ON A CONJECTURE OF STEINHAUS

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Steinhaus conjectured that if all the partial sums of

$$1+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx),$$

are everywhere non negative then $\lim a_k=\lim b_k=0$. Von Neumann[1] and Schur[1] proved that $\lim\inf a_k=\lim\inf b_k=0$. Sidon[2] proved that

$$\lim\frac{1}{n}\sum_{k\leq n}|a_k|+|b_k|=0.$$

In the present paper we are going to sharpen these results by proving the following Theorem. Let each partial sum of

$$1+\sum(a_k\cos kx+b_k\sin kx)$$

be everywhere non negative; then the number of indices $k\leq n$ for which the inequality $a_k^2+b_k^2>c^2$ holds is not greater than

$$(\log_2 n)^{4/c^2+1}\ [3].$$

Proof. Our chief tool will be the following classical theorem of Fejér[4]: Let

$$0\leq 1+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)\quad\text{for } 0\leq x<2\pi,$$

[1] Unpublished.

[2] *Acta Litt. ac Scient. Szeged*, Tom. II, pp. 43-47.

[3] $\log_2 n$ denotes the logarithm of n to the base 2.

[4] *Journal für reine und angewandte Mathematik*, Vol. 146, 1916, p. 63.

then real numbers $x_0, x_1 \dots x_n, y_0, y_1 \dots y_n$ exist such that

$$1 = \sum_{i=0}^{n} x_i^2 + \sum_{i=0}^{n} y_i^2, \qquad a_k = 2\sum_{l=0}^{n-k} (x_{k+l}x_l + y_{k+l}y_l),$$

$$b_k = 2\sum_{l=0}^{n-k} x_{k+l}y_l - y_{k+l}x_l. \qquad (1)$$

Conversely if we choose $2n+2$ real numbers $x_0, x_1 \dots x_n, y_0, y_1, \dots y_n$ with

$$\sum_{i=0}^{n} (x_i^2 + y_i^2) = 1$$

arbitrarily and determine the a's and b's by (1), the resulting trigonometric polynomial will be everywhere non negative.

If we replace x_i by $\sqrt{x_i^2 + y_i^2}$ and y_i by 0 we obtain a non negative pure cosine polynomial

$$1 + \sum_{k=1}^{n} A_k^2 \cos kx, \qquad A_k^2 = a_k^2 + b_k^2.$$

Thus $A_k \geq \max(|a_k|, |b_k|)$ which shows that it suffices to prove Lemma 1 for pure cosine polynomials. For these polynomials [1] we obtain from (1)

$$1 = \sum_{i=0}^{n} x_i^2, \qquad a_k = 2\sum_{l=0}^{n-k} x_l x_{k+l}.$$

First we prove two lemmas.

Lemma 1[2]. Let $i_1 < i_2 < \dots i_p < \frac{n}{2}$ be a sequence of integers such that $i_r > 2i_{r-1}$ $(r = 2, 3 \dots p)$ then

$$\sum_{j=1}^{p} a_{n-i_j}^2 < 4$$

Proof. We have by Schwartz's inequality

$$a_{n-i_j}^2 \leq 4\sum y_j^2 \sum z_j^2 < 4\sum y_j^2$$

[1] Fejér, *ibid.*, p. 64.

[2] S. Sidon, *Journal London Math. Soc.*, Vol. VIII, 1938, p. 181.

where the y_j's and z_j's run over $x_0, x_1, \dots x_{i_j}, x_{n-i_j} \dots x_n$ such that if x_k is a y_j then x_{k+n-i_j} is a z_j. Now we can choose the y_{j+1}'s belonging to $a_{n^2-i_{j+1}}$ in such a way that no y_{j+1} equals any of the y_r's $(r = 1, 2 \dots j)$. For if this were not possible then for a certain k x_k would be a y_{r_1} and $x_{k+n-i_{j+1}}$ would be a y_{r_2} $(r_1, r_2 \leq j)$. But by definition the y_r's $(r \leq j)$ are a subset of the x_l's with $0 \leq l \leq i_j$, or $n - i_j \leq l \leq n$. But from $i_{j+1} > 2i_j$ it follows that x_k and $x_{k+n-i_{j+1}}$ can not both satisfy one of these inequalities, which completes the proof. Hence

$$\sum_{j=1}^{s} a_{n^2-i_j} < 4 \sum_{j=1}^{s} \sum y_j^2 \leq 4 \sum_{i=0}^{n} x_i^2 = 4$$

which proves the Lemma[1].

Lemma 2. Let $i_1 < i_2 < \dots < i_p \leq n$ be a sequence of integers with $p > (\log_2 n)^t$ (t integer) then there exist $t + 2$ i's, $i_1' < i_2' < \dots i'_{t+2}$ such that for every $r < t + 2$ $i'_{r+1} - i_r' > 2(i_r' - i_1')$.

Proof. The Lemma holds for $t = 0$, we use induction. Suppose the Lemma holds for $t - 1$.

If the interval[2] $i_1, \frac{i_1 + i_p}{2}$ contains more than $(\log n)^{t-1}$ i's our Lemma holds; for then we can find $t + 1$ i''s $i_1' < i_2' < \dots i'_{t+1} < \frac{i_1 + i_p}{2}$ satisfying the Lemma and we can choose $i'_{t+2} = i_p$. If the interval does not contain more than $(\log n)^{t-1}$ integers then we take the least i not less than $\frac{i_1 + i_p}{2}$, say $i^{(2)}$, and consider the interval $i^{(2)}, \frac{i^{(2)} + i_p}{2}$; if it contains more than $(\log n)^{t-1}$ i's the Lemma is proved; if not we consider the least i not less than $\frac{i^{(2)} + i_p}{2}$ say $i^{(3)}$ etc. But the number of the intervals of the form $i^{(q)}, \frac{i^{(q)} + i_p}{2}$ is at most $\log n$ since the length of each of them is not greater than half the length of the preceeding one. Hence at least one of them contains more than $(\log_2 n)^{t-1}$ i's which proves the Lemma.

[1] It would be easy to show that the Lemma remains true if only $i_r > \sum_{j<r} i_j$ for $r = 1, 2 \dots p$.

[2] Our intervals are closed from below and open from above.

We can now prove our Theorem. Suppose it is false and let a_{n-i_1}, $a_{n-i_2} \dots a_{n-i_p}$, $p > (\log_2 n)^{4/c^2+1}$ be the a's which are greater than c. By Lemma 2, there exist $\left[\frac{4}{c^2}\right] + 3 = \tau$ a's, $a_{n-i'_1}$, $a_{n-i'_2}, \dots a_{n-i'_\tau}$ satisfying $i'_{r+1} - i'_r > 2(i'_r - i'_1)$ $(r < \tau)$ By the hypothesis

$$1 + \sum_{k=1}^{n-i_1} a_k \cos kx \geq 0, \quad \text{for} \quad 0 \leq x < 2\pi,$$

thus by Lemma 1.

$$\sum_{r=1}^{\tau-1} a^2_{n-i'_r} < 4 \left(\text{for } i'_{\tau-1} - i'_1 < \frac{i'_\tau - i'_1}{2} \leq \frac{n - i'_1}{2}\right)$$

which does not hold since $a^2_{n-i'_r} \geq c^2 > \frac{4}{\tau - 1}$; this completes the proof.

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