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THE DIMENSION OF THE RATIONAL POINTS IN HILBERT SPACE

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Let H denote the Hilbert-space consisting of all sequences of real numbers

$$\xi = (x_1, x_2, \cdots) \tag{1}$$

such that

$$\sum_{i=1}^{\infty} x_i^2 < \infty$$

with the distance defined as usual. R will denote the set of points of H having all coordinates rational. R_0 will denote the set of points of H of the form

$$\nu = \left(\frac{1}{n_1}, \frac{1}{n_2}, \cdots\right) \tag{2}$$

where n_i are positive integers.

Let $R_1 = \bar{R}_0$. Clearly $R_0 \subset R_1 \subset R$.

THEOREM.[1] Dim $R_0 =$ dim $R_1 =$ dim $R = 1$.

Before we proceed with the proof let us remark that the cartesian product $R_1 \times R_1$ is homeomorphic to R_1. Hence we obtain that

There exists a metric separable complete space X such that dim $X =$ dim $X \times X = 1$.

This seems to be a new contribution to the "product problem"[2] of the theory of dimensions. It might also be worth noticing that R_1 is disconnected between any two of its points.

Proof that dim $R_0 > 0$. Let U be an open subset of H of diameter less than $\frac{1}{2}$ and such that $UR_0 \neq 0$. Let therefore (2) belong to U.

We shall define a sequence of natural numbers $m_1, m_2, \cdots$ such that

$$\nu_k = \left(\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_{k-1}}, \frac{1}{m_k}, \frac{1}{n_{k+1}}, \frac{1}{n_{k+2}}, \cdots\right) \in U \tag{3}$$

$$\mu_k = \left(\frac{1}{m_1}, \frac{1}{m_2}, \cdots, \frac{1}{m_{k-1}}, \frac{1}{m_k - 1}, \frac{1}{n_{k+1}}, \frac{1}{n_{k+2}}, \cdots\right) \notin U. \tag{4}$$

Suppose that the m_i are already defined for $i < k$. Let m_k be the smallest integer such that (3) holds. Since the diameter of U is less than $\frac{1}{2}$ it is clear that $m_k > 1$ and that (4) holds.

[1] The problem of computing the dimension of R was proposed to me by Professor W. Hurewicz.

[2] See L. Pontrjagin, C.R. Paris 190 (1930), p. 1105 and W. Hurewicz, Ann. of Math. 36 (1935), p. 194.

Since U is bounded there is a number N such that $|\nu_k| < N$. Therefore $\sum_{i=1}^{\infty}\left(\frac{1}{m_i}\right)^2 < \infty$ and the point

$$\mu = \left(\frac{1}{m_1}, \frac{1}{m_2}, \cdots\right)$$

is in R. Clearly $\mu = \lim \nu_k$. Since $|\nu_k - \mu_k| = \frac{1}{m_k(m_k - 1)}$ and $m_k \to \infty$ we have also $\mu = \lim \mu_k$. In view of (3) and (4) μ is then on the boundary of U.

This proves that R_0 has positive dimension at every one of its points. The same holds for R_1 and R.

Proof that $\dim R \leqq 1$. Let S be the sphere consisting of all points (1) such that

$$\sum_{i=1}^{\infty} x_i^2 = 1.$$

It is clearly sufficient to prove that $\dim R \cdot S = 0$. Let

$$\rho = (r_1, r_2, \cdots)$$

be any point of $R \cdot S$. Given any positive irrational number δ choose n so that

$$\sum_{i=n+1}^{\infty} r_i^2 < \delta. \tag{5}$$

Let V_δ be the set of all points (1) such that

$$\sum_{i=1}^{n} r_i x_i > 1 - \delta.$$

Clearly V_δ is an open set. If a point (1) is on the boundary of V_δ then

$$\sum_{i=1}^{n} r_i x_i = 1 - \delta$$

hence $x_1, x_2, \cdots, x_n$ cannot all be rational. We have proved therefore that the boundary of V_δ contains no point of R.

Since $\rho \in S$ we have $\sum_{i=1}^{\infty} r_i^2 = 1$, therefore by (5) $\sum_{i=1}^{n} r_i^2 > 1 - \delta$ and $\rho \in V_\delta$.

To finish the proof it is therefore sufficient to prove that the diameter of $S \cdot V_\delta$ tends to zero as δ tends to zero. Let

$$\xi = (x_1, x_2, \cdots)$$

be a point if $S \cdot V_\delta$. We have then

$$\sum_{i=1}^{\infty} (r_i - x_i)^2 = \sum_{i=1}^{\infty} r_i^2 + \sum_{i=1}^{\infty} x_i^2 - 2\sum_{i=1}^{n} r_i x_i - 2\sum_{i=n+1}^{\infty} r_i x_i.$$

Since ρ and ξ are in S and ξ is in V_δ therefore

$$\sum_{i=1}^{\infty} (r_i - x_i)^2 < 1 + 1 - 2(1 - \delta) - 2 \sum_{i=n+1}^{\infty} r_i x_i.$$

Using Schwarz's inequality and (5) we have

$$\sum_{i=n+1}^{\infty} r_i x_i \leqq \left(\sum_{i=1}^{\infty} r_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} x_i^2\right)^{\frac{1}{2}} < \delta^{\frac{1}{2}}.$$

Therefore

$$\sum_{i=1}^{\infty} (r_i - x_i)^2 < 2\delta + 2\delta^{\frac{1}{2}}$$

which completes the proof.

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