SOME SET-THEORETICAL PROPERTIES OF GRAPHS

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Let $i \leq k \leq l$ be any integers. A theorem of Ramsey [1] states that there exists a function $f(i, k, l)$ such that if $n \geq f(i, k, l)$, and if we select, from each combination of order k of n elements, a combination of order i, then there exists a combination of order l all of whose combinations of order i have been selected.

All the proofs give very bad estimates for $f(i, k, l)$. If $i = 2$ the theorem of Ramsey can be formulated as theorem about graphs: Let $n \geq \varphi(k, l)$, and consider any graph having n points; then either the number of independent points is $\geq k$ or the graph contains a complete graph of order [2] l.

Szekeres' proof gives $\varphi(k, l) \leq \binom{k+l-2}{k-1}$. This is probably very far from the best possible value. We do not even know whether or not $\lim \frac{\varphi(3, l)}{l} < \infty$ is true. Perhaps even the following stronger result holds: There exists an integer c (independent of n) such that, given a graph without a triangle, we can number its vertices with the integers 1, 2, ... c, in such a way that no two vertices numbered with the same integer are connected. It is easy to see that $c \geq 4$.

Ramsey [3] also proved that if G is an infinite graph, then either G

[1] F. P. RAMSEY, *Collected papers. On a problem of formal logic*, 82-111. See also SKOLEM, *Fundamenta Math.*, 20 (1933), 254-261, and P. ERDÖS and G. SZEKERES, *Compositio Math.* 2 (1935), pp. 463-470.

[2] In a graph G a set A of points is called independent if no two points of A are joined by a line. A graph is complete if any two of its points are joined by a line.

[3] RAMSEY, *ibid.*

contains an infinite set of independent points or G contains an infinite complete graph.

If the number of vertices of G is not countable, Duschnik, Miller and I proved the following theorem [4] : Let the power of the points of G be m ; then either G contains an infinite complete graph, or G contains a set of m independent points. We can also state this theorem as follows : If we split the complete graph of m points into two subgraphs G_1 and G_2, then if G_1 does not contain an infinite complete graph, G_2 contains a set of m independent points.

In the present note we prove the following results :

Theorem I : Let a and b be infinite cardinals such that $b > a^a$. If we split the complete graph of power b into a sum of a subgraphs at least one of them contains a complete graph of power $> a$.

In particular: If $b > c$ (the power of the continuum) and we split the complete graph of power b into a countable sum of subgraphs; at least one subgraph contains a non denumerable complete graph.

Theorem I is best possible. As a matter of fact, if $b = a^a = 2^a$ we can split the complete graph of power b into the sum of a subgraphs, such that no one of them contains a triangle. For the sake of simplicity we show this only in the case $b = c = 2^{\aleph_0}$. We write

$$G = \sum_{k=1}^{\infty} G_k$$

where G is a graph connecting every two points of the interval $(0, 1)$, and the edges of G_k connect two points x and y if $\frac{1}{2^{k-1}} > y - x = \frac{1}{2^k}$.
Clearly none of the G_k's contains any triangles.

Let us now assume that the generalized continuum hypothesis is true, i.e. $2^{\aleph_x} = \aleph_{x+1}$. Let $m = \aleph_{x+2}$, and let G be the complete graph containing m points, then we prove

Theorem II : Put $G = G_1 + G_2$; if G_1 does not contain a complete graph of power m, then G_2 contains a complete graph of power $\aleph_{x+1}$. From theorem I it would only follow that either G_1 or G_2 contains a complete graph of power $\aleph_{x+1}$. By using results of paper of Sierpinski [5] it is not difficult to find a decomposition $G = G_1 + G_2$ such that

[4] Ben DUSCKNIK and E. W. MILLER, *Partially ordered sets*, *Amer. Journal of Math.*, 63 (1941), p. 606.

[5] W. SIERPINSKI, *Fundamenta Math.*, 5 (1924), p. 179.

neither G_1 nor G_2 contains a complete graph of power m, which shows that theorem II can not be improved. (We have to assume that m is accesible).

Tukey and I have shown by using a result of Sierpinski [6] that the complete graph of power $\aleph_1$ can be decomposed into the countable sum of trees. Without assuming the continuum hypothesis we can not decide whether this also holds for the complete graph of power $\aleph_2$.

Proof of theorem I. Let G be the complete graph of power b; write

$$G = \sum_{\alpha} G_\alpha, \qquad \alpha < \Omega_a,$$

where Ω_a denotes the least ordinal corresponding to the power a.

Let p be any point of G. We split the remaining points of G into a classes Q_{α_1}, $\alpha_1 < \Omega_a$, by the rule; — a point q is in Q_{α_1} if the line pq is in G_{α_1}. Take now an arbitrary point $p_{\alpha_1} \subset Q_{\alpha_1}$ ($\alpha_1 = 1, 2 ..., \alpha_1 < \Omega_a$) and split the remaining points of Q_{α_1} into classes Q_{α_1, α_2}, $\alpha_2 < \Omega_a$, by the rule: — q belongs to Q_{α_1, α_2} if the line $p_{\alpha_1} q$ belongs to G_{α_2}. Next we take an arbitrary point p_{α_1, α_2} in Q_{α_1, α_2} and split the remaining points of Q_{α_1, α_2} into classes $Q_{\alpha_1, \alpha_2, \alpha_3}$, etc. If k is not a limit ordinal we define the classes $Q_{\alpha_1, \alpha_2 \dots \alpha_k}$ in the obvious way from the classes $Q_{\alpha_1, \alpha_2 \dots \alpha_{k-1}}$ ($\alpha_k < \Omega_a$). If k is a limit ordinal, we define the classes $Q_{\alpha_1, \alpha_2 \dots \alpha_i \dots}$ ($i < k$) as $\prod_{i<k} Q_{\alpha_1, \alpha_2 \dots \alpha_i}$. Our construction can stop only if for some k all the classes $Q_{\alpha_1, \alpha_2, \dots \alpha_k}$ become empty; in other words if all the points of G become $p_{\alpha_1, \alpha_2 \dots \alpha_i}$'s ($i < k$). Denote now by a^+ the smallest power $> a$, and by Ω_{a^+} the smallest ordinal belonging to a^+. We shall prove that not all the sets $Q_{\alpha_1, \alpha_2 \dots \alpha_i}$ ($i < \Omega_{a^+}$) can be empty. Clearly the power of the points $p_{\alpha_1, \alpha_2, \dots \alpha_i}$ ($i < \Omega_{a^+}$) does not exceed $a^+ . a^a = a^a$ (*i.e.* $a^a \geq a^+$). But the power of the points of G is by assumption $> a^a$; thus not all the points of G are $p_{\alpha_1, \alpha_2 \dots \alpha_i}$'s ($i < \Omega_{a^+}$). Let r be such a point, and consider the sets $Q_{\alpha_1, \alpha_2 \dots \alpha_i}$ ($i < \Omega_{a^+}$) with $r \subset Q_{\alpha_1, \alpha_2 \dots \alpha_i}$. Clearly $r \subset \prod_{i < \Omega_{a^+}} Q_{\alpha_1, \alpha_2 \dots \alpha_i}$ thus $Q_{\alpha_1, \alpha_2 \dots \alpha_i \dots}$ ($i < \Omega_{a^+}$) is non empty. If i is not a limit ordinal, α_i runs at most through a values ($\alpha_i < \Omega_a$) thus there must be an index j ($j < \Omega_a$) which occurs in $Q_{\alpha_1, \alpha_2 \dots}$ a^+ times. Clearly G_j contains a complete graph of power a^+. For let $j = \alpha_{i_1} = \alpha_{i_2} = \dots \alpha_{i_k} = \dots$. and consider the points $p_{\alpha_1, \alpha_2, \dots \alpha_{i_k - 1}}$. It is clear from our construction that the complete graph determined by these points is in G_j, this completes the proof of theorem I.

[6] W. Sierpinski, *ibid.*

Proof of theorem II. We state theorem II as follows : Let G be a graph containing $\aleph_{x+2}$ points. Then if each set of independent points has power $< \aleph_{x+2}$, our graph contains a complete graph of power $\aleph_{x+1}$.

Let $p_1, p_2, \ldots p_{\alpha_1} \ldots$ be a complete set of independent points $(\alpha_1 < \Omega_{\aleph_{x+1}}$. Clearly every other point of G is connected with at least one of the p's. The point q of G will belong to class Q_{α_1} if p_{α_1} is the p with smallest index with which q is connected. In each Q_{α_1} consider now a maximal system of independent points. Thus we obtain the points p_{α_1, α_2}, $\alpha_1, \alpha_2 < \Omega_{\aleph_{x+1}}$, and we split the remaining points of Q_{α_1} into classes as before ; the point $q \subset Q_{\alpha_1}$ belongs to Q_{α_1, α_2} if p_{α_1, α_2} is the point of lowest index with which q is connected. We can continue this process as in the proof of theorem I. We claim that this process can not stop in $\aleph_{x+1}$ steps, in other words, the sets $Q_{\alpha_1, \alpha_2, \ldots \alpha_j \ldots}, j < \Omega_{\aleph_{x+1}}$, can not all be empty. For if these sets were all empty, all points of G would be $p_{\alpha_1, \alpha_2, \ldots \alpha_j}$'s for some $j < \Omega_{\aleph_{x+1}}$. But the number of these points does not exceed $\aleph_{x+1} \aleph_{x+1}^{\aleph_x} = \aleph_{x+1}$, by the generalized hypothesis of the continuum.

Consider, then, a sequence of sets, $Q_{\alpha_1}, Q_{\alpha_1, \alpha_2}, \ldots, Q_{\alpha_1, \alpha_2, \ldots \alpha_j}, j < \Omega_{\aleph_{x+1}}$ whose intersection is non empty. Clearly our graph contains the complete graph determined by the points $p_{\alpha_1}, p_{\alpha_1, \alpha_2}, \ldots p_{\alpha_1, \alpha_2 \ldots \alpha_j}, j < \Omega_{\aleph_{x+1}}$ and this completes the proof of theorem II.

I do not know whether theorem II remains true if the power of the points of G is $\aleph_{x+1}$, where $\aleph_x$ is a limit cardinal.

If the power of the points of G is a limit cardinal e. g. $\aleph_\omega$ the theorem is certainly false. Let M be the set of points of G and write $M = \sum_{i=1}^{\infty} M_i$ where the power of M is $\aleph_\omega$. We define G as follows : Two points of G are connected if and only if they belong to the same M_i. Then clearly G does not contain a complete graph of power M, and every system of independent points is countable.

In general, let m be a limit cardinal, which is the sum of $\aleph_k$ sets of power $< m$, but not the sum of fewer than $\aleph_k$ such sets. Then we can construct a graph G the power of whose points is m, such that G does not contain a complete graph of power m, and every set of independent points has power $< \aleph_k$. On the other hand, perhaps the following result holds : If such a graph G does not contain a complete graph of power m, then it contains a set of independent points of power $\aleph_{k-1}$.

Let A be a set of power m, and let $n < m$. To every point $x \varepsilon A$, we correspond a subset $f(x)$ of A such that $x \bar{\epsilon} f(x)$, and the power of $f(x)$ is $< n$. A subset B of A is called *independent* if $B \frown f(B)$ is empty. If we assume the generalized continuum hypothesis we can prove that there always exists an independent set of power m. This result has been proved previously, without using the continuum hypothesis, in the cases: (I) m is not a limit cardinal; (II) m is a countable sum of smaller cardinals [7].

[7] D. LÁZÁR, *Fundamenta Math.*, 3 (1936), p. 304. SOPHIE PICCARD, *Fundamenta Math.*, 29 (1937), pp. 5-8, *C. R. Soc. Sc. Varsovie*, 30 (1937).