G}) \leqq \mathfrak{c}(\mathfrak{G}_1) + \cdots + \mathfrak{c}(\mathfrak{G}_l) + \cdots < \mathfrak{n}.$$

Finally we can show that

(22) Every family $\mathfrak{G} \subseteq \mathfrak{F}$ of mutually exclusive sets has 0 power $< \mathfrak{n}$.

For, by (11), every set $Y \in \mathfrak{G}$ is a union of finite intersections of sets $X \in \mathfrak{H}$ and their complements $N - X$. Furthermore, from (5) and (10) it follows that the complement $N - X$ of a set $X \in \mathfrak{H}$ is a union (not necessarily finite) of sets of $\mathfrak{H}$. Therefore the set Y can be represented as a union of finite intersections of sets $X \in \mathfrak{H}$. Hence we can correlate with every set $Y \in \mathfrak{G}$ (which is not empty) a non-empty subset $Y^* \subseteq Y$ of finite order. The family $\mathfrak{G}^*$ of all the sets Y^* thus obtained has clearly the same power as $\mathfrak{G}$, and by (21) this power is $< \mathfrak{n}$.

From the definition of $\mathfrak{d}$, (22) implies:

$$\mathfrak{d}(\mathfrak{G}) \leqq \mathfrak{n};$$

and this formula together with (12) gives:

$$(23) \qquad \mathfrak{c}(\mathfrak{F}) = \mathfrak{d}(\mathfrak{F}) = \mathfrak{n}.$$

Thus our theorem has been proved for a particular set N of power $\mathfrak{n}$. Now consider an arbitrary set S of power $\geqq \mathfrak{n}$. The set S contains a subset N_1 of power $\mathfrak{n}$. We can establish a one-to-one correspondence between the subsets of N and those of N_1, and by means of this correspondence we can construct a field $\mathfrak{F}_1$ of sets $X \subseteq N_1 \subseteq S$ which satisfies (23). This brings the proof to an end.

COROLLARY 1. *If* $\mathfrak{n}$ *is a regular number* $>\aleph_0$, *then for every number* $\aleph \geqq \mathfrak{n}$ *there exists a topological space* S *of power* $\aleph$ *such that the family* $\mathfrak{G}$ *of all open sets (or of all sets which are both open and closed, or of all regular open sets) satisfies the condition*: $\mathfrak{d}(G) = \mathfrak{n}$.

PROOF. Consider any set S of power $\aleph$. By Theorem 2 there is a field $\mathfrak{F}$ of subsets of S such that $\mathfrak{d}(\mathfrak{F}) = \mathfrak{n}$ and that every family $\mathfrak{H} \subseteq \mathfrak{F}$ of mutually exclusive sets has power $<\mathfrak{n}$. We can assume that $S \in \mathfrak{F}$; for otherwise, we could replace $\mathfrak{F}$ by the field $\mathfrak{F}'$ consisting of all the sets $X \in \mathfrak{F}$ and their complements $S - X$, and we could easily show that $\mathfrak{F}'$ still satisfies the conclusion of Theorem 2. Now we correlate with every subset $X \in S$ a new subset $\bar{X} \in S$, the closure of X, by defining $\bar{X}$ as the intersection of all the sets $Y \in \mathfrak{F}$ which contain $X (Y \supseteq X)$. It is easily seen that with this definition S becomes a topological space.

In this space, F is contained in the family $\mathfrak{G}'$ of all those sets which are both open and closed, and the family G of all open sets is constituted by all the unions of the sets $X \in \mathfrak{F}$. Hence it follows that $\mathfrak{d}(G) = \mathfrak{d}(F) = \mathfrak{n}$, and that $\mathfrak{G}$, like $\mathfrak{F}$, does not contain any family $\mathfrak{H}$ of mutually exclusive sets with power $\mathfrak{c}(\mathfrak{H}) = \mathfrak{n}$. Finally it may be noticed that $\mathfrak{G}'$ and the family $\mathfrak{G}''$ of all regular open sets also have these two properties, for we clearly have

$$F \subset \mathfrak{G}' \subset \mathfrak{G}'' \subset \mathfrak{G}.$$

COROLLARY 2. *If* $\mathfrak{n}$ *is a regular number* $>\aleph_0$, *then there exists a complete Boolean algebra* B *such that* $\mathfrak{d}(B) = \mathfrak{n}$.

PROOF. This corollary follows directly from Corollary 1 since, as is well known, the regular open sets of an arbitrary topological space form a complete Boolean algebra,[3] and any two elements of this algebra are disjoint if, and only if, they are disjoint in the usual set-theoretic sense.

Corollary 2 formulates a condition which is necessary for a cardinal number $\mathfrak{n}$ to be regular and $>\aleph_0$. From Theorem 1 it follows that this condition is at the same time a sufficient one. It is easily seen that in this necessary and sufficient condition the term "Complete Boolean algebra" can be replaced by "partially ordered set", "ring (or field) of sets", "family of all open sets of topological space," and so on. If we restrict ourselves to the case of limit numbers, we obtain a necessary and sufficient condition for a number $\mathfrak{n} > \aleph_0$ to be weekly inaccessible.

---

[3] This result was first stated in A. Tarski, Les fondements de la géometrie de corps, Commemoration of the first Polish Math. Congress, Kracow 1929, p. 42; see also G. Birkhoff op. cit. p. 102.

Finally we give a result of a more special nature:

COROLLARY 3. *The following two sentences are equivalent:*

(*i*) *In every topological space of power* $\leqq 2^{\aleph_0}$ *there exists a family of mutually exclusive open sets with a maximal power.*

(*ii*) *There is no weakly inaccessible number* $\mathfrak{n}$ *which is* $> \aleph_0$ *and* $\leqq 2^{\aleph_0}$.

PROOF. From Corollary 1 it follows immediately that (i) implies (ii); the implication in the opposite direction can be easily derived from Theorem 1.

GENERAL REMARKS ON INACCESSIBLE NUMBERS

In connection with the last corollary it should be noticed that the problem as to whether there exist weakly inaccessible numbers i.e. regular limit numbers which are $> \mathfrak{n}$ and $< 2^{\mathfrak{n}}$ for an infinite number $\mathfrak{n}$ is so far unsolved and probably can not be solved at all within the present systems of general set theory. By definition the weakly inaccessible numbers can not be obtained from smaller ones by such operations as those of infinite addition or of passage from one number to the next greater number. However it is by no means settled that they can not be obtained from smaller ones by means of the other arithmetical operations, namely multiplication and exponentiation. For this reason we single out among the weakly inaccessible numbers a more special class the so called strongly inaccessible numbers, i.e., the numbers which can not be obtained from smaller ones by an arithmetical operation. While e.g. $2^{\aleph_0}$ is clearly not a strongly inaccessible number, it is not known whether this number is weakly inaccessible.

If we enrich the axiom system of set theory by adding the so-called generalized hypothesis of Cantor (which asserts that there is no cardinal number $> \mathfrak{n}$ and $< 2^{\mathfrak{n}}$ for any infinite number $\mathfrak{n}$), we can easily show that the two kinds of inaccessible numbers coincide. However nothing compels us to regard the generalized hypothesis of Cantor as the only possible basis for set-theoretic investigations, and we can equally well consider the possibility of enriching the axioms of set theory by other axioms which contradict the hypothesis of Cantor. For instance it seems quite plausible that the following hypothesis would constitute a consistent and fertile addition to the set theoretical axioms:

Hypothesis of inaccessible numbers: For every infinite number n, $2^{\mathfrak{n}}$ is the smallest weakly inaccessible number $> \mathfrak{n}$.

Furthermore we should like to point out that many set theoretical problems are known at present whose solution involves the notion of an inaccessible number. The first problems of this kind were formulated more than thirty years ago; their number has however considerably increased in recent years. Like the problem solved in the present paper most of these problems can be presented in the following form: Is it true that a certain cardinal number $\mathfrak{n}$ has property P?

We want to give here a few examples of such problems:

PROBLEM 1. (*The representation problem.*) *Is it true that every* $\mathfrak{n}$*-additive and* $\mathfrak{n}$*-distributative Boolean algebra is isomorphic with an* $\mathfrak{n}$*-additive field of sets?* (A Boolean algebra B is called $\mathfrak{n}$-additive if for every set $X \subseteq B$ with $\mathfrak{c}(X) < \mathfrak{n}$

there is an element $y \in B$ such that $y = \bigcup_{x \in X} x$. An $\mathfrak{n}$-additive Boolean algebra is called $\mathfrak{n}$-distributive if

$$\bigcap_{i \in \mathfrak{H}} \bigcup_{j \in \mathfrak{G}_i} x_{i,j} = \bigcup_{f} \bigcap_{i \in \mathfrak{H}} x_{i,f(i)},$$

where $\mathfrak{H}$ is any non-empty set with $\mathfrak{c}(\mathfrak{H}) < \mathfrak{n}$; $\mathfrak{G}_i$ (for $i \in \mathfrak{H}$) are any non-empty sets with $\mathfrak{c}(\mathfrak{G}_i) < \mathfrak{n}$; $x_{i,j}$ is always an element of B, and f runs through all functions which correlate with every element $i \in \mathfrak{H}$ an element $j_i \in \mathfrak{G}_i$. The number of functions f is in general $\geqq \mathfrak{n}$, but it is assumed that the existence of $\bigcap_{i \in H} \bigcup_{j \in j_i} X_{i,j}$ implies that of $\bigcup_f \bigcap_{i \in H} x_{i,f(i)}$.)

PROBLEM 2. (*The prime ideal problem.*) *Is it true that the field of all subsets of a set N with power $\mathfrak{c}(N) = \mathfrak{n}$ contains an $\mathfrak{n}$-additive ideal which is not a principal ideal?* (A family $\mathfrak{F}$ is called n-additive if for every family $\mathfrak{G} \subseteq \mathfrak{F}$ with $\mathfrak{c}(\mathfrak{G}) < \mathfrak{n}$ the union $\bigcup_{X \in \mathfrak{G}} X$ also belongs to $\mathfrak{F}$.)

This problem can also be formulated as that of the existence of an $\mathfrak{n}$-additive non trivial two-valued measure defined over all the subsets of a set N with $\mathfrak{c}(N) = \mathfrak{n}$.

PROBLEM 3. (*The set-function problem.*) *Is it true that there exists an $\mathfrak{n}$-additive and $\mathfrak{n}$-multiplicative set-function defined over all sybsets of a set N of power $\mathfrak{n}$, which is not absolutely additive and absolutely multiplicative?* (By a set function we mean here a function G which correlates with every set X of a certain family $\mathfrak{F}$ another set $G(X)$ which need not belong to the same family. A set function G is called $\mathfrak{n}$-additive or $\mathfrak{n}$-multiplicative, if for every family $\mathfrak{H} \subseteq \mathfrak{F}$ with $\mathfrak{c}(\mathfrak{H}) < \mathfrak{n}$ we have:

$$G(\bigcup_{X \in \mathfrak{H}} X) = \bigcup_{X \in \mathfrak{H}} (G(X)) \quad \text{or} \quad G(\bigcap_{X \in \mathfrak{H}} X) = \bigcap_{X \in \mathfrak{H}} (G(X)),$$

respectively. If these formulas hold for every family $\mathfrak{H} \leqq \mathfrak{F}$, the set function is called absolutely additive or absolutely multiplicative.)

PROBLEM 4. (*The graph problem.*) *Is it true that if a complete graph G of power $\mathfrak{n}$ is split into two graphs G_1 and G_2, at least one of them contains a subgraph of power $\mathfrak{n}$?* (A graph is to be defined as an arbitrary set of non-ordered couples (x, y) with $x \neq y$. By a complete graph of power $\mathfrak{n}$ we mean the set of all such couples formed from the elements of a set N of power $\mathfrak{n}$.)

PROBLEM 5. (*The ordering problem.*) *Is it true that every ordered set N of power $\mathfrak{n}$ contains a subset X of power $\mathfrak{n}$, which is either well ordered, or becomes well ordered if we invert the ordering relation.*

PROBLEM 6. (*Ramification problem.*) *Let ν be the smallest ordinal number such that the power of all ordinals $\xi < \nu$ is $\mathfrak{n}$. Is it true that every ramification system of the ν^{th} order, in which the set of all elements of the ξ^{th} order has power $< \mathfrak{n}$ for every $\xi < \nu$, contains a well-ordered subset of the type ν.* (By a ramification system S we understand a partially ordered set which has the property that, for every $x \in S$, the set $S(x)$ of all elements $y \leqq x$ is well ordered; If the set $S(x)$ is of the type ξ the element x is said to be of the ξ^{th} order. The order of the whole set S is the smallest ordinal number greater than the order of all elements of S.)

None of these six problems has yet been entirely solved. It can be shown that the solution of these problems is positive for $\mathfrak{n} = \aleph_0$ and is negative for every infinite number $\mathfrak{n} > \aleph_0$ which is not strongly inaccessible, in the case of problems 1–5. (In the case of problem 6 it has been only shown that the solution is negative if $\mathfrak{n}$ is not inaccessible and the generalized Cantor hypothesis holds.) All the problems remain open in case of strongly inaccessible numbers $>\aleph_0$.[4]

This situation is rather typical of the problems involving the notion of an inaccesssible number, which we have here in mind. Most of them so far have resisted all attempts at solution in the case in which $\mathfrak{n}$ is an inaccessible number $>\aleph_0$; it depends, however, on the nature of the problem whether strongly or weakly inaccessible numbers are involved. Ths situation differs slightly in connection with certain problems from the theory of ordered sets. Here the solution is positive for $\aleph_0$, and for all regular numbers which are not weakly inaccessible, and is negative for infinite singular numbers; but the problem again remains open in the case of weakly inaccessible numbers $>\aleph_0$.

The difficulties which we meet in attempting to solve the problems under consideration do not seem to depend essentially on the nature of inaccessible

---

[4] The solution of Problem 1 was given for $\mathfrak{n} = \aleph_0$ by M. H. Stone (see G. Birkhoff op. cit. p. 89.) The solution for numbers which are not inaccessible and $> \aleph_0$ was recently found by A. Tarski and has not yet been published.

For the solution of Problem 2. see A. Tarski, Fund. Math. Vol. 15, p. 42–50. (Une contribution à la théorie de la mesure) (the case $\mathfrak{n} = \aleph_0$). For the case when $\mathfrak{n}$ is not inaccessible, see A. Tarski, Fund. Math. Vol. 30 (1938) p. 150 (Dritter überderkungnatz.)

The solution of Problem 3 for $\mathfrak{n} = \aleph_0$ was given by S. Ulam, Fund. Math. Vol. 16 p. 140–150. (Zur Masstheorie in der allgemeinen Mengenlehre.) The solution for numbers which are not inaccessible follows from a general theorem of A. Tarski; C. R. Soc. Varsovie, Vol. 30 p. 158 (Theorem 2.18).

The solution of Problem 4 was given for $\mathfrak{n} = \aleph_0$ by Ramsay on a problem of formal logic, Proc. London Math. Soc. (2), 30; and for the numbers $\mathfrak{n} > \aleph_0$ which are not inaccessible by P. Erdös, appear in Revista de Tucuman).

The solution of Problem 5 is obvious for $\mathfrak{n} = \aleph_0$; for the numbers $\mathfrak{n} > \aleph_0$ which are not inaccessible the solution was given by Hausdorff, Mengenlehre (1914) p. 145–146. He does not state the solution explicitly, but it can be deduced easily from his results.

The solution of Problem 6 was given for $\mathfrak{n} = \aleph_0$ by D. König, Über eine Schlussweise aus dem endlichen ins unendliche, Acta Szeged, 3, p. 121–130. For the numbers $\mathfrak{n} > \aleph_0$ which are not inaccessible it was given by Aronsajn. It can be shown that the positive solution of Problem 1 for inaccessible numbers $>\aleph_0$ would imply the positive solution of Problem 2; the positive solution of Problem 2 implies that of Problems 3, 4, and 5; also the positive solution of 3 implies that of 2, so that 2 and 3 are equivalent. Further, the positive solution of 4 implies that of 5, and the positive solution of Problem 6 for strongly inaccessible numbers can be deduced from that of Problem 2 (however this solution can also be obtained from weaker hypotheses and can be extended to all inaccessible numbers). Also the positive solution of Problem 6 implies that of Problems 4 and 5. Finally the positive solution of Problem 6 implies the positive solution of Problem 1 in the special case when the Boolean algebra contains only $\mathfrak{n}$ elements. The proof of these equivalences is as yet unpublished.

numbers. In most cases the difficulties seem to arise from lack of devices which enable us to construct maximal sets which are closed under certain infinite operations. It is quite possible that a complete solution of these problems would require new axioms which would differ considerably in their character not only from the usual axioms of set theory, but also from those hypotheses whose inclusion among the axioms has previously been discussed in the literature and mentioned previously in this paper (e.g., the existential axioms which secure the existence of inaccessible numbers, or from hypotheses like that of Cantor which establish arithmetical relations between the cardinal numbers.)

If we now compare the problem which has been actually solved in this paper with those which we have recently discussed, we see that the peculiarity of our problem consists in two facts. First, our problem has been solved for all cardinal numbers, although the inaccessible numbers are essentially involved in the solution. And secondly the number $\aleph_0$ behaves in the discussion of the problem like a singular limit number, and in a directly opposite way to the other regular or inaccessible numbers.

UNIVERSITY OF PENNSYLVANIA AND INSTITUTE FOR ADVANCED STUDY