ON SEQUENCES OF INTEGERS NO ONE OF WHICH DIVIDES THE PRODUCT OF TWO OTHERS AND ON SOME RELATED PROBLEMS.

Paul Erdös (Manchester).

Introduction.

In § 1 we consider sequences—say A sequences—such that no member of them divides the product of any two other members. We prove hat the number of integers not exceeding n of an A sequence is less than $\pi(n)+O\left(\frac{n^{2/3}}{\log n}\right)^2$, $\pi(n)$ denotes, as usual, the number of primes not exceeding n, and we show that the error term is best possible. Sequences no term of which divides any other may be much denser [1]).

In § 2 we deal with B cequences, which are such that the product f any two of their memders is different. Here we prove that the number of integers not exceeding n contained in a B sequence is less than $\pi(n)+O(n^{3/4})$ and we show that the error term cannot be better than

$$O\left(\frac{n^{3/4}}{(\log n)^{3/2}}\right).$$

The sequence of primes is both an A and a B sequence. Our A and B sequences seem ot be very much more general, but our theorems show that they cannot be very much more dense than the sequence of the primes.

In § 3 we chow by using the results of § 2, that if $p_1 < p_2 < \ldots < p_z \leqslant n$ is an arbitrary sequence ofprimes such that $z > \frac{c_1 n \log\log n}{(\log n)^2}$, where c_1 is a sufficiently large absolute constant, then the products $(p_i-1)(p_j-1)$ cannot all be different.

In this connection I proved in a previons paper that for an infinity of n the number of solutions of the equation $n=(p-1)(q-1)$ (p, q primes) is greater than $e^{(\log n)^{1/2-\varepsilon}}$.

§ 1.

In order to make our method more intelligible we first prove that the number of integers not exceeding n of an A sequence is less than $\pi(n)+2n^{2/3}$.

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*) Such a sequence may obviously contain $n/2$ numbers not exceeding n.

We denote by $b_1, b_2, \dots$ the integers not exceeding $n^{2/3}$ and the primes of the interval $(n^{2/3}, n)$, further by $d_1, d_2, \dots$ the integers $\leqq n^{2/3}$, so that every d is at the same time a b.

Now we prove

Lemma I.

Any integer $m \leqq n$ may be written in the form $b_i\, d_j$.

Proof of the Lemma.

We may evidently suppose $m > n^{2/3}$.

If m has a prime factor $p > n^{1/3}$ we write $m = p \cdot {}^{m}/_{p}$, where $p = b_i$ and ${}^{m}/_{p} = d_j$. If, on the other hand, all prime factors of m are less than $n^{1/3}$ then we write $m = p_1\, p_2 \dots p_y$ where all p 's are less than $n^{1/3}$ but not necessarily different. Hence at least one of the integers $p_1, p_1 p_2, p_1 p_2 p_3, \dots$ say $p_1 p_2 \dots p_\lambda$ lies between $n^{1/3}$ and $n^{2/3}$. Then we write $b_i = p_1\, p_2 \dots p_\lambda$ and $d_j = m / p_1\, p_2 \dots p_\lambda$.

Now we write every a in the form $b_i\, d_j$ so that every a is represented by the segment connecting the points b_i and d_j. If b_i is connected with two or more d 's say $d_{j_1}, d_{j_2}, \dots$ then these d 's cannot be connected with any other b 's. For if e. g. d_{j_1} would be connected with b_i then in contradiction with the definition of our sequence, $b_i\, d_{j_1}$ would divide the product $(d_{j_1}\; b'_i)\,(b_j\; d_{j_2})$.

We assert that the number of these segments is less than $\beta + \gamma$ where β denotes the number of b 's and γ the number of d 's i. e. the number of a 's is less than $\pi(n) + 2\, n^{2/3}$. To prove this we split the b 's into two classes. In the first class are the b 's connected with only a single d and in the second class all the other b 's. The number of segments starting from the b 's of the first class is evidently less then or equal to the total number of b 's. In consequence of our previous remark a d cannot be connected with two b 's of the second class. Hence the number of segments starting from the b 's of the second class is evidently less than the number of d 's. Hence the result.

Now we improve the error term to $O\left(\frac{n^{2/3}}{(\log n)^2}\right)$.

First we improve our Lemma.

Lemma II.

Any integer not exceeding n may be written in the form $b_i\, d_j$ where $b_1, b_2, \dots$ represent four classes of integers:

(a) **the integers not exceeding $n^{2/3}$,**

(b) **the primes of the interval $(n^{2/3}, n)$,**

(c) **the integers of the form pq with $p, q \leqq n^{1/3}$,**

(d) the integers of the form qr with $n^{1/6} \leqq q \leqq n^{2/3}$ and $r < \frac{n}{q^2}$ (p, q, r primes).

The d 's denote the integers not exceeding $n^{3/5}$ (class (a) of the b's).

Now be an integer $m \leqq n$ we have the following 6 possibilities:

1. $m \leqq n^{9/10}$. This case is settled by Lemma I., if we replace n of the lemma by $n^{9/10}$.

2. m has a prime factor $p > n^{2/5}$; then we write $b_i = p$, $d_j = \frac{m}{p}$.

3. All prime factors of m are less than $n^{1/5}$. Let $m = p_1 p_2 \ldots p_y$. At least one of the integers $p_1, p_1 p_2, \ldots, p_1 p_2 \ldots p_y$ say $p_1 p_2 \ldots p_\lambda$ lies between $n^{2/5}$ and $n^{3/5}$. Hence $b_i = p_1 p_2 \ldots p_\lambda$, $d_i = m / p_1 p_2 \ldots p_\lambda$.

4. Only one prime factor p of m is greater than $n^{1/5}$ (but of course $p \leqq n^{2/5}$). We then write $m = p p_1 p_2 \ldots p_\mu$. This case may be settled as the previous one since at least one of the integers $p p_1, p p_1 p_2 \ldots, p p_1 p_2 \ldots p_\mu$ lies between $n^{2/5}$ and $n^{3/5}$.

5. Exactly two prime factors of m, say p, q are greater than $n^{1/5}$. Then $m = p q p_1 p_2 \ldots p_\nu$, $p p_1 p_2 \ldots p_\nu = \frac{m}{q} > n^{9/10 - 2/5} > n^{2/5}$ hence at least one of the integers $p p_1$, $p p_1 p_2, \ldots, p p_1 p_2 \ldots p_\nu$ lies between $n^{2/5}$ and $n^{3/5}$. This settles 5.

6. At least three prime factors of m say $p > q > r$ are greater than $n^{1/5}$. If q and r are both less than $n^{1/3}$ we write $b_i = qr$, $d_j = \frac{m}{qr}$; if on the other hand $q > n^{1/3}$, we have from $p q r \leqq n$ $r < \frac{n}{pq} < \frac{n}{q^2}$, thus again $b_i = qr$, $d_j = \frac{m}{qr}$.

Thus Lemma II. is proved.

To prove that the error term is $O\left(\frac{n^{2/3}}{(\log n)^2}\right)$ we have only to show that the number of the b's is $\pi(n) + O\left(\frac{n^{2/3}}{(\log n)^2}\right)$.

For the first 3 classes of the b's this is immediately clear. The number of b's of class d equals:

$$\sum_{n^{2/3} \geqq q \geqq n^{1/3}} \pi\left(\frac{n}{q^2}\right) < c_2 \frac{n}{\log n} \sum_{q \geqq n^{1/3}} \frac{1}{q^2} < \frac{c_3 n}{\log n} \sum_{k > c_4 \frac{n^{1/3}}{\log n}} \frac{1}{k^2 \log k^2} =$$

$$O\left(\frac{n^{2/3}}{(\log n)^2}\right).$$

Now we prove that the error term is best possible.

Let $p_1 < p_2 < \ldots < p_s$ be the primes not exceeding $n^{1/3}$. From the elements $1, 2, \ldots, s$ we form combinations taken 3 at a time such that no two of them have two common elements. We estimate the number of these combinations.

For sake of shortness we call any combination taken 2 at time a pair and any combination taken 3 at a time a triplet. Let now $(i_1 j_1 k_1)$ $(i_2 l_2 k_2), \ldots, (i_w l_w k_w)$ be a complete triplet system of having no common pair, which means that if the triplet (IJK) does not occur in the system then there exists at least one triplet of the system having two common elements with (IJK). The number of pairs contained in the complete system of triplets is evidently $3w$, and since there are $s-2$ triplets containing a given pair we have

$$(s-2)\ 3w \geqslant \binom{s}{3},$$

hence

$$w \geqslant \frac{1}{q}\binom{s}{2}.$$

Now we define a sequence which consists of the primes of the interval $(n^{1/2}, n)$ and of the products $p_{i_1} p_{j_1} p_{k_1}, p_{j_2} p_{i_2} p_{k_2}, \ldots, p_{i_w} p_{j_w} p_{k_w}$. It is evident that this is an A sequence and the number of its elements is greater than

$$\pi(n) - s + \frac{1}{q}\binom{s}{2} > \pi(n) + \frac{n^{2/3}}{80 (\log n)^2},$$

since by the prime number theorem $s > \dfrac{n^{1/3}}{2 \log n}$.

Hence the result.

§ 2.

Here we deal with the B sequences.

Let $a_1 < a_2 < \ldots < a_x \leqslant n$ be a B sequence. We write all a's in the form $b_i\, d_j$ where the b's and d's are defined as in Lemma I. Here we represent again the a's by segments connecting the b's and the d's. No two b's can be connected with the same two d's. For if they were, let $b_{i_1}\ b_{i_2},\ d_{j_1}\ d_{j_2}$ be the b's and d's in question. $b_{i_1}\ d_{j_1} = a_{i_1 j_1}$, $b_{i_1}\ d_{j_2} = a_{i_1\, j_2}$, $b_{i_2}\ d_{j_1} = a_{i_2\, j_1}$, $b_{i_2}\ d_{j_2} = a_{i_2\, j_2}$ and $a_{i_1\, j_1} . a_{i_2\, j_2} = a_{i_1\, j_2}\, a_{i_2\, j_1}$: an evident contradiction. We may suppose in the representation of any a that $b_i > d_j$.

We split the a's into 3 classes. The first class contains the a's for which $b_i \leqslant n^{1/2}$, the second contains the a's for which $n^{1/2} \leqslant b_i < n^{2/3}$ and the third class the other a's.

To estimate the number of a's of the first class we split the b's not exceeding $n^{1/2}$ into two groups. Into the first group we put the b's connected with more than $n^{1/4}$ d's and into the second group all the other b's. Let $j_1, j_2, \ldots, j_\nu$ be the numbers of segments starting from the first, second,... b's of the first group. Taking in consideration that no b's can be connected with the same two d's we have

$$\binom{j_1}{2} + \binom{j_2}{2} + \ldots + \binom{j_\nu}{2} \leqslant \binom{[n^{1/2}]}{2},$$

since the d's are $< n^{1/2}$, so that the number of pairs of d's is $\leqq \binom{[n^{1/2}]}{2}$.

Since all j's are greater than $n^{1/4}$ we have

$$\frac{n^{1/4}-1}{2}(j_1+j_2+\ldots+j_\nu)\leqq\frac{n}{2}$$

so that

$$j_1+j_2+\ldots+j_\nu\leqq 2\,n^{3/4}$$

On the other hand it is evident that the number of segments starting from the b's of the second group does not exceed $n^{3/4}$. Hence the number of a's of the first class does not exceed $3\,n^{3/4}$.

The argument was really based upon the following theorem for graphs. Let $2\,k$ points be given. We split them into two classes each containing k of them. The points of the two classes are connected by segments such that the segments form no closed quadrilateral. Then the number of segments is less than $3\,k^{3/2}$. Putting $k=n^{1/2}$ we obtain our result.

To estimate the number of a's of the second class, we split them into several subclasses. In the first subclass are the a's for which the corresponding b's lie between $n^{1/2}$ and $2\,n^{1/2}$. For the second subclass the b's lie between $2\,n^{1/2}$ and $4\,n^{1/2}$ and for the $(k+1)^{th}$ subclass $2^k n^{1/2} < b_i < 2^{k+1} n^{1/2}$. It is evident that the d's belonging to the b's of the $(k+1)^{th}$ subclass are all less than $\dfrac{n^{1/2}}{2^k}$.

To estimate the number of a's of the $(k+1)^{th}$ subclass, we split the corresponding b's into two groups. In the first group are the b's connected with more than $n^{1/4}\,2^{\frac{3k}{2}}$ d's and in the second group are all the other b's. Let $h_1, h_2, \ldots, h_z$ be the numbers of segments starting from the b's of the first group. Taking again into consideration that no two b's can be connected with the same two d's, we have

$$\binom{h_1}{2}+\binom{h_2}{2}+\ldots+\binom{h_z}{2}\leqq\binom{\left[\frac{n^{\frac{1}{2}}}{2^k}\right]}{2}$$

Since all h's are greater than $\dfrac{n^{1/4}}{2^{3k/2}}$

we have

$$\frac{n^{1/4}-1}{2^{\frac{3k}{2}}}(h_1+h_2+\ldots+h_z)\leqq\frac{n}{2^{2k}}$$

Hence finally

$$h_1+h_2+\ldots h_z\leqq\frac{2\,n^{3/4}}{2^{k/2}}.$$

The number of a's starting from the b's of the second group is evidently less that $2^k n^{1/2} \frac{n^{1/4}}{2^{\frac{3k}{2}}} = \frac{n^{3/4}}{2^{k/2}}$.

Hence the number of a's belonging to the $(k+1)^{th}$ subclass is less than $\frac{2n^{3/4}}{2^{k/2}}$. From this we obtain that the number of a's belonging to the second class is less than

$$3n^{3/4} \sum_{k=0}^{\infty} \frac{1}{2^{k/2}} = 3n^{3/4} \frac{1}{\frac{1}{\sqrt{2}} - 1} < 9n^{3/4}.$$

The d's belonging to the a's of the third class are all less than $n^{1/4}$. We split the b's belonging to the a's of the third class into two groups. In the first group are the b's connected with only a single d. The number of these segments equals at the utmost the number of the b's greater than $n^{3/4}$ which is less than $\pi(n)$.

Taking again into consideration that no two b's are connected with the same two d's we obtain that the number of segments starting from the b's of the second group is less than $n^{1/2}$. Hence the number of a's of the third class is less than $\pi(n) + n^{1/2}$.

Thus finally the number of a's not exceeding n is less than

$$\pi(n) + 8n^{3/4} + n^{1/2} = \pi(n) + O(n^{3/4}).$$

Hence the result.

Now we prove that the error term cannot be better than $O\left[\frac{n^{3/4}}{(\log n)^{3/2}}\right]$. First we prove the following lemma communicated to me by Miss E. Klein.

Lemma.

Given $p(p+1)+1$ elements (p a prime), we can construct $p(p+1)+1$ combinations taken $(p+1)$ at a time having no two elements in common.

Remark.

Since $\left[\frac{p(p+1)+1}{2}\right] = [p(p+1)+1]\left[\frac{p+1}{2}\right]$ each pair will be contained once and only once in the above combinations.

Proof of the lemma.

We construct the combinations taken $p+1$ at a time as follows. The first $p+1$ combinations are:

$$\begin{array}{llll}
1 & 2 & 3 & \dots\ p+1 \\
1 & p+2 & p+3 & \dots\ 2p+1 \\
1 & 2p+2 & 2p+3 & \dots\ 3p+1 \\
\dots & \dots & \dots & \dots \\
p & p.p+2 & p.p+3 & \dots (p+1)p+1
\end{array}$$

For sake of shortness we denote the matrix

$$\begin{matrix} p+2 & p+3 & \dots & 2p+1 \\ 2p+2 & 2p+3 & \dots & 3p+1 \\ \cdot & \cdot & \cdot & \cdot \\ pp+2 & pp+3 & \dots & (p+1)p+1 \end{matrix}$$

by

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ a_{p1} & a_{p2} & \dots & a_{pp} \end{matrix}$$

The next p^2 combinations are the following

$$\begin{matrix} 2 & a_{11} & a_{21} & a_{31} & \dots & & a_{p1} \\ 2 & a_{12} & a_{22} & a_{32} & \dots & & a_{p2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2 & a_{1p} & a_{2p} & a_{3p} & \dots & & a_{pp} \\ 3 & a_{12} & a_{22} & a_{33} & \dots & a_{pp-1} & a_{pp} \\ 3 & a_{12} & a_{23} & a_{34} & \dots & a_{p-1,p} & a_{p1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r & a_{11} & a_{2,1+r-2} & a_{3,1+(2r-2)} & \dots & & a_{p,1+(p-1)(r-2)} \\ r & a_{12} & a_{2,r-2} & a_{3,2+2(r-2)} & \dots & & a_{p,2+(p-1)(r-2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ r & a_{1i} & a_{2,i+r-2} & a_{3,i+2(r-2)} & \dots & & a_{p,i+(p-1)(r-2)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix}$$

where $r \leq p$ and the index $i+k(r-2)$ is to be reduced mod p.

It is easy to see that no two of these $p(p+1)+1$ combinations have two elements in common, which proves our lemma.

Let now $q_1, q_2, \dots, q_\tau$ be the primes not exceeding $1/_3\, n^{1/2}$. We consider the greatest prime p for which $\rho = p(p+1)+1$ does not exceed τ. By the prime-number-theorem $\rho > \frac{\tau}{2}$. From the elements $q_1, q_2, \dots, q_\tau$ we now form ρ combinations taken $p+1$ at a time and having no two common elements; in consequence of our lemma this is always possible. Let these combinations be $C_1, C_2, \dots, C_\rho$.

Further let $r_1, r_2, \dots$ be the primes of the interval $(1/_3\, n^{1/2}, n^{1/2})$. By the prime-number-theorem, their number is greater than ρ.

Now we define a B sequence as follows.

We multiply r_1 by the q's contained in C_1,

r_2 „ „ q's „ „ C_2,

. . . .

r_ρ „ „ q's „ „ C_ρ.

Our B sequence is formed by these products and by the primes of the interval $(n^{1/2}, n)$.

By the prime-number-theorem

$$\rho > \frac{n^{1/2}}{6 \log n}.$$

Hence the number of elements of our B sequence is greater than

$$\pi(n) - n^{1/2} + \rho^{3/2} > \pi(n) + \frac{n^{3/4}}{36 (\log n)^{3/2}}.$$

Hence the result.

§ 3.

Let now $p_1 < p_2 < \ldots p_t \leqslant n$ be a sequence of primes such that $t > \frac{c_1 n \log\log n}{(\log n)^2}$, where c_1 is sufficiently large and will be determined later. We have to prove that the products $(p_i - 1)(p_j - 1)$ cannot all be different.

We split the primes p_i into two classes. In the first class are the primes for which $p-1$ has a prime factor $q > \frac{n}{\log n}$, in the second class are all the other p's. The primes of the first class are all of the form $aq+1$ with $a < \log n$. But the number of primes of the form $aq+1$ for any a is by Brun's method [1]) less than

$$\frac{c_5\, n \prod_{p|a}\left(1+\frac{1}{p}\right)}{a (\log n)^2},$$

hence the number of primes of the first class is less than

$$\frac{c_5\, n}{(\log n)^2} \sum_{a < \log n} \frac{\prod_{p|a}\left(1+\frac{1}{p}\right)}{a} \leqslant \frac{c_5\, n}{(\log n)^2} \sum_{d < \log n} \frac{1}{d} \sum_{\substack{ad \\ a|\vee \log n}} \frac{1}{a} <$$

$$< \frac{n}{(\log n)^2} \sum_{d < \log n} \frac{c_6 \log\log n}{d^2} < \frac{c_7\, n \log\log n}{(\log n)^2}.$$

Suppose now $c_1 > c_7$ i. e. the number of primes of the second class be greater than $\frac{2n}{(\log n)^2}$. Now we prove that for the primes of the second class the products $(p_i - 1)\ (p_j - 1)$ cannot all be different.

More generally we prove: let $a_1 < a_2 < \ldots < a_s \leqslant n$ be a sequence of positive integers, $s > \frac{2n}{(\log n)^2}$, and no $a_i - 1$ be divisible by a prime $> \frac{n}{\log n}$, then the products $a_i a_j$ cannot all be different.

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1) P. Erdös. On the normal number of prime factors of $p-1$ and on some related problems concerning Euler's Φ function. Quarterly Journal fo Mathematics. Vol. 6. (1935) 205—213.

As in § 2, we write the a's in the form $b_i d_j$ (but here $b_i < \frac{n}{\log n}$). Where no two b's can be connected with the same two d's and split them just as in § 2. into 3 classes, and obtain by the same argument that:

1) the number of a's of the first class is less than $2n^{3/4}$,
2) the number of a's of the second class is less than $6n^{3/4}$,
3) the number of a's of the third class is less then $\pi\left(\frac{n}{\log n}\right) + n^{1/2}$.

Hence the number of a's is less than

$$\pi\left(\frac{n}{\log n}\right) + 8n^{3/4} + n^{1/2} < \frac{2n}{(\log n)^2},$$

which establishes the result.

Заметка о некоторых свойствах целочисленных последовательностей.

П. ЭРДЁШ, МАНЧЕСТЕР.

Пусть $a_1 < a_2 < \ldots < a_x \leqslant n$ означает последовательность целых чисел, таких, что ни одно из произведений любых двух чисел из последовательности не делится ни на одно из остальных.

Тогда

$$x < \pi(n) + O\left(\frac{n^{2/3}}{(\log n)^2}\right)$$

при чем оценка не может быть улучшена. Доказательство будет яснее, если я в начале докажу только, что

$$x < \pi(n) + 2n^{2/3}.$$

В этом случае доказательство основано на лемме:

Каждое целое число $m \leqslant n$ может быть записано в форме $b_i c_j$, где b_i означает некоторое целое число, не превосходящее $n^{2/3}$, или простое число, интервала $(n^{2/3}, n)$, и c_j означает некоторое целое число, не превосходящее $n^{2/3}$.

Чтобы вывести для x более точную оценку необходима тонкая и довольно сложная форма леммы.

Пусть будет $a_1 < a_2 < \ldots < a_y \leqslant n$ другая последовательность целых положительных чисел, такая что все произведения $a_i a_j$ различны между собой. Тогда

$$y < \pi(n) + O(n^{3/4}).$$

Доказательство основано на предыдущей лемме.

Здесь оценочный член не может быть сделан лучше чем $O\left(\frac{n^{3/4}}{(\log n)^{3/4}}\right)$.