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ON THE UNIFORMLY-DENSE DISTRIBUTION OF CERTAIN SEQUENCES OF POINTS

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Let

$$\mathfrak{M}_1 \equiv \left\{ \begin{matrix} \varphi_1^{(1)} & & \\ \varphi_1^{(2)}, \varphi_2^{(2)} & & \\ \vdots & & \\ \varphi_1^{(n)}, & \cdots & \varphi_n^{(n)} \\ \vdots & & \ddots \end{matrix} \right\} \tag{1}$$

be a sequence of *numbers*, where for all n

$$0 \leqq \varphi_1^{(n)} < \varphi_2^{(n)} < \cdots < \varphi_n^{(n)} \leqq \pi.$$

Weyl[1] calls such a sequence uniformly-dense in $[0, 2\pi]$, if for every subinterval $[\alpha, \beta]$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{\nu \\ \alpha \leqq \varphi_\nu^{(n)} \leqq \beta}} 1 = \frac{\beta - \alpha}{\pi}. \tag{2}$$

Weyl proved that the sequence $\mathfrak{M}_1$ is uniformly dense in $[0, 2\pi]$ if and only if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} e^{ki\varphi_\nu^{(n)}} = 0 \tag{3}$$

for every integer k.

Suppose we are given on the unit circle of the complex z-plane a sequence of *points*

$$\mathfrak{M}_2 \equiv \left\{ \begin{matrix} z_1^{(1)} & & & \\ z_1^{(2)}, z_2^{(2)} & & & \\ \vdots & & & \\ z_1^{(n)}, & z_2^{(n)}, & \cdots & z_n^{(n)} \\ \vdots & \vdots & & \vdots \ddots \end{matrix} \right\} \equiv \left\{ \begin{matrix} e^{i\varphi_1^{(1)}} & & \\ e^{i\varphi_1^{(2)}}, e^{i\varphi_2^{(2)}} & & \\ \vdots & & \\ e^{i\varphi_1^{(n)}}, & \cdots & e^{i\varphi_n^{(n)}} \\ \vdots & & \ddots \end{matrix} \right\}$$

with

$$0 \leqq \varphi_1^{(n)} < \varphi_2^{(n)} < \cdots < \varphi_n^{(n)} \leqq 2\pi$$

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[1] H. Weyl: *Über die Gleichverteilung von Zahlen mod Eins*, Math. Ann., 1916, Bd. 77, pp. 313–352.

for all n. The sequence is called uniformly dense on the unit circle, if for any arc of the length l

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\substack{\nu \\ z_\nu^{(n)} \prec l}} 1 = \frac{l}{2\pi}, \tag{4}$$

holds. It is satisfied if and only if the sequence of *numbers*

$$\left\{\begin{matrix} \varphi_1^{(1)} & & \\ \vdots & & \\ \varphi_1^{(n)} & \cdots & \varphi_n^{(n)} \\ \vdots & & \ddots \end{matrix}\right\}$$

is, in the sense of (2), uniformly dense in $[0, 2\pi]$.

Let a closed Jordan-curve l of the complex ζ-plane be given. The sequence of points

$$\mathfrak{M}_3 \equiv \left\{\begin{matrix} \zeta_1^{(1)} & & & \\ \vdots & & & \\ \zeta_1^{(n)}, & \zeta_2^{(n)}, & \cdots & \zeta_n^{(n)} \\ \vdots & \vdots & & \ddots \end{matrix}\right\}$$

lying on l is called uniformly dense if, mapping the exterior of l schlicht-conformally and the periphery continuously on the closed exterior of the unit circle of the z-plane, we obtain on the circumference of the unit circle a sequence of points

$$\left\{\begin{matrix} z_1^{(1)} & & & \\ \vdots & & & \\ z_1^{(n)}, & z_2^{(n)}, & \cdots & z_n^{(n)} \\ \vdots & \vdots & & \ddots \end{matrix}\right\} \tag{5}$$

uniformly dense in the sense of (4).[2]

We have to explain the case, when l degenerates into an open arc on the ζ-plane. In this case, in the same say as in (5), we obtain two $z_\nu^{(n)}$ belonging to one $\zeta_\nu^{(n)}$. Let l be e.g. the interval $[-1, +1]$; in this case the mapping function being $\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, the connection between the $\zeta_\nu^{(n)}$ and $z_\nu^{(n)}$, i.e. between $\zeta_\nu^{(n)}$ and $\varphi_\nu^{(n)}$, respectively, is

$$\zeta_\nu^{(n)} = \tfrac{1}{2}(e^{i\varphi_\nu^{(n)}} + e^{-i\varphi_\nu^{(n)}}) = \cos \varphi_\nu^{(n)}$$

$$\nu = 1, 2, \cdots n, \qquad n = 1, 2, \cdots, \qquad 0 \leqq \varphi_\nu^{(n)} < 2\pi.$$

[2] The uniform-dense distribution of points lying on ρ, but with potential theoretic characterization, occurs at first in the investigations of Hilbert; see D. Hilbert: *Über die Entwickelung einer beliebigen analytischen Function einer Variabeln* ⋯ , Nachrichten von der Königlichen Gesellschaft Göttingen, 1897, pp. 63–70. The above formalization, which is equivalent to Hilbert's, is due to L. Fejèr, see L. Fejèr: *Interpolation und konforme Abbildung*, Nachrichten van der Kön. Ges. Göttingen, 1918, pp. 319–331.

For fixed ν and n, $\varphi_\nu^{(n)}$ has two values, which lie symmetrically with respect to $\varphi = \pi$; so *we call the sequence of points*

$$(6) \qquad \begin{Bmatrix} \zeta_1^{(1)} & & \\ \vdots & & \\ \zeta_1^{(n)}, & \cdots & \zeta_n^{(n)} \\ \vdots & & \ddots \end{Bmatrix}$$

lying on $(-1, +1]$, *with*

$$1 \geqq \zeta_1^{(n)} > \zeta_2^{(n)} > \cdots > \zeta_n^{(n)} \geqq -1, \qquad n = 1, 2, \cdots,$$

uniformly dense, if for the sequence of numbers

$$\begin{Bmatrix} \varphi_1^{(1)} & & & \\ \vdots & & & \\ \varphi_1^{(n)}, & \varphi_2^{(n)}, & \cdots & \varphi_n^{(n)} \\ \vdots & \vdots & & \ddots \end{Bmatrix}$$

defined by[3]

$$\zeta_\nu^{(n)} = \cos \varphi_\nu^{(n)}, \; 0 \leqq \varphi_\nu^{(n)} \leqq \pi, \; \nu = 1, 2, \cdots n, \; n = 1, 2, \cdots,$$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\substack{\nu \\ \alpha \leqq \varphi_\nu^{(n)} \leqq \beta}} 1 = \frac{\beta - \alpha}{\pi},$$

holds, where $[\alpha, \beta]$ *means an arbitrary subinterval of* $[0, \pi]$.

To this definition, appearing artificial at the first moment, we may give geometric sense in the following way: Let us draw upon $[-1, +1]$ a semicircle and project the point $x_\nu^{(n)}$ upon the circle and obtain $A_\nu^{(n)}$. Then $\varphi_\nu^{(n)}$ means clearly the angle between the positive real axis and $0A_\nu^{(n)}$. So the above definition means that a sequence of points in $[-1, +1]$ is here called uniformly dense if, *projected upon the unit circle, the projections* are uniformly distributed. According to this definition the most uniform distribution is the case $\varphi_\nu^{(n)} = \nu\pi/(n + 1)$, $(\nu = 1, 2, \cdots n, \; n = 1, 2, \cdots)$, attained when the sequence of points in $[-1, +1]$ is $\zeta_\nu^{(n)} = \cos \nu\pi/(n + 1)$ $(\nu = 1, 2, \cdots, n, \; n = 1, 2, \cdots)$. Let us observe that for this sequence

$$\omega_n(\zeta) = \prod_{\nu=1}^{n} (\zeta - \zeta_\nu^{(n)}) = U_n(\zeta),$$

where $U_n(\zeta)$ are the Tchebysheff-polynomials of second kind; U_n $(\cos \vartheta)$ differs only in a factor from $\sin (n + 1)\theta/\sin \theta$, which is independent of ϑ; these polynomials are well known by their many important extremal properties. This

[3] Clearly $0 \leqq \varphi_1^{(n)} < \varphi_2^{(n)} < \cdots < \varphi_n^{(n)} \leqq \pi$.

holds also in the important case $\varphi_\nu^{(n)} = \frac{2\nu - 1}{2n}\pi$. In both cases exactly $\left[\frac{\beta - \alpha}{\pi} n\right] \varphi_\nu^{(n)}$ (n fixed), fall in any subinterval $[\alpha, \beta]$ of $[0, \pi]$; the bracket in the last expression means the largest integer contained in $\frac{\beta - \alpha}{\pi} n$.

According to Weyl's criterion the uniformly dense distribution of a sequence of points is assured by the asymptotic behaviour of certain sequences associated with it. Fekete[4] gives another criterion of the uniform dense distribution; he forms with the sequence of points $\mathfrak{M}_3$ the sequence of polynomials

$$\omega_n(z) = \prod_{\nu=1}^{n} (z - z_\nu^{(n)}) \qquad (n = 1, 2, \cdots)$$

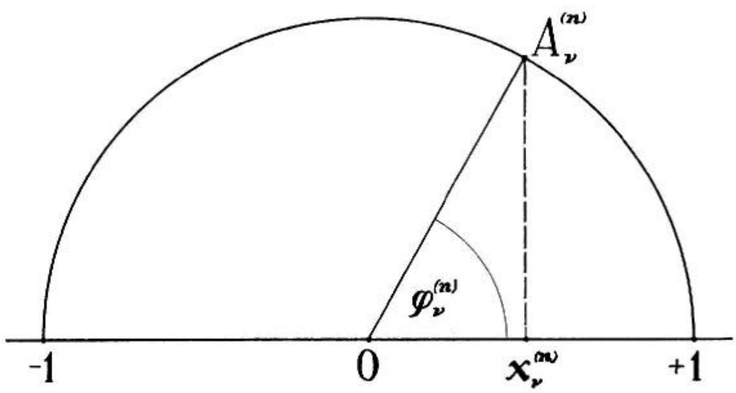

FIG. 1

and shows that $\mathfrak{M}_3$ is uniformly distributed upon l when and only when at every fixed point z_0 of l

$$\overline{\lim_{n \to \infty}} \left(| \omega_n(z_0) |\right)^{1/n} \leqq M,$$

where M is the so-called transfinite diameter[5] of l. If l is the interval $[-1 +1]$ then $M = \frac{1}{2}$; if l is a circle, the radius of which is 1, then $M = 1$. Thus Fekete's criterion requires instead of asymptotic equalities only inequalities.

Let us now consider the special case when l is the interval $[-1, +1]$. In this case we stated that the ideal case is obtained when $\left[\frac{\beta - \alpha}{\pi} n\right]$ numbers $\varphi_\nu^{(n)}$, (n fixed), fall in any subinterval $[\alpha, \beta]$ of $[0, \pi]$. The above mentioned theorems give no account of the measure of the deviation from this ideal case. In this direction we prove the following

[4] We know this theorem only from an oral communication.

[5] This notion was introduced by M. Fekete: *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen, etc.*, Math. Zeit., 1923, pp. 228–249.

THEOREM: *Let the sequence of points*

$$\mathfrak{M} \equiv \begin{Bmatrix} \zeta_1^{(1)} & & & \\ \vdots & \ddots & & \\ \zeta_1^{(n)}, & \zeta_2^{(n)}, & \cdots & \zeta_n^{(n)} \\ \vdots & \vdots & & \ddots \end{Bmatrix}$$

with

$$1 \geqq \zeta_1^{(n)} > \zeta_2^{(n)} > \cdots > \zeta_n^{(n)} \geqq -1$$

be given. Let us construct the matrix

$$\mathfrak{M}' \equiv \begin{Bmatrix} \varphi_1^{(1)} & & & \\ \vdots & \ddots & & \\ \varphi_1^{(n)}, & \varphi_2^{(n)}, & \cdots & \varphi_n^{(n)} \\ \vdots & \vdots & & \ddots \end{Bmatrix}$$

with

$$\zeta_\nu^{(n)} = \cos \varphi_\nu^{(n)}, \qquad 0 \leqq \varphi_\nu^{(n)} \leqq \pi, \qquad \nu = 1, 2, \cdots n, \qquad n = 1, 2, \cdots$$

If for the polynomials $\omega_n(\zeta) = \prod_{\nu=1}^{n} (\zeta - \zeta_\nu^{(n)})$ *the inequality*

$$(7) \qquad | \omega_n(\zeta) | \leqq \frac{A(n)}{2^n}, \qquad -1 \leqq \zeta \leqq +1, n = 1, 2, \cdots,$$

holds, then for every subinterval $[\alpha, \beta]$ *of* $[0, \pi]$ *we have*

$$(8) \qquad \left| \sum_{\substack{\nu \\ \alpha \leqq \varphi_\nu^{(n)} \leqq \beta}} 1 - \frac{\beta - \alpha}{\pi} n \right| < \frac{8}{\log 3} (n \log A(n))^{\frac{1}{2}}.$$

Our proof differs thoroughly from that of Fekete and besides it is elementary. $A(n)$ in (7) denotes any function of n tending monotonically to infinity and for which, following Tchebysheff, $A(n) \geqq 2$.

The proof requires a theorem of M. Riesz,[6] the proof of which is so short that for sake of completeness we may reproduce it as a

LEMMA. *Let the trigonometric polynomial* $f(\varphi)$ *of order* n *take its absolute maximum in* $[0, 2\pi]$ *at* $\varphi = \varphi_0$; *then the distance of the next root from this* φ_0 *is at least* $\pi/2n$ *to the right or to the left. Thus a fortiori: if* $f(\varphi)$ *takes its absolute maximum between two real roots, then the distance between these roots is* $\geqq \pi/n$.

PROOF. Suppose that the theorem is false. Without any loss of generality we may assume $\varphi_0 = 0$, there is a maximum at $\varphi = \varphi_0$ and the value of this maximum is 1. Suppose, that the next root lies to the right. Then for $\varphi = 0$

[6] M. Riesz: *Eine trigonometrische Interpolations-formel etc.*, Jahresbericht der deutschen Mathematikerver, 1915, Bd. 23, pp. 354–368.

the curves $y = f(\varphi)$ and $y = \cos n\varphi$ would have at least one point of intersection and according to the supposition at least three in $[0, \pi/n]$. There is evidently at least one point of intersection at each of the intervals $[\pi/n, 2\pi/n]$, $[2\pi/n, 3\pi/n]$, $\cdots$ $[(2n-2)\pi/n, (2n-1)\pi/n]$ too; thus the trigonometric polynomial $f(\varphi) - \cos n\varphi$, of order n, would have in $[0, 2\pi]$ $(2n+1)$ roots, an obvious impossibility.

Now we proceed to prove our theorem. It will be sufficient to prove the upper estimate, for if we have proved

$$\sum_{\substack{\nu \\ \alpha \leq \varphi_\nu^{(n)} \leq \beta}} 1 - \frac{\beta - \alpha}{\pi} n < \frac{4}{\log 3} (n \log A(n))^{\frac{1}{2}}$$

for every subinterval $[\alpha, \beta]$ between 0 and π, then

$$\sum_{\substack{\nu \\ 0 \leq \varphi_\nu^{(n)} \leq \alpha}} 1 - \frac{\alpha}{\pi} n < \frac{4}{\log 3} (n \log A(n))^{\frac{1}{2}} \tag{9a}$$

$$\sum_{\substack{\nu \\ \beta \leq \varphi_\nu^{(n)} \leq \pi}} 1 - \frac{\pi - \beta}{\pi} n < \frac{4}{\log 3} (n \log A(n))^{\frac{1}{2}} \tag{9b}$$

i.e. from (9a) and (9b)

$$\sum_{\substack{\nu \\ \alpha \leq \varphi_\nu^{(n)} \leq \beta}} 1 = n - \sum_{\substack{\nu \\ 0 \leq \varphi_\nu^{(n)} \leq \alpha}} 1 - \sum_{\substack{\nu \\ \beta \leq \varphi_\nu^{(n)} \leq \pi}} 1 > \frac{\beta - \alpha}{\pi} - \frac{8}{\log 3} (n \log A(n))^{\frac{1}{2}},$$

which implies the lower estimate.

Let us now consider the upper estimate. Let $[\alpha, \beta]$ be any subinterval of $[0, \pi]$, but now we regard it as fixed. Let $k = \left[\frac{\beta - \alpha}{\pi} n\right]$, let l be any positive integer, and consider the following extremum problem of the Tchebysheff type: determine the minimum of the absolute maxima of the polynomials $f(\zeta) = \zeta^n + a_1\zeta^{n-1} + \cdots + a_n$ taken in $[-1, +1]$ with the restriction, that $f(\zeta)$ has in the interval $[\cos \beta, \cos \alpha] \equiv [a, b]$, $k + 2l$ roots, (counted by their multiplicity), where as a matter of fact $k + 2l \leq n$. By a well known argument the existence of this polynomial is assured. We shall prove that the extremum polynomial $f_1(\zeta)$ takes its absolute maximum value with respect to $[-1, +1]$ in each of its root-intervals $[\zeta_{\nu+1}^{(n)}, \zeta_\nu^{(n)}]$ which is in the interior of $[a, b]$. For let the absolute value of the maximum of $f_1(\zeta)$ in $[-1, +1]$ be M and, in $[\zeta_{\nu+2}^{(n)}, \zeta_\nu^{(n)}]$, $|f_1(\zeta)| \leq M - \eta$, with $\eta > 0$. Then according to a fundamental theorem, for the polynomial

$$f_1^+(\zeta) \equiv \frac{f_1(\zeta)}{(\zeta - \zeta_\nu^{(n)})(\zeta - \zeta_{\nu+1}^{(n)})} (\zeta - \zeta_\nu^{(n)} - \epsilon)(\zeta - \zeta_{\nu+1}^{(n)} + \epsilon)$$

(where $\epsilon > 0$), we have in $[\zeta_{\nu+1}^{(n)} - \epsilon^{\frac{1}{3}}, \zeta_\nu^{(n)} + \epsilon^{\frac{1}{3}}]$ $|f_1^+(\zeta)| \leqq M - \frac{1}{2}\eta$, if ϵ sufficiently small. On the other hand in the exterior of $[\zeta_{\nu+1}^{(n)} - \epsilon^{\frac{1}{3}}, \zeta_\nu^{(n)} + \epsilon^{\frac{1}{3}}]$, since

$$f_1^+(\zeta) = f_1(\zeta)\left(1 - \frac{\epsilon}{\zeta - \zeta_\nu^{(n)}}\right)\left(1 + \frac{\epsilon}{\zeta - \zeta_{\nu+1}^{(n)}}\right)$$

$$= f_1(\zeta)\left\{1 - \frac{\epsilon(\zeta_\nu^{(n)} - \zeta_{\nu+1}^{(n)})}{(\zeta - \zeta_\nu^{(n)})(\zeta - \zeta_{\nu+1}^{(n)})} - \frac{\epsilon^2}{(\zeta - \zeta_\nu^{(n)})(\zeta - \zeta_{\nu+1}^{(n)})}\right\},$$

we have, for $\epsilon < \dfrac{\zeta_\nu^{(n)} - \zeta_{\nu+1}^{(n)}}{2}$,

$$|f_1^+(\zeta)| < |f_1(\zeta)| \left|1 - \frac{\epsilon}{2} \frac{\zeta_\nu^{(n)} - \zeta_{\nu+1}^{(n)}}{|\zeta - \zeta_\nu^{(n)}||\zeta - \zeta_{\nu+1}^{(n)}|}\right| < M \left|1 - \frac{\epsilon^{\frac{1}{3}}}{2}(\zeta_\nu^{(n)} - \zeta_{\nu+1}^{(n)})\right|.$$

This is less than M for sufficiently small ϵ, which contradicts the fact that $f_1(\zeta)$ is an extremum polynomial. Thus $f_1(\cos\vartheta)$ is a polynomial of order n the $k + 2l$ roots of which in $[\alpha, \beta]$ determine intervals such that in each of these intervals $f(\cos\vartheta)$ takes its absolute maximum.

Now we apply the theorem of M. Riesz formulated in the Lemma. According to this $f(\cos\vartheta)$ cannot have in the interior of $[\alpha, \beta]$ more than $\left[\dfrac{\beta - \alpha}{\pi} n\right] = k$ roots i.e. $2l$ roots must be located at the borders and consequently their multiplicity must be l at least at one of the borders.

By the premise, by the definition of $f_1(\zeta)$, and by the above, we obtain that if $\omega_n(\zeta)$ has $k + 2l$ roots in $[a, b]$, then

$$\frac{A(n)}{2^n} \geqq \max_{-1 \leqq \zeta \leqq +1} |\omega_n(\zeta)| \geqq \max_{-1 \leqq \zeta \leqq +1} |f_1(\zeta)| \geqq \max_{-1 \leqq \zeta \leqq +1} |\psi_1(\zeta)|, \tag{10}$$

where $\psi_1(\zeta) = \zeta^n + b_1\zeta^{n+1} + \cdots + b_n$ denotes the polynomial of degree n the absolute maximum of which is a minimum for polynomials of degree n having *somewhere* in $[-1, +1]$ one root with multiplicity l. As

$$\max_{-1 \leqq \zeta \leqq +1} |\psi_1(\zeta)| \geqq \left(\frac{1}{\pi}\int_{-1}^{1} \frac{|\psi_1(\zeta)|^2}{(1 - \zeta^2)^{\frac{1}{2}}} d\zeta\right)^{\frac{1}{2}}$$

we have, evidently, (by (10)),

$$\frac{A(n)}{2^n} \geqq \min_{\psi_2} \left(\frac{1}{\pi}\int_{-1}^{1} \frac{|\psi_2(\zeta)|^2}{(1 - \zeta^2)^{\frac{1}{2}}} d\zeta\right)^{\frac{1}{2}},$$

where $\psi_2(\zeta) = \zeta^n + \cdots$ runs over the polynomials of order n having somewhere in $[-1, +1]$ a root of the multiplicity l. Let $I_n(\zeta_0)$ denote the minimum value of $\displaystyle\int_{-1}^{1} \frac{|\psi_3(\zeta)|^2}{(1 - \zeta^2)^{\frac{1}{2}}} d\zeta$, if $\psi_3(\zeta) = \zeta^n + \cdots$ runs over the polynomials of degree n having *at a fixed* ζ_0 in $[-1, +1]$ a root of multiplicity l. Then we have

$$\frac{A(n)}{2^n} \geqq \min_{\substack{\psi_3 \\ -1 \leqq \zeta \leqq +1}} \left(\frac{1}{\pi} I_n(\zeta_0)\right)^{\frac{1}{2}}. \tag{11}$$

Let us now consider $I_n(\zeta_0)$. Every $\psi_3(\zeta)$ can be written in the form $(\zeta - \zeta_0)^l$ $\psi_4(\zeta)$, where $\psi_4(\zeta) = \zeta^{n-l} + \cdots$. Thus

$$I_n(\zeta_0) = \min_{\psi_4(\zeta)=\zeta^{n-l}+\cdots} \int_{-1}^{1} \frac{|\psi_4(\zeta)|^2 |\zeta - \zeta_0|^{2l}}{(1-\zeta^2)^{\frac{1}{2}}} \, d\zeta.$$

Let $\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$, which transforms $[-1, +1]$ into the upper part of the unit circle of the z-plane. Then—for $z = e^{i\varphi}$—

$$I_n(\zeta_0) = \min_{\psi_4(\zeta)=\zeta^{n-l}+\cdots} \frac{1}{2^{2l}} \int_0^{\pi} \left| \psi_4\left(\frac{z+\frac{1}{z}}{2}\right)\right|^2 \left| z + \frac{1}{z} - 2\zeta_0 \right|^{2l} d\varphi$$

$$= \min_{\psi_4(\zeta)=\zeta^{n-l}+\cdots} \frac{1}{2^{2l+1}} \int_0^{2\pi} \left| \psi_4\left(\frac{z+\frac{1}{z}}{2}\right)\right|^2 \left| z + \frac{1}{z} - 2\zeta_0 \right|^{2l} d\varphi$$

as $|z| = 1$, we have evidently for $\zeta_0 = \cos \alpha_0$

$$I_n(\zeta_0) = \frac{1}{2^{2n+1}} \min_{\psi_5(z)=z^{2n-2l}+\cdots} \int_0^{2\pi} |\psi_5(z)|^2 |z - e^{i\alpha_0}|^{2l} |z - e^{-i\alpha_0}|^{2l} d\varphi$$

$$= \frac{1}{2^{2n+1}} \min \int_0^{2\pi} |\psi_6(z)|^2 d\varphi,$$

where the last minimum is to be taken amongst the polynomials $\psi_6(z) = z^{2n} + \cdots$ of degree $2n$ having at $z = e^{i\alpha_0}$ and $z = e^{-i\alpha_0}$ roots of multiplicity l. Thus

$$I_n(\zeta_0) > \frac{1}{2^{2n+1}} \min_{\psi_7} \int_{|z|=1} |\psi_7(z)|^2 d\varphi,$$

where the minimum relates to the polynomials of degree $2n$ $\psi_7(z) = z^{2n} + \cdots$ having an l-fold root only at $z = e^{i\alpha_0}$. But in this case, since $\psi_7(z) = (z - e^{i\alpha_0})^l \psi_8(z)$, we have

$$I_n(\zeta_0) > \frac{1}{2^{2n+1}} \min_{\psi_8} \int_{|z|=1} |\psi_8(z)|^2 |z - e^{i\alpha_0}|^{2l} d\varphi,$$

where $\psi_8(z)$ runs over the polynomials of degree $(2n - l)$ beginning with z^{2n-l}. Finally by replacing φ by $\alpha_0 + \varphi + \pi$ we obtain

$$I_n(\zeta_0) > \frac{1}{2^{2n+1}} \min \int_{|z|=1} |\psi_9(z)|^2 |1 + z|^{2l} d\varphi, \tag{12}$$

the minimum being taken for all polynomials $\psi_9(z) = z^{2n-l} + \cdots$ of degree $(2n - l)$.

If $p(\varphi)$ defines in $[0, 2\pi]$ a non negative and L-integrable function then, after Szegö we may define a sequence of polynomials $\phi_0(z)$, $\phi_1(z)$, $\cdots$ such that

$$\phi_m(z) = z^m + \cdots \qquad m = 1, 2, \cdots \tag{13a}$$

$$\int_{|z|=1} \phi_m(z)(\bar{z})^{\nu} p(\varphi)\, d\varphi = 0 \qquad \nu = 0, 1, \cdots, (m-1); \; m = 1, 2, \cdots. \tag{13b}$$

In this case $\phi_m(z)$ minimizes, for polynomials $U(z)$ of degree m of the form $U(z) = z^m + \cdots$, the integral $\int_{|z|=1} |U(z)|^2 p(\varphi)\, d\varphi$. For any other such polynomial may be reduced the form $\phi_m(z) + \pi_{m-1}(z)$, where $\pi_{m-1}(z)$ is a polynomial of degree $(m - 1)$. Then by (13b)

$$\int_{|z|=1} |\phi_m(z) + \pi_{m-1}(z)|^2 p(\varphi)\, d\varphi$$
$$= \int_{|z|=1} |\phi_m(z)|^2 p(\varphi)\, d\varphi + 2\Re \int_{|z|=1} \phi_m(z)\overline{\pi_{m-1}(z)} p(\varphi)\, d\varphi + \int_{|z|=1} |\pi_{m-1}(z)|^2 p(\varphi)\, d\varphi$$
$$= \int_{|z|=1} |\phi_m(z)|^2 p(\varphi)\, d\varphi + \int_{|z|=1} |\pi_{m-1}(z)|^2 p(\varphi)\, d\varphi \geqq \int_{|z|=1} |\phi_m(z)|^2 p(\varphi)\, d\varphi,$$

and equality holds only for $\pi_{m-1}(z) \equiv 0$. The expression on the right side of (12) takes its minimum value for the $(2n - l)^{\text{th}}$ polynomial orthogonal to the weight-function $p(\varphi) = |1 + z|^{2l}$. According to a theorem of Szegö these polynomials may be expressed in terms of Jacobi polynomials but we prefer to present them in the form of an explicit integral interesting in itself. For $m = 2n - l$ we write

$$(14) \quad \phi_{2n-l}(z) = \frac{l\binom{2n+l}{l}}{(1+z)^{2l}} \int_{-1}^{z} (z-t)^{l-1}(1+t)^l t^{2n-l}\, dt \equiv \frac{l\binom{2n+l}{l}}{(1+z)^{2l}} F_{2n+l}(z).$$

This expression is a polynomial; we prove it by showing that $F_{2n+l}(-1) = F'_{2n+l}(-1) = \cdots = F^{(2l-1)}_{2n+l}(-1) = 0$. The first of these equations is an immediate consequence of (14). Since for $1 \leqq \nu \leqq l - 1$ we have by (14)

$$F^{(\nu)}_{2n+l}(z) = (l-1)(l-2)\cdots(l-\nu)\int_{-1}^{z} (z-t)^{l-\nu-1}(1+t)^l t^{2n-l}\, dt,$$

it is evident that the assertion holds for $1 \leqq \nu \leqq l - 1$. On the other hand

$$F^{(l)}_{2n+l}(z) = (l-1)!\,(1+z)^l z^{2n-l},$$

i.e. evidently $F^{(l)}_{2n+l}(-1) = \cdots = F^{(2l-1)}_{2n+l}(-1) = 0$. Hence the expression in (14) is a polynomial.

We now prove that the coefficient of z^{2n-l} in (14) equals 1. The coefficient in question is

$$l\binom{2n+l}{l} \lim_{z\to\infty} \frac{\int_{-1}^{z} (z-t)^{l-1}(1+t)^l t^{2n-l}\, dt}{z^{2n+l}}$$
$$= l\binom{2n+l}{l} \lim_{z\to\infty} \frac{\int_{-1}^{z} (z-t)^{l-1} t^{2n}\, dt}{z^{2n+l}},$$

which by the substitution $t = zw$ can be transformed into

$$l\binom{2n+l}{l}\lim_{z\to\infty}\int_{-1/z}^{\infty}(1-w)^{l-1}w^{2n}\,dw = l\binom{2n+l}{l}\int_0^{\infty}(1-w)^{l-1}w^{2n}\,dw = 1.$$

And now we have to verify the relation of orthogonality. Let

$$\int_{|z|=1}\phi_{2n-l}(z)(\bar z)^{\nu}\,|1+z|^{2l}\,d\varphi \equiv A_{\nu}. \qquad \nu = 0, 1, \cdots (2n-l-1).$$

Since, for $|z| = 1$,

$$|1+z|^{2l} = \frac{(1+z)^{2l}}{z^l}, \qquad \bar z = \frac{1}{z},$$

we have by (14)

$$A_{\nu} = l\binom{2n+l}{l}\int_{|z|=1}\frac{F_{2n+l}(z)}{z^{\nu+l}}\,d\varphi = \frac{l\binom{2n+l}{l}}{i}\int_{|z|=1}\frac{F_{2n+l}(z)}{z^{\nu+l+1}}\,dz.$$

Hence if in $F_{2n+l}(z)$ the coefficient of $z^l, z^{l+1}, \cdots, z^{2n-1}$ equals 0, (13b) is verified. But according to the definition of $F_{2n+l}(z)$

$$(15)\quad F_{2n+l}(z) = \int_{-1}^{0}(z-t)^{l-1}(1+t)^l t^{2n-l}\,dt + \int_0^z (z-t)^{l-1}(1+t)^l t^{2n-l}\,dt.$$

Here the first integral is a polynomial of z of degree $(l-1)$; thus it has no influence upon our assertion. The second one we transform by the substitution $t = zw$ into

$$z^{2n}\int_0^1(1-w)^{l-1}(1+2w)^l w^{2n-l}\,dw.$$

Thus the second term is a polynomial the *lowest* term of which is $2n$, which establishes the orthogonality.

The minimum value is given by

$$\int_{|z|=1}|\phi_{2n-l}(z)|^2\,|1+z|^{2l}\,d\varphi = \int_{|z|=1}\phi_{2n-l}(z)\{(\bar z)^{2n-l}+\cdots\}\,|1+z|^{2l}\,d\varphi$$

$$= \int_{|z|=1}\phi_{2n-l}(z)(\bar z)^{2n-l}\,|1+z|^{2l}\,d\varphi = \frac{l\binom{2n+l}{l}}{i}\int_{|z|=1}\frac{F_{2n+l}(z)}{z^{2n+1}}\,dz$$

$$= 2\pi l\binom{2n+l}{l}\times(\text{the coefficient of } z^{2n} \text{ in } F_{2n+l}(z)),$$

which by the form of $F_{2n+l}(z)$ in (15) equals

$$2\pi\binom{2n+l}{l}\int_0^1(1-w)^{l-1}w^{2n-l}\,dw = 2\pi\,\frac{\binom{2n+l}{l}}{\binom{2n}{l}}.$$

By this and (12)

$$I_n(\zeta_0) > \frac{\pi}{2^{2n}} \frac{\binom{2n+l}{l}}{\binom{2n}{l}},$$

i.e., by (11),

$$\frac{A(n)}{2^n} > \frac{1}{2^n}\left(\frac{\binom{2n+l}{l}}{\binom{2n}{l}}\right)^{\frac{1}{2}} = \frac{1}{2^n}\left(\prod_{\nu=0}^{l-1}\left(1+\frac{l}{2n-\nu}\right)\right)^{\frac{1}{2}}. \tag{16}$$

Since $\frac{l}{2n-\nu} \leqq 2$ and $\log(1+x) \geqq \frac{\log 3}{2}x$, (if $0 \leqq x \leqq 2$), we have, (by (16)),

$$A(n) > \operatorname{Exp}\left[\frac{\log 3}{4} l \sum_{\nu=2n-l+1}^{2n} \frac{1}{\nu}\right]$$

$$> \operatorname{Exp}\left[\frac{\log 3}{4} l \log\left(1+\frac{l}{2n-l+1}\right)\right] > \operatorname{Exp}\left[\left(\frac{\log 3}{4}\right)^2 \frac{l^2}{n}\right],$$

$$l < \frac{4}{\log 3}(n \log A(n))^{\frac{1}{2}},$$

which establishes the result.

NOTE I. For the Tchebysheff-polynomial $T_n(x)$, where $T_n(\cos\vartheta) = \frac{\cos n\vartheta}{2^{n-1}} = \cos^n\vartheta + \cdots$, we have in $[-1, +1]$ evidently $|T_n(x)| \leqq \frac{1}{2^{n-1}}$, i.e. $T_n(x)$ approximates the function $y \equiv 0$ in Tchebysheff's sense, the error being less then $\frac{2}{2^n}$. By the above argument it can be seen that the function $y \equiv 0$ is to be approximated not essentially worse, in Bessel's sense, by a polynomial of the form $x^n + \cdots$, even when the polynomial has somewhere in $[-1, +1]$ a root the multiplicity of which is less than $[\sqrt{n}]$. We are of the opinion that this very probably holds also for the Tchebysheff approximation; i.e. there exists a polynomial of degree n $f(x) = x^n + \cdots$, which has somewhere in $[-1, +1]$ a root of the multiplicity $[\sqrt{n}]$, and yet in $[-1, +1]$

$$|2^n f(x)| < B,$$

where B is independent of n. By this it is clear that in general, the above theorem is not to be improved.

NOTE II. Let $\omega_n(x) = x^n + \cdots$ be the polynomial of degree n minimizing for polynomials of degree n of the form $f(x) = x^n + \cdots$ the integral

$$I_k(f) \equiv \int_{-1}^{1} |f(x)|^k p(x)\,dx, \tag{17}$$

where k is a fixed positive number, $p(x)$ is L-integrable, and in $[-1, +1] p(x) \geqq m(> 0)$. According to a theorem of Fejér all roots of $\omega_n(x)$ are in $[-1, +1]$. Denote the absolute maximum of $|\omega_n(x)|$ by M; if this maximum is taken at $x = x_0$ and one of the intervals $\left[x_0 - \frac{1}{2n^2}, x_0\right], \left[x_0, x_0 + \frac{1}{2n^2}\right]$, suppose the latter, lies in $[-1, +1]$, then

$$\int_{-1}^{1} |\omega_n(x)|^k p(x)\,dx > m \int_{x_0}^{x_0+\frac{1}{2}n^{-2}} |\omega_n(x)|^k\,dx.$$

But by Markoff's theorem $|\omega_n(x)| > \frac{M}{2}$ in $\left[x_0, x_0 + \frac{1}{2n^2}\right]$, i.e.

$$\int_{-1}^{1} |\omega_n(x)|^k p(x)\,dx > m \left(\frac{M}{2}\right)^k \frac{1}{2n^2}. \tag{18}$$

On the other hand, if $f(x) = T_n(\cos\vartheta) = \frac{\cos n\vartheta}{2^{n-1}}$, we obtain by the minimum-property of $\omega_n(x)$

$$\int_{-1}^{1} |\omega_n(x)|^k p(x)\,dx < \frac{1}{2^{k(n-1)}} \int_{-1}^{1} p(x)\,dx, \tag{19}$$

i.e. by (18) and (19), for $[-1, +1]$ we have

$$|\omega_n(x)| \leqq M < \frac{2^{2+k^{-1}}}{m^{k^{-1}}} \left(\int_{-1}^{1} p(t)\,dt\right)^{k^{-1}} \frac{n^{2k^{-1}}}{2^n}.$$

Thus by the theorem mentioned above we obtain for the roots of polynomials minimizing the expressions in (17) that if the roots on the n^{th} polynomials are $x_1^{(n)}, x_2^{(n)} \cdots, x_n^{(n)}$, and $x_\nu^{(n)} = \cos\theta_\nu^{(n)}$, $0 \leqq \vartheta_\nu^{(n)} \leqq \pi$, then for any fixed subinterval $[\alpha, \beta]$

$$\left| \sum_{\substack{\nu \\ \alpha \leqq \varphi_\nu^{(n)} \leqq \beta}} 1 - \frac{\beta - \alpha}{\pi} n \right| < c(p, k)(n \log n)^{\frac{1}{2}}$$

i.e., roughly speaking, the distribution of the roots of the minimizing polynomials is uniformly dense. Analogous theorems are to be deduced for the polynomials solving extremum problems of the Tchebysheff-type.

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