Annals of Mathematics
Vol. 43, No. 1, January, 1942

ON THE UNIFORM DISTRIBUTION OF THE ROOTS OF CERTAIN POLYNOMIALS

By P. Erdös

(Received July 21, 1941)

Let

$$\begin{array}{c} x_1^{(1)} \\ x_1^{(2)} \quad x_2^{(2)} \\ \cdots\cdots\cdots\cdots\cdots \\ x_1^{(n)} \quad x_2^{(n)} \cdots x_n^{(n)} \end{array}$$

be a triangular matrix, where, for each n,

$$1 \geqq x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)} \geqq -1.$$

Since $x_i^{(n)}$ may be written in the form $x_i^{(n)} = \cos(\vartheta_i^{(n)})$, where $0 \leqq \vartheta_i^{(n)} \leqq \pi$, we may define another triangular matrix

$$\begin{array}{c} \vartheta_1^{(1)} \\ \vartheta_1^{(2)} \quad \vartheta_2^{(2)} \\ \cdots\cdots\cdots\cdots\cdots \\ \vartheta_1^{(n)} \quad \vartheta_2^{(n)} \cdots \vartheta_n^{(n)} \end{array}$$

with

$$0 \leqq \vartheta_1^{(n)} < \vartheta_2^{(n)} < \cdots < \vartheta_n^{(n)} \leqq \pi.$$

Put $\omega_n(x) = \prod(x - x_i)$.[1] Suppose $0 \leqq A < B \leqq \pi$. We denote by $N_n(A, B)$ the number of the ϑ_i in (A, B). Let $-1 \leqq a < b \leqq 1$. Then we denote by $M_n(a, b)$ the number of the x_i in (a, b). It does not matter whether the intervals (A, B) and (a, b) are open or closed.

In a previous paper[2] Turán and the author proved that if

$$|\omega_n(x)| < \frac{f(n)}{2^n}$$

then

$$N_n(A, B) = \frac{B - A}{\pi} n + O(n^{\frac{1}{2}}(\log f(n))^{\frac{1}{2}}).$$

---

[1] We omit the upper index n where there is no danger of confusion.

[2] *On the uniformly dense distribution of certain sequences of points*, Annals of Math. Vol. 41 (1940), pp. 162–173.

In another paper[3] we proved that if $| l_k^{(n)}(x) | < c_1$ then

$$N_n(A, B) = \frac{B - A}{\pi} n + O[(B - A)n]^{\frac{1}{2}+\epsilon}.$$

($l_k^{(n)}(x)$ denotes the fundamental polynomials, i.e. $l_k^{(n)}(x) = \omega(x)/[\omega'(x_k)(x - x_k)]$ is of degree $n - 1$, and $l_k(x_k) = 1$, $l_k(x_i) = 0$, $i \neq k$.)

In the present paper we are going to improve these results. First we prove

THEOREM 1. *Put* $x_0 = -1$, $x_{n+1} = 1$, *and let*

$$(1) \qquad \max_{-1 \leq x \leq 1} | \omega_n(x) | < \frac{c_2}{2^n} \quad \text{and} \quad \max_{x_k \leq x \leq x_{k+1}} | \omega_n(x) | > \frac{c_3}{2^n}, \qquad k = 0, 1 \cdots n.$$

Then

$$N_n(A, B) = \frac{B - A}{\pi} n + O[\log n(B - A)].$$

This result is the best possible.

Next we prove

THEOREM 2. *Let* $| l_k(x) | < c_4$ *; then*

$$N_n(A, B) = \frac{B - A}{\pi} n + O[(\log n)(\log n(B - A))]$$

if $| l_k(x) | < n^{c_5}$, *then*

$$N_n(A, B) = \frac{B - A}{\pi} n + O[(\log n)^2].$$

Theorem 2 is also the best possible. Theorems 1 and 2 can be generalized to

THEOREM 3. *Let* $\omega(x)$ *be such that*

$$\frac{c_6 f(n)}{2^n} < \max_{x_k < x \leq x_{k+1}} | \omega_n(x) | < \frac{c_7 f(n)}{2^n} \qquad k = 0, 1, 2 \cdots n;$$

then

$$N_n(A, B) = \frac{B - A}{\pi} n + O[(\log n)(\log f(n))].$$

Similarly, if $| l_k(x) | < c_8 f(n)$ *then*

$$N_n(A, B) = \frac{B - A}{\pi} n + O[(\log n)(\log nf(n))].$$

To prove Theorem 1 we first have to prove two lemmas.

LEMMA 1. *Suppose that* (1) *holds; then*

$$(2) \qquad \frac{c_9}{n} < \vartheta_{k+1} - \vartheta_k < \frac{c_{10}}{n}, \qquad k = 0, 1 \cdots n.$$

[3] *On interpolation iii*, ibid. pp. 510–553.

PROOF. A theorem of M. Riesz states that if $h(x)$ is a polynomial of degree n which assumes its absolute maximum in $(-1, 1)$ at the point x_0, and if $y_1, \cdots, y_r$ are the roots of $h(y) = 0$ in the interval $(-1, 1)$, then $|\theta_i - \theta_0| \geqq \pi/2n$, where $\cos\theta_i = y_i$ and $\cos\theta_0 = x_0$. Thus if x_0 lies between the roots y_i and y_{i+1}, then $\theta_{i+1} - \theta_i \geqq \pi/n$. Also if $\max_{y_i \leqq x \leqq y_{i+1}} h(x)$ assumes its smallest value for $i = k$ then

$$\theta_{k+1} - \theta_k \leqq \pi/n.$$

Suppose that (2) does not hold, for example assume that

$$\vartheta_{k+1} - \vartheta_k > r(n)/n,$$

where $\overline{\lim}\, r(n) = \infty$. Take $\epsilon > 0$, and define u and v by the relations: u and v are symmetric with respect to $(x_k + x_{k+1})/2$, and $\arccos u - \arccos v = \pi/n + \epsilon$. Consider the polynomial $\phi(x) = \omega(x)\cdot(x - u)\cdot(x - v)/(x - x_k)(x - x_{k+1})$. It can be seen that if $u \leqq x \leqq v$ then

$$\frac{(x-u)(x-v)}{(x-x_k)(x-x_{k+1})} < c_{11}/r(n);$$

hence

$$\max_{u \leqq x \leqq v} |\phi(x)| < (c_{11}/r(n)) \max_{x_k \leqq x \leqq x_{k+1}} \omega_n(x). \tag{3}$$

Also, since the sum of two quantities whose sum is fixed increases as they tend to equality, we have, in the intervals $(-1, x_k)$ and $(x_{k+1}, 1)$,

$$|\phi(x)| > |\omega(x)|. \tag{4}$$

We have $\arccos v - \arccos u > \pi/n$; and a simple calculation shows that, if $r(n)$ is large enough, $\vartheta_{k+1} - \arccos v > \pi/n$ and $\arccos u - \vartheta_k > \pi/n$; thus it follows from the lemma of M. Riesz (applied to $\phi(x)$) that $\max |\phi(x)|$ between two consecutive roots of $\phi(x)$, assumes its smallest value between the roots x_i and x_{i+1}, where either $i \leqq k - 2$ or $i \geqq k + 2$. Thus, from (3) and (4),

$$\max_{x_k \leqq x \leqq x_{k+1}} |\omega(x)| > \frac{r(n)}{c_{11}} \cdot \min_{j=0,1,\cdots,n} \max_{x_j \leqq x \leqq x_{j+1}} |\omega(x)|.$$

This contradicts (1), which completes the proof. By the same argument we could prove the other inequality in (2).

COROLLARY. *We obtain from Lemma 1, by a simple computation, that*

$$\frac{c_{12}}{n}\cdot(1 - x_k^2)^{\frac{1}{2}} < x_{k+1} - x_k < \frac{c_{13}}{n}(1 - x_k^2)^{\frac{1}{2}}, \qquad (k = 1, 2, \cdots, n-1).$$

LEMMA 2. *Suppose that* (1) *holds; then for* $-1 \leqq x \leqq 1$,

$$|l_k(x)| < c_{14}\frac{(1 - x_k^2)^{\frac{1}{2}}}{(x - x_k)n}.$$

PROOF. We have $l_k(x) = \omega(x)/[\omega'(x_k)(x - x_k)]$; thus by (1) it suffices to show that

$$\omega'(x_k) > c_{15}\frac{n}{2^n(1 - x_k^2)^{\frac{1}{2}}}.$$

Consider the polynomial $\psi(x) = \omega(x)/(x - x_k)$. It is clear that either $|\omega'(x_k)| = \psi(x_k) \geqq |\psi(y)|$ if $x_{k-1} \leqq y \leqq x_k$, or $|\omega'(x_k)| = \psi(x_k) \geqq \psi(y)$ if $x_k \leqq y \leqq x_{k+1}$. Without loss of generality we can assume that the first inequality holds. Then by (1) and the corollary to lemma 1 we have

$$\omega'(x_k) = \frac{\max\limits_{x_{k-1} \leqq x \leqq x_k} |\omega(x)|}{x_k - x_{k-1}} > c_{15}\frac{n}{2^n(1 - x_k^2)^{\frac{1}{2}}},$$

which completes the proof.

Now we can prove Theorem 1. To simplify the calculations we assume that $a = 0$, $b = 1$. Then we have to show that, assuming (1)

$$\frac{n}{2} - c_{16} \log n < M_n(0, 1) < \frac{n}{2} + c_{17} \log n.$$

It will be sufficient to prove the first inequality. Suppose that it does not hold; then

$$M_n(0, 1) < \frac{n}{2} - r(n) \log n, \qquad \overline{\lim}\, r(n) = \infty.$$

Consider the polynomial $g(x)$ whose roots are defined as follows: In the interval $(-1, \log n/n)$, $g(x)$ has the same roots as $T_{n-1}(x)$($T_n(x)$ denotes the n^{th} Tchebicheff polynomial); at the points $(\frac{3}{2})^r \log n/n$, $r = 1, 2, \cdots s$ where s is such that

$$\left(\frac{3}{2}\right)^s \frac{\log n}{n} \leqq 1 < \left(\frac{3}{2}\right)^{s+1} \frac{\log n}{n},$$

$g(x)$ has a root of multiplicity $\left[\frac{r(n)}{10}\right]$; and finally $g(x)$ vanishes at the roots of $\omega(x)$ in the interval $(0, 1)$. Clearly the degree of $g(x)$ does not exceed

$$\frac{n}{2} + \log n + \frac{3 \log n}{10} r(n) + \frac{n}{2} - r(n) \log n < n - 1$$

if $r(n) > 10$. Thus, by the lemma of M. Riesz, $g(x)$ assumes its absolute maximum in the interval $(\log n/n, 1)$. Suppose that it assumes its absolute maximum at x_0, $\log n/n \leqq x_0 \leqq 1$. We have for some r

$$\left(\frac{3}{2}\right)^r \frac{\log n}{n} < x_0 < \left(\frac{3}{2}\right)^{r+1} \frac{\log n}{n}.$$

(If $(\frac{3}{2})^{r+1} \log n/n > 1$, we replace it by 1.) Put $(\frac{3}{2})^r \log n/n = q$; we consider the polynomial

$$g_1(x) = \frac{g(x)}{(x-q)^p}, \qquad p = \left[\frac{r(n)}{10}\right].$$

By the Lagrange interpolation formula we evidently have

$$g_1(x) = \sum_{k=1}^{n} g_1(x_k) l_k(x)$$

where the x_k are the roots of $\omega(x)$. Thus

$$g_1(x_0) = \sum_{k=1}^{n} g_1(x_k) l_k(x_0). \tag{5}$$

Now $g_1(x_k) = 0$ for $0 \leqq x_k \leqq 1$; and since x_0 was the place where $g(x)$ takes its absolute maximum, we have

$$g_1(x_0) \geqq [2(t+1)]^p g_1(x)$$

if x satisfies

$$-\frac{\log n}{n} t\left(\frac{3}{2}\right)^r \geqq x \geqq -\frac{\log n}{n}(t+1)\left(\frac{3}{2}\right)^r; \quad t = 0, 1, 2 \cdots \tag{6}$$

(6) may be verified by noting that

$$g_1(x_0) = \frac{g(x_0)}{(x_0-q)^p} \geqq \frac{g(x)}{(x_0-q)^p} = g_1(x)\left(\frac{x-q}{x_0-q}\right)^p.$$

Hence from (5) and (6), by putting

$$-\frac{\log n}{n} t\left(\frac{3}{2}\right)^r = u_t,$$

we obtain

$$1 \leqq \sum_{t \geqq 0} \frac{M_n(u_t, u_{t+1}) \max\limits_{u_t \geqq x_k \geqq u_{t+1}} |l_k(x_0)|}{[2(t+1)]^p} = \sum\nolimits_1 + \sum\nolimits_2$$

where in $\sum_1$ t is restricted by $u_t \geqq -\frac{1}{2}$. Now by the corollary to Lemma 1, and Lemma 2.

$$\sum\nolimits_1 < \sum_{t \geqq 0} c_{18} n(u_{t+1} - u_t) c_{19} \frac{1}{n_{t+1}} \frac{1}{[2(t+1)]^p} < c_{20} \sum_{t \geqq 0} \frac{1}{[2(t+1)]^p} < \frac{1}{2}$$

for sufficiently large p.

For the x_k in $\sum_2$ we clearly have $x_k < -\frac{1}{3}$. Thus by lemma 2.

$$\sum\nolimits_2 < c_{21} \frac{n}{2^p} \max_{x_k < -\frac{1}{3}} |l_k(x_0)| < \frac{1}{2}$$

for sufficiently large p. Thus $\sum_1 + \sum_2 < 1$, and this contradiction establishes the proof.

In the proof we did not use the full strength of Lemma 2; in fact we only used

$|l_k(x)| < c_{22}\dfrac{1}{n|x - x_k|}$. We would have had to use the sharper estimate if we had not restricted ourselves to the interval (0, 1) but had considered a "small" interval near -1 or $+1$.

Now we have to prove that the error term in Theorem 1 is the best possible. Put

$$\vartheta_0 = \frac{\pi}{2}, \qquad \vartheta_k = \frac{\pi}{2} + \frac{k\pi}{n} + \sum_{i=1}^{k} \frac{1}{i}, \qquad \vartheta_l = \frac{\pi}{2} - \frac{l\pi}{n} - \sum_{i=1}^{l} \frac{1}{i}$$

where k and l take all positive integral values such that $\vartheta_k < \pi - n^{-2}$, and $\vartheta_l > n^{-2}$ it is easy to see that the number of the ϑ's is $n + O(1)$. Consider the polynomial $\omega(x)$ whose roots are the cos ϑ's. It can be shown by elementary computations that $\omega(x)$ satisfies (1). We do not give the details. On the other hand it is easy to see that

$$M_n(0, 1) < \frac{n}{2} - c_{23} \log n$$

which shows that the error term in Theorem 1 is the best possible.

The proof of Theorem 2 is very similar to that of Theorem 1. The difference is that, in defining $g(x)$, $g(x)$ now has roots of order $\left[\dfrac{r(n) \log n}{10}\right]$ at the points $(\frac{3}{2})^r \log n/n$. The proof of Theorem 3 also runs along the same lines.

UNIVERSITY OF PENNSYLVANIA