Annals of Mathematics
Vol. 40, No. 3, July, 1939

ON SUMS OF POSITIVE INTEGRAL k^{th} POWERS

By H. Davenport and P. Erdös

(Received February 20, 1939)

1. Let $k \geqq 3$ be an integer. Let $N_s^{(k)}(n)$ denote the number of integers not exceeding n that are representable as the sum of s positive integral k^{th} powers. We are concerned in this paper with inequalities of the form

$$N_s^{(k)}(n) > n^{\alpha}, \qquad \alpha = \alpha(k, s),$$

valid for all large n. Such inequalities are of great importance in the application of the Hardy-Littlewood method to Waring's Problem.

Until recently, practically the only such inequality known was that of Hardy and Littlewood[1]

$$(1) \qquad N_s^{(k)}(n) > n^{\alpha_1 - \epsilon}, \qquad \alpha_1 = 1 - \left(1 - \frac{2}{k}\right)\left(1 - \frac{1}{k}\right)^{s-2},$$

valid for any $\epsilon > 0$ and $n > n_0(\epsilon)$.[2]

This has been improved upon independently by the two authors of the present paper, using different methods. We give here an account of the method of Erdös, with modifications due to Davenport.[3]

2. Definition. *s positive numbers $\lambda_1, \cdots, \lambda_s$ satisfying $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_s$ are called admissible exponents for k^{th} powers if the number of solutions of*

$$(2) \qquad x_1^k + \cdots + x_s^k = y_1^k + \cdots + y_s^k$$

in integers subject to

$$(3) \qquad P^{\lambda_1} < x_1, y_1 < 2P^{\lambda_1}; \cdots; \qquad P^{\lambda_s} < x_s, y_s < 2P^{\lambda_s}$$

is $O(P^{\lambda_1 + \cdots + \lambda_s + \epsilon})$ as $P \to \infty$, for any $\epsilon > 0$.

It is easily seen[4] that if $\lambda_1, \cdots, \lambda_s$ are admissible exponents for k^{th}s, power then

$$(4) \qquad N_s^{(k)}(n) > n^{\alpha - \epsilon}, \qquad \alpha = \frac{\lambda_1 + \cdots + \lambda_s}{k\lambda_1},$$

for any $\epsilon > 0$ and $n > n_0(\epsilon)$.

---

[1] See, for example, Landau, *Vorlesungen über Zalentheorie*, (Leipzig, 1927), 1, Satz 348.

[2] For $s = 2$, Erdös and Mahler (*Journal London Math. Soc.*, 13 (1938), pp. 134–139) have proved that $N_2^{(k)}(n) > cn^{2/k}$, where $c > 0$ depends only on k.

[3] The results of Davenport are in course of publication in the *Proc. Royal Soc.* An account of the particular case $k = 4$, $s = 3$ has appeared in the *Comptes Rendus*, 207 (1938), p. 1366.

[4] A formal proof is given in Davenport's paper in *Proc. Royal Soc.*

In what follows, $c_1, c_2, \cdots$ denote positive numbers depending only on k and s.

THEOREM 1. *Let* $\theta = 1 - k^{-1}$. *Then* $1, \lambda, \lambda\theta, \lambda\theta^2, \cdots, \lambda\theta^{s-2}$ *are admissible exponents for* k^{th} *powers, provided* λ *satisfies*

$$k\lambda - (k - 1) \leqq \lambda\theta^{s-2}. \tag{5}$$

PROOF. (1) Suppose that $s = 2$. It is well known that the number of representations of an integer m as $y_1^k + y_2^k$ with positive integral y_1, y_2 is $O(m^\epsilon)$, for any $\epsilon > 0$. Hence any *two* positive numbers are admissible exponents for k^{th} powers. Thus the theorem is true for $s = 2$.

(2) Suppose that $s \geqq 3$, and that the theorem is true with $s - 1$ in place of s.

Let $x_1, \cdots, x_s, y_1, \cdots, y_s$ satisfy (2) and (3), where $\lambda_1 = 1$, $\lambda_2 = \lambda$, $\lambda_3 = \lambda\theta, \cdots, \lambda_s = \lambda\theta^{s-2}$. Since

$$| x_1^k - y_1^k | > P^{k-1} | x_1 - y_1 |,$$

we have

$$| x_1 - y_1 | < c_1 \frac{P^{k\lambda}}{P^{k-1}} \leqq c_1 P^{\lambda\theta^{s-2}},$$

by (5). Hence the number of possible pairs x_1, y_1 is $O(P^{1+\lambda\theta^{s-2}})$. The number of possible sets of values for $x_1, y_1, x_2, x_3, \cdots, x_{s-1}$ is therefore $O(P^{1+\lambda+\lambda\theta+\cdots+\lambda\theta^{s-2}})$. For fixed values of these variables, (2) has the form

$$\begin{aligned} A &= y_2^k + y_3^k + \cdots + y_s^k - x_s^k \\ &= y_2^k + O(P^{k\lambda\theta}), \end{aligned}$$

where A is fixed. This has only $O(1)$ solutions y_2 with $P^\lambda < y_2 < 2P^\lambda$, since

$$(y_2 + 1)^k - y_2^k > (P^\lambda)^{k-1} = P^{k\lambda\theta}.$$

For fixed y_2, the argument can be repeated (if $s \geqq 4$) and shows that there are only $O(1)$ possibilities for y_3, and so on generally. Finally, (2) takes the form

$$B = y_s^k - x_s^k,$$

which has only $O(P^\epsilon)$ solutions, unless $B = 0$. Hence the total number of solutions of (2), subject to (3), with $x_s \neq y_s$ is $O(P^{1+\lambda+\lambda\theta+\cdots+\lambda\theta^{s-2}+\epsilon})$.

For the solutions with $x_s = y_s$, we observe that, since

$$k\lambda - (k - 1) \leqq \lambda\theta^{s-2} \leqq \lambda\theta^{s-3},$$

$1, \lambda, \lambda\theta, \cdots, \lambda\theta^{s-3}$ are admissible exponents for k^{th} powers (by the case $s - 1$ of the theorem). Hence the number of solutions with $x_s = y_s$ is

$$O(P^{1+\lambda+\lambda\theta+\cdots+\lambda\theta^{s-3}+\epsilon+\lambda\theta^{s-2}}).$$

This establishes Theorem 1.

COROLLARY. *For any* $\epsilon > 0$, *and* $n > n_0(\epsilon)$,

$$(6)\quad N_s^{(k)}(n) > n^{\alpha_2 - \epsilon}, \qquad \alpha_2 = 1 - \frac{1 - \frac{1}{k}}{1 - \frac{1}{k}\left(1 - \frac{1}{k}\right)^{s-2}}\left(1 - \frac{2}{k}\right)\left(1 - \frac{1}{k}\right)^{s-2}.$$

PROOF. Taking $\lambda = (k - 1)/(k - \theta^{s-2})$, we have

$$\frac{1}{k}(1 + \lambda + \lambda\theta + \cdots + \lambda\theta^{s-2}) = \frac{1}{k}\left(1 + \frac{k-1}{k - \theta^{s-2}}\,\frac{1 - \theta^{s-1}}{1 - \theta}\right)$$

$$= 1 - \frac{1 - \frac{1}{k}}{1 - \frac{1}{k}\left(1 - \frac{1}{k}\right)^{s-2}}\left(1 - \frac{2}{k}\right)\left(1 - \frac{1}{k}\right)^{s-2},$$

on simplification.

Comparing (1) with (6), it is plain that $\alpha_2 > \alpha_1$ (provided $s > 2$).

3. THEOREM 2. $1, 1 - \frac{1}{k^2}, 1 - \frac{1}{k} - \frac{1}{k^2}$ *are admissible exponents for* k^{th} *powers.*

PROOF. Let $x_1, y_1, z_1, x_2, y_2, z_2$ satisfy

$$(7)\qquad x_1^k + y_1^k + z_1^k = x_2^k + y_2^k + z_2^k,$$

$$(8)\qquad \begin{gathered} P < x_1, x_2 < 2P, \qquad P^{1-k^{-2}} < y_1, y_2 < 2P^{1-k^{-2}}, \\ P^{1-k^{-1}-k^{-2}} < z_1, z_2 < 2P^{1-k^{-1}-k^{-2}}. \end{gathered}$$

The number of solutions of (7), (8) with $x_1 = x_2$ is

$$O(P^{1+1-k^{-2}+1-k^{-1}-k^{-2}+\epsilon}).$$

Hence we need only consider solutions with $x_2 > x_1$. By a similar argument, we can also suppose that $z_1 \neq z_2$.

Writing $x_2 = x_1 + t$, (7) becomes

$$(9)\qquad (x_1 + t)^k - x_1^k + y_2^k - y_1^k = z_1^k - z_2^k.$$

Plainly

$$tP^{k-1} < (x_1 + t)^k - x_1^k < y_1^k + z_1^k < 2(2P^{1-k^{-2}})^k,$$

whence

$$(10)\qquad 0 < t < c_2 P^{1-k^{-1}}.$$

For fixed t, y_1, y_2, (9) has the form

$$(11)\qquad (x_1 + t)^k - x_1^k + A = O(P^{k-1-k^{-1}}).$$

Since

$$\{(x_1 + 1 + t)^k - (x_1 + 1)^k\} - \{(x_1 + t)^k - x_1^k\} > c_3 t P^{k-2},$$

the number of values of x_1 satisfying (11) is

$$O\left(\frac{P^{k-1-k^{-1}}}{tP^{k-2}} + 1\right) = O\left(\frac{P^{1-k^{-1}}}{t} + 1\right).$$

Hence the number of sets x_1, t, y_1, y_2 satisfying (9), (10), (8) is

$$O\left(P^{2(1-k^{-2})} \sum_{0<t<c_2P^{1-k^{-1}}} \left(\frac{P^{1-k^{-1}}}{t} + 1\right)\right) = O(P^{2(1-k^{-2})+(1-k^{-1})+\epsilon}).$$

Also, for fixed values of these variables, (9) has only $O(P^\epsilon)$ solutions in z_1, z_2 with $z_1 \neq z_2$.

Hence the number of solutions of (7), (8) is

$$O(P^{1+(1-k^{-2})+(1-k^{-1}-k^{-2})+\epsilon}),$$

which proves the theorem.

COROLLARY. *For any* $\epsilon > 0$ *and* $n > n_0(\epsilon)$,

$$N_3^{(k)}(n) > n^{\alpha_3-\epsilon}, \qquad \alpha_3 = \frac{3}{k} - \frac{k+2}{k^3}.$$

The value of α_2 given by the Corollary to Theorem 1 in the case $s = 3$ is

$$\alpha_2 = \frac{3}{k} - \frac{k+1}{k(k^2-k+1)}.$$

It is easily verified that $\alpha_3 > \alpha_2$.

4. The results above obtained for $N_s^{(k)}(n)$ are not as precise as those of Davenport when $k \leqq 5$, but are at any rate more precise when $k \geqq 6$ and $s = 3$. It may be remarked that the case $k = 4$, $s = 4$ of the Corollary to Theorem 1, namely

$$N_4^{(4)}(n) > n^{(83/110)-\epsilon},$$

is sufficient for the proof that $G(4) = 16$, though not for the further results announced by Davenport.[5]

VICTORIA UNIVERSITY, MANCHESTER, ENGLAND, AND
INSTITUTE FOR ADVANCED STUDY, PRINCETON, N. J.

---

[5] *Comptes Rendus*, loc. cit.