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ON INTERPOLATION II

ON THE DISTRIBUTION OF THE FUNDAMENTAL POINTS OF LAGRANGE AND HERMITE INTERPOLATION[1]

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1. Introduction.

Let

$$A \equiv \left\{ \begin{matrix} x_1^{(1)} & & & \\ x_1^{(2)}, & x_2^{(2)} & & \\ \vdots & & & \\ x_1^{(n)}, & x_2^{(n)}, & \cdots & x_n^{(n)} \\ \vdots & \vdots & & \vdots & \ddots \end{matrix} \right\}$$

be a triangular matrix, or shortly matrix, where for every line

$$-1 \leqq x_n^{(n)} < x_{n-1}^{(n)} < \cdots < x_1^{(n)} \leqq +1.$$

We define the n^{th} Lagrange interpolation parabola belonging to the function $f(x)$ as the polynomial $L_n(f)$ of degree $(n-1)$ at most taking at $x_1^{(n)}, \cdots, x_n^{(n)}$ the values $f(x_1^{(n)}), \cdots f(x_n^{(n)})$. The explicit form of this polynomial is

$$L_n(f) \equiv \sum_{\nu=1}^{n} f(x_\nu^{(n)}) l_\nu^{(n)}(x) \equiv \sum_{\nu=1}^{n} f(x_\nu) l_\nu(x),$$

where

$$l_\nu(x) = \frac{\omega(x)}{\omega'(x_\nu)(x - x_\nu)},$$

and

$$\omega(x) \equiv \omega_n(x) = \prod_{\nu=1}^{n} (x - x_\nu^{(n)}) \equiv \prod_{\nu=1}^{n} (x - x_\nu).$$

The polynomials $l_\nu(x)$, the "fundamental functions" are independent of $f(x)$. We give explicit indication of the dependence of $l_\nu(x)$ upon n only when we want to emphasize this dependence or when a misunderstanding may arise.

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[1] We reported part of these results to the Math. and Phys. Assoc. Budapest, 12. XII. 1935.

We evidently have

$$(1a) \qquad L_n(1) \equiv \sum_{\nu=1}^{n} l_\nu(x) \equiv 1,$$

or more generally, if $\psi(x)$ is a polynomial of degree k

$$(1b) \qquad L_{n+k}(\psi) = \psi \qquad n = 1, 2, \cdots .$$

In the theory of the Lagrange-interpolation we shall consider the two sequences $L_n(f)_{x=x_0}$ and $\int_{-1}^{1} L_n(f)\,dx$ for $n \to \infty$. The behavior of the first sequence is determined[2] by

$$(2a) \qquad B_1(n, x_0) = \sum_{k=1}^{n} | l_k(x_0) |,$$

that of the second one[3] by

$$(2b) \qquad B_2(n) = \sum_{\nu=1}^{n} | \lambda_\nu^{(n)} | \equiv \sum_{\nu=1}^{n} \left| \int_{-1}^{1} l_\nu(x)\,dx \right|.$$

These expressions $B_1(n, x_0)$ and $B_2(n)$ are evidently independent of the function $f(x)$; they depend only upon the matrix A of the fundamental points and (as in (2a)) upon the value of x_0; they are the analogues of the Lebesgue-constants in the theory of Fourier-series.

The examination of the second problem is particularly easy when the so called "Cotes numbers" are all greater than or equal to 0,[4] i.e.

$$(3) \qquad \lambda_k = \lambda_k^{(n)} \geqq 0, \qquad k = 1, 2, \cdots n, \quad n = 1, 2, \cdots .$$

For in this case, by (1a) we have

$$\sum_{k=1}^{n} | \lambda_k | = \sum_{k=1}^{n} \lambda_k = 2,$$

i.e. we immediately obtain by Pólya (l.c.) that if $f(x)$ is R.-integrable,[2] then for the Lagrange parabolas taken on such an A

$$\lim_{n \to \infty} \int_{-1}^{1} L_n(f)\,dx = \int_{-1}^{1} f(x)\,dx.$$

Thus (3) implies an important interpolation property of the matrix A. Pólya proved that the necessary and sufficient condition for quadrature convergence for continuous functions is

$$(4) \qquad B_2(n) < c_1$$

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[2] H. Hahn: *Über das Interpolationsproblem*, Math. Zeitschrift 1918. The notations "bounded," "R-integrable" "L-integrable" "continuous" refer always, to the interval $[-1, +1]$.

[3] G. Pólya, *Über die Konvergenz von Quadraturverfahren*, Math. Zeitschrift 1933. Bd. **37**. pp. 264–286.

[4] L. Fejér: *Mechanische Quadraturen mit positiven Cotes-Zahlen*, Math. Zeitschrift 1933. Bd. 37. p. 287–309. See (2b).

where c_1 (and later $c_2 \cdots$) are positive constants independent of n. Also (4) depends only upon A and implies an important interpolation property.

To obtain a new and important interpolation property of A in the theory of the *Lagrange* interpolation we are forced according to Fejér[5] to consider the *Hermite*-interpolation. The n^{th} step parabola of the bounded and integrable function $f(x)$ is defined as the polynomial $H_n(f)$ of degree $(2n - 1)$ at most taking at $x_1^{(n)}, \cdots, x_n^{(n)}$ the values $f(x_1^{(n)}), \cdots, f(x_n^{(n)})$ with $\dfrac{dH_n(f)}{dx}_{x=x_\nu^{(n)}} = 0$. $(\nu = 1, 2, \cdots n.)$ The explicit form of this polynomial is given by

$$H_n(f) = \sum_{\nu=1}^{n} f(x_\nu)\left[1 - \frac{\omega''(x_\nu)}{\omega'(x_\nu)}(x - x_\nu)\right] l_\nu(x)^2 \equiv \sum_{\nu=1}^{n} f(x_\nu) h_\nu(x), \tag{5a}$$

where

$$\omega(x) = \prod_{\nu=1}^{n} (x - x_\nu). \tag{5b}$$

Then the above mentioned property is

$$\begin{gathered} \theta_k(x) \equiv 1 - \frac{\omega''(x_k)}{\omega'(x_k)}(x - x_k) \geqq c_2, \\ -1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots, n, \quad n = 1, 2, \cdots \end{gathered} \tag{5c}$$

The matrices with the property (5c) are called by Fejér "strongly normal" matrices and he deduces for their Lagrange parabolas convergence criteria of great generality. The identity $\sum_{k=1}^{n} h_k(x) \equiv 1$ plays an important rôle here. By this identity and (5c) we have for strongly normal matrices $\sum_{k=1}^{n} l_k(x)^2 \leqq \dfrac{1}{c_2}$ i.e. a fortiori

$$\begin{gathered} | l_k(x) | \leqq c_2^{-\frac{1}{2}} \\ -1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots n, \quad n = 1, 2, \cdots. \end{gathered} \tag{6}$$

Thus the strongly normal matrices satisfy (6), but the converse is not true. (6) implies an important interpolation property, too.

The importance of the Hermite interpolation is also shown by the following fact. As Bernstein[6] proved, there exists for every matrix A a continuous $f(x)$ and an abscissa x_0 such, that $\limsup_{n\to\infty} | L_n(f) |_{x=x_0} = +\infty$. On the other hand Fejér[7] proved, that for certain special matrices the Hermite parabolas $H_n(f)$ of

[5] L. Fejér, *On the Characterization of some remarkable systems, etc.* Amer. Math. Monthly. 1934.

[6] Bernstein, *Sur la limitation des valeurs d'un polynom etc.* Bull. de l'Acad. de Sciences de l'URSS, 1931.

[7] Fejér, *Die Abschätzung eines Polynoms* etc. Math. Zeitsch. 1930, Bd. 32, pp. 426–457.

any continuous $f(x)$ converge uniformly to $f(x)$ in $[-1, +1]$; e.g. the "matrix T", the n^{th} row of which consists of the n roots of $T_n(x)$, the Tschebyscheff polynomial ($T_n(\cos\theta) = \cos n\theta$), displays this property. The question now arises, which matrices possess this property? Or if uniformity of convergence is not required: what is the necessary and sufficient condition, that for *any* continuous $f(x)$ and at any fixed point x_0

$$\lim_{n\to\infty} H_n(f)_{x=x_0} = f(x_0)? \tag{7}$$

For our purpose it will be sufficient to know, that a necessary condition for (7) is

$$\sum_{k=1}^{n} |h_k(x)| \leqq c_3, \qquad -1 \leqq x \leqq 1, \qquad n = 1, 2, \cdots. \tag{8}$$

This condition follows immediately from the theorem of Hahn (l.c.). The sum in (8) evidently depends only upon A; thus it expresses an interesting interpolation property.

In (3), (4), (5c), (6) and (8) we enumerated some interpolation properties. As far as we know, the whole literature on interpolation[8]—with the exception of two papers—is deducing convergence—and divergence—properties from given suppositions for the matrices. Fejér[9] was the first to invert the problem, deducing distribution-properties from given interpolation properties. He proved e.g. that from (3) or from (5c) it follows, that for $n \to \infty$ the difference of the consecutive elements of the n^{th} row of the matrix tends to 0. The importance of the new idea is shown by the fact, that the required interpolation properties are sometimes quite easily verified. An interesting example is given by the "matrix P", the n^{th} row of which is given by the n roots of the n^{th} Legendre polynomial $P_n(x)$. In consequence of the orthogonality we evidently have

$$\int_{-1}^{1} l_\nu(x)\,dx = \int_{-1}^{1} l_\nu(x)^2\,dx > 0, \qquad \nu = 1, 2, \cdots, n, \quad n = 1, 2, \cdots$$

which means that the matrix P satisfies (3).

In this paper we are concerned with analogous investigations; we deduce the distribution of the fundamental-abscissas from given interpolation properties. In §2 we show the effect of the condition (6).

THEOREM I. *Let*

$$x_\nu^{(n)} = \cos\theta_\nu^{(n)}, \qquad 0 = \theta_0^{(n)} \leqq \theta_1^{(n)} < \theta_2^{(n)} < \cdots < \theta_n^{(n)} \leqq \theta_{n+1}^{(n)} = \pi. \tag{9}$$

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[8] As a matter of fact, we do not mean here complex interpolation.

[9] See footnote 4 and Fejér: *Lagrangesche Interpolation* etc., Math. Ann. (1932).

Then, if matrix A satisfies (6), *we have*

$$\text{(10a)} \qquad \frac{c_4}{n} \leqq \theta_{\nu+1}^{(n)} - \theta_{\nu}^{(n)} \leqq \frac{c_5}{n}, \qquad \nu = 1, 2, \cdots (n-1)$$

The upper bound is valid for $\nu = 0$ *and* $\nu = n$, *the lower one is not.*[10]

The theorem generalizes Fejér's second result in two respects; the assumption is weaker and the result stronger. The theorem means, that the distribution of the roots on the circle with the radius 1 is quasi-uniform i.e.

$$\text{(10b)} \quad \begin{cases} \dfrac{c_4}{n} < |\operatorname{arc} Q_{\nu}^{(n)} Q_{\nu+1}^{(n)}| < \dfrac{c_5}{n}, \qquad \nu = 1, 2, \cdots (n-1) \\ \text{and} \\ |\operatorname{arc} Q_0^{(n)} Q_1^{(n)}| \leqq \dfrac{c_5}{n}, \qquad |\operatorname{arc} Q_n^{(n)} Q_{n+1}^{(n)}| \leqq \dfrac{c_5}{n}. \end{cases}$$

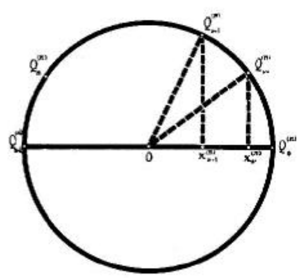

FIG. 1

As stated above, from theorem I it follows a fortiori, that the distribution of the fundamental points Fig. 1. of a strongly normal matrix is quasi-uniform.[11]

There is an application of theorem I for the roots of some classical polynomials. The Jacobi-polynomials $J_n(x, \alpha, \beta)$ corresponding to the parameters α, β ($\alpha \geqq 0$, $\beta \geqq 0$, α and β fixed, $n = 1, 2, \cdots$) may be characterised as the polynomial solutions of the differential equation

$$\text{(11)} \quad (1 - x^2) \frac{d^2 J_n}{dx^2} + 2[(\alpha - \beta) - (\alpha + \beta)x] \frac{dJ_n}{dx} + n[n + 2(\alpha + \beta) - 1] J_n = 0.$$

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[10] E.g. the Π-matrix, the n^{th} row of which is given by the n roots of $\Pi_n(x) = \int_{-1}^{x} P_{n-1}(t)dt$ ($P_n(t)$ the Legendre-polynomial), satisfies (6) and $\theta_1^{(n)} = \theta_0^{(n)} = 0$, $\theta_n^{(n)} = \theta_{n+1}^{(n)} = \pi$. We remark that the lower bound in (10a) is implicitly contained in Fejér's paper: *Bestimmung* etc. Annali della R. Scuole Norm. Sup. Pisa, 1932.

[11] It is not uninteresting to note, that the weaker supposition $\frac{1}{n} \sum_{k=1}^{n} |l_k(x)| \leqq c_6$ is *not* sufficient to assure a quasi-uniform distribution.

We reproduce the proof of Fejér, that the matrices given by the roots of $J_n(x, \alpha, \beta)$ are strongly normal, if $0 \leqq \alpha, \beta < \frac{1}{2}$. Replacing x in (11) by a root $x_k^{(n)}$ of $J_n(x, \alpha, \beta) = 0$, we obtain

$$-\frac{\omega_n''(x_k)}{\omega_n'(x_k)} = -\frac{J_n''(x_k, \alpha, \beta)}{J_n'(x_k, \alpha, \beta)} = \frac{2(\alpha - \beta) - 2(\alpha + \beta)x_k}{1 - x_k^2}$$

further from (5c)

$$\theta_k(1) = 1 - 2(\alpha + \beta) + \frac{4\alpha}{1 + x_k} \geqq 1 - 2\beta \tag{12a}$$

and similarly

$$\theta_k(-1) = 1 - 2(\alpha + \beta) + \frac{4\beta}{1 - x_k} \geqq 1 - 2\alpha \tag{12b}$$

and (5c) immediately follows from (12a) and (12b).

Now applying theorem I we see, that the roots of $J_n(x, \alpha, \beta) = 0$ are quasi-uniformly distributed on the unit-circle in the sense of (10 b), if $0 \leqq \alpha, \beta < \frac{1}{2}$.[12] For $\alpha = \beta$ we obtain the so called ultraspherical polynomials. If $\alpha = \beta = 0$, we have the polynomial $\Pi_n(x)$ (see footnote[10]). Hence by Rolle's theorem we obtain, that the roots of the Legendre-polynomial $P_n(x)$ are quasi-uniformly distributed. Further as $J_n(x, \alpha, \beta)$ differs form $J'_{n+1}(x, \alpha - \frac{1}{2}, \beta - \frac{1}{2})$ only by a constant factor, we conclude by repeatedly employing Rolle's theorem, that the distribution of the roots of the ultraspherical polynomials is for *any* $\alpha \geqq 0$ quasi-uniform.

By this method we can obtain general results concerning the distribution of the roots of certain polynomials satisfying suitable differential equations of the second order.

In §3 we infer the structure of the matrix from properties of the Cotes-numbers. Theorem II states that *if the Cotes-numbers are non negative, then*

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_7}{n}, \qquad \nu = 0, 1, \cdots n, \quad n = 1, 2, \cdots. \tag{13}$$

Theorem III states that *if there exists an integrable $s(x)$ lying between two positive bounds and such that*

$$\int_{-1}^{1} l_k(x)s(x)\,dx \geqq 0, \qquad k = 1, \cdots n, \quad n = 1, 2, \cdots, \tag{14}$$

[12] In addition to (11) and theorem I we must know here that each of the roots lies in $[-1, 1]$; but this is a well known elementary consequence of the orthogonality of the Jacobi polynomials with the weight function $(1 - x)^{2\alpha-1}(1 + x)^{2\beta-1}$.

then (13) *holds*. It will be sufficient to prove theorem III since it is more general than theorem II.[13]

Let us give an application of theorem III. Let $p(x)$ be an R-integrable function, lying between two positive bounds. Consider the orthogonal polynomials with respect to $p(x)$. As it is known, the n^{th} polynomial has in $[-1, +1]$ n different roots. Since

$$\int_{-1}^{1} l_\nu(x)p(x)\,dx = \int_{-1}^{1} l_\nu(x)^2 p(x)\,dx > 0,$$

the hypothesis of theorem III are satisfied and we obtain the

COROLLARY. *Let $p(x)$ be the weight function defined above; then denoting by $\cos\theta_\nu^{(n)}$ ($\nu = 1, 2, \cdots n$) the roots of the n^{th} orthogonal polynomials with respect to $p(x)$, we have*

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq c_8/n, \qquad \nu = 0, 1, \cdots n.$$

Combining theorem III with lemma III of §3 and replacing the η of this lemma by a $\theta_j^{(n)}(4/n < \theta_j^{(n)} < c_9/n)$ we see that, if (14) holds for a matrix, then

$$\mu_1^{(n)} \equiv \mu_1 = \int_{-1}^{1} l_1(x)s(x)\,dx < \frac{c_9}{n^2}.$$

Since

$$\sum_{k=1}^{n} \int_{-1}^{1} l_k(x)s(x)\,dx \geqq 2 \min_{|x|\leqq 1} s(x),$$

it is immediately clear, that for $n > c_{10}$ the $\mu_k^{(n)}$ cannot all be equal. For $s(x) = 1$ Bernstein[14] proved this for any $n > 9$. It would be easy to estimate $c_{10} = c_{10}(s)$ for general $s(x)$, but this we omit for the present. It is essential that $s(x)$ should be bounded; for if $p(x) = (1 - x^2)^{-\frac{1}{2}}$, then $\mu_k^{(n)} = \pi/n$, $k = 1, 2, \cdots n$.

Let us examine the effect of the interpolation property (4), or more simply that of the weaker hypotheses

$$(15) \qquad |\lambda_k^{(n)}| \equiv \left| \int_{-1}^{1} l_k(x)\,dx \right| \leqq c_{11} n^{c_{12}}, \qquad k = 1, 2, \cdots n$$

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[13] It is easy to see that from the fact that the Cotes-numbers are non-negative we cannot obtain a lower estimate for the consecutive θ's. For consider the matrix such that its $(2\nu + 1)^{\text{th}}$ row is given by the roots of the $(2\nu + 1)^{\text{th}}$ Legendre-polynomial, $P_{2\nu+1}(x)$ ($\nu = 0, 1, \cdots$) and its $2\nu^{\text{th}}$ row by the roots of $P_\nu(x)\, P_\nu[(1 + \frac{1}{2}\epsilon_\nu)x + \frac{1}{2}\epsilon_\nu]$, where $\epsilon_\nu > 0$ and so small that $(1 - \frac{1}{2}\epsilon_\nu)/(1 + \frac{1}{2}\epsilon_\nu)$ is greater than the greatest root of $P_\nu(x)$. It is easy to prove that the Cotes-numbers belonging to the matrix are all non-negative, but the difference of certain pairs of consecutive roots of the $2\nu^{\text{th}}$ row are less than ϵ_ν (and the difference of the θ's belonging to this pair $< 2\epsilon_\nu$).

[14] S. Bernstein, Comptes Rendus 1936, 1305-6.

Theorem IV asserts that

$$(16) \qquad \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_{13}\log(n+1)}{n};^{15} \qquad \nu = 0, \cdots n, \quad n = 1, 2, \cdots.$$

c_{13} depends *only* upon c_{11} and c_{12}. This estimate is the best possible, (16) cannot be improved even, if (4) holds. Take e.g. the matrix T, but multiply the elements of its n^{th} row by $1 - a\dfrac{\log^2(n+1)}{n^2}$. Then the elements of the n^{th} row will be

$$\left[1 - a\frac{\log^2(n+1)}{n^2}\right]\cos\frac{2l-1}{2n}\pi, \qquad l = 1, 2, \cdots n,$$

where $a < \frac{1}{2}$. It can be shown by simple computation that (4) holds and

$$|\theta_0^{(n)} - \theta_1^{(n)}| > c_{14}\frac{\log(n+1)}{n}, \qquad n = 1, 2, \cdots,$$

where c_{14} depends only upon a. We omit the details.

A Corollary of Theoem IV. *Consider a sequence of polynomials orthogonal with respect to the weight-function $p(x)$, where we only suppose that $p(x) \geqq 0$ and $\int_{-1}^{1} p(x)\,dx$ and $\int_{-1}^{1}[p(x)]^{-1}\,dx$ (in Riemann or Lebesgue-sense) exist. In this case, as we proved in I.,* (4) *is satisfied, hence for the roots of the n^{th} polynomial* (16) *holds.*

General results about the distribution of the roots of the orthogonal-polynomials—as far as we know—are due to Szegö[16] and Bernstein.[14] They give asymptotic formulae for the orthogonal-polynomials but, as a matter of fact, they are compelled to make strong restrictions with regard to the weight-function. For these weight functions they determine asymptotically the roots, whereas our corollaries deduced from theorem III and IV give weaker, but more general results.[17]

As we saw in footnote,[13] it is impossible to give an estimate from below in theorem IV, but for essentially positive and R-integrable weights we have[18]

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} > \frac{c_{15}}{n^2}. \qquad \nu = 1, 2, \cdots (n-1)$$

[15] From [13] and (4) it follows, that here too $\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}$ can be arbitrarily small.

[16] G. Szegö: *Über den asymptotischen Ausdruck von Polynomen* etc. Math. Ann. 1922, Bd. 86, pp. 114–139.

[17] Since, then, we have proved by another method, that if $p(x) \geqq m > 0$ is L or R-integrable, then it is possible to cover the points of infinity of $p(x)$ by intervals of total length less than ϵ so that on the remaining set the roots of the polynomials orthogonal with respect to $p(x)$ are quasi-uniformly distributed in the sense of (10b). We intend to publish in another paper this result together with others concerning the uniformly dense distribution of the fundamental points.

[18] G. Grünwald and P. Turán: *Über Interpolation.* This paper will appear in Annali di Pisa. In $[-1+\epsilon, 1-\epsilon]$ we can replace the exponent 2 by $\frac{3}{2}$.

We do not give the proof of this result here.

We have already seen that the property given by (3) does not secure the quasi-uniform distribution of the roots; but we may assert a property connected with the integral of the fundamental functions by which it is involved. The property in question is

$$(17) \qquad \int_{-1}^{1} \frac{l_k(x)^2}{\sqrt{(1-x^2)}}\,dx \leqq \frac{c_{16}}{n}, \qquad k = 1, 2, \cdots n, \quad n = 1, 2, \cdots.$$

The proof of this statement is so simple, that we give it immediately. From (17) it follows, that the fundamental functions are uniformly bounded with respect to n i.e. if $1 \leqq \nu \leqq n$,

$$\max_{-1 \leqq x \leqq +1} |\, l_\nu(x) \,| = |\, l_\nu(\xi_0) \,| = |\, l_\nu(\cos \varphi_0) \,|,$$

where without any loss of generality

$$0 \leqq \varphi_0 \leqq \frac{\pi}{2},$$

by Bernstein's theorem

$$\frac{c_{16}}{n} \geqq \int_{-1}^{1} \frac{l_\nu(x)^2}{\sqrt{(1-x^2)}}\,dx = \int_0^\pi l_\nu(\cos\theta)^2\,d\theta > \int_{\varphi_0}^{\varphi_0+\frac{1}{2n}} l_\nu(\cos\theta)^2\,d\theta > \frac{c_{17}}{n}\, l_\nu(\cos\varphi_0)^2,$$

which means, that our assertion is an immediate consequence of theorem I.

In §4, we shall be concerned with the interpolation property (8) and with the consequences of the much more general supposition

$$(18) \qquad |\, h_k(x) \,| \leqq c_{18}, \; -1 \leqq x \leqq 1, \qquad k = 1, 2, \cdots n, \quad n = 1, 2, \cdots.$$

In our theorem V we show, that even (18) implies quasi-uniform distribution as does (6) in theorem I. It is probable that (18) implies (6), but we cannot prove it.

2.

THEOREM I. *If*

$$|\, l_k(x) \,| \leqq c_{19}, \qquad -1 \leqq x \leqq +1, \qquad k = 1, \cdots n, \quad n = 1, 2, \cdots,$$

then we have

$$\frac{c_{20}}{n} \leqq \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_{21}}{n} \qquad \nu = 1, 2, \cdots, (n-1),$$

and the upper bound is valid for $\nu = 0$ *and* $\nu = n$.

First we prove the lower estimate. For any $1 \leqq \nu \leqq n$ we have by Rolle's theorem

$$(19) \qquad \frac{1}{|\, \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \,|} = \left| \frac{l_\nu(\cos\theta_\nu^{(n)}) - l_\nu(\cos\theta_{\nu+1}^{(n)})}{\theta_\nu^{(n)} - \theta_{\nu+1}^{(n)}} \right| = \left| \frac{dl_\nu(\cos\theta)}{d\theta} \right|_{\theta=\theta'},$$

where θ' lies between $\theta_\nu^{(n)}$ and $\theta_{\nu+1}^{(n)}$. But by the hypothesis $l_\nu(\cos\theta)$ is a trigonometric polynomial of degree $(n-1)$, for which

$$| l_\nu(\cos\theta) | \leqq c_{19}.$$

Thus according to a well-known theorem of Bernstein-Fejér

$$\left| \frac{dl_\nu(\cos\theta)}{d\theta} \right| \leqq c_{19}(n-1).$$

Putting this into (19) we obtain

$$(20) \qquad | \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} | \geqq \frac{1}{c_{19}(n-1)} \geqq \frac{c_{20}}{n}.$$

We now prove the upper estimate. Let

$$(21) \qquad \max_{i=0,1,\cdots n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} = \frac{2D(n)c_{20}}{n}.$$

We must prove that $D(n) \leqq c_{22}$; we can suppose $D(n) \geqq 2$. Let $\frac{1}{2}(\theta_\nu^{(n)} + \theta_{\nu+1}^{(n)}) = \delta$ and

$$(22) \qquad \varphi(\theta) = \frac{1}{n^2}\left(\frac{\sin n \dfrac{\theta+\delta}{2}}{\sin \dfrac{\theta+\delta}{2}} \right)^2 + \frac{1}{n^2}\left(\frac{\sin n \dfrac{\theta-\delta}{2}}{\sin \dfrac{\theta-\delta}{2}} \right)^2, \qquad 0 \leqq \theta \leqq \pi.$$

This expression is in consequence of a well known identity, a pure cosine polynomial in θ of degree $(n-1)$. Evidently

$$(23) \qquad \varphi(\delta) \geqq 1.$$

Further

$$(24) \qquad | \varphi(\theta) | \leqq \frac{4}{n^2(\theta+\delta)^2}\left[\frac{\dfrac{\theta+\delta}{2}}{\sin\dfrac{\theta+\delta}{2}} \right]^2 + \frac{4}{n^2(\theta-\delta)^2}\left[\frac{\dfrac{\theta-\delta}{2}}{\sin\dfrac{\theta-\delta}{2}} \right]^2.$$

Suppose first that $0 \leqq \delta \leqq \frac{\pi}{2}$; then $0 \leqq \frac{\theta+\delta}{2} \leqq \frac{3\pi}{4}$, $-\frac{\pi}{4} \leqq \frac{\theta-\delta}{2} \leqq \frac{\pi}{2}$. As $\left| \frac{\sin\alpha}{\alpha} \right| \geqq \frac{4}{3\pi}\frac{\sqrt{2}}{2}$ when $|\alpha| \leqq \frac{3\pi}{4}$, by (24) we have for $0 \leqq \theta \leqq \pi$

$$(25) \qquad | \varphi(\theta) | \leqq \frac{9\pi^2}{2n^2}\left[\frac{1}{(\theta+\delta)^2} + \frac{1}{(\theta-\delta)^2} \right].$$

As $\varphi(\theta)$ is a pure cosine polynomial, we have

$$(26) \qquad \sum_{\mu=1}^{n} \varphi(\theta_\mu^{(n)}) l_\mu(\cos\theta) \equiv \varphi(\theta)$$

i.e. for $\theta = \delta$ in consequence of (23), (26), (25) and the hypotheses

$$
\begin{aligned}
1 &\leqq |\varphi(\delta)| = \left| \sum_{\mu=1}^{n} \varphi(\theta_\mu^{(n)}) l_\mu(\cos\delta) \right| \leqq \sum_{\mu=1}^{n} |\varphi(\theta_\mu^{(n)})| \, |l_\mu(\cos\delta)| \\
&\leqq \frac{9\pi^2}{2n^2} c_{19} \left[\sum_{\mu=1}^{n} \frac{1}{(\theta_\mu^{(n)} + \delta)^2} + \sum_{\mu=1}^{n} \frac{1}{(\theta_\mu^{(n)} - \delta)^2} \right] \equiv \frac{9 c_{19} \pi^2}{2n^2} [S_1 + S_2].
\end{aligned} \tag{27}
$$

But by (20)

$$
\theta_\mu^{(n)} \geqq (\mu - 1) \frac{c_{20}}{n}, \qquad \mu = 1, 2, \cdots n \tag{28a}
$$

$$
\delta \geqq \frac{D(n) c_{20}}{n} \tag{28b}
$$

thus

$$
S_1 \equiv \sum_{\mu=1}^{n} \frac{1}{(\theta_\mu^{(n)} + \delta)^2} \leqq \sum_{\mu=1}^{n} \frac{1}{\left[(\mu - 1) \dfrac{c_{20}}{n} + \dfrac{D(n) c_{20}}{n} \right]^2} < \frac{n^2}{c_{20}^2} \sum_{\mu=0}^{\infty} \frac{1}{(\mu + D(n))^2},
$$

and from $D(n) \geqq 2$ we have

$$
S_1 < \frac{n^2}{c_{20}^2} \int_{D(n)-1}^{\infty} \frac{dt}{t^2} = \frac{n^2}{c_{20}^2} \frac{1}{D(n) - 1}. \tag{29a}
$$

Further if $1 \leqq \nu \leqq n$,

$$
\begin{aligned}
S_2 &= \sum_{\mu=1}^{\nu} \frac{1}{(\theta_\mu^{(n)} - \delta)^2} + \sum_{\mu=\nu+1}^{n} \frac{1}{(\theta_\mu^{(n)} - \delta)^2} \leqq \sum_{\mu=1}^{n} \frac{1}{\left[\dfrac{D(n) c_{20}}{n} + (\nu - \mu) \dfrac{c_{20}}{n} \right]^2} \\
&+ \sum_{\mu=\nu+1}^{n} \frac{1}{\left[\dfrac{D(n) c_{20}}{n} + (\mu - \nu - 1) \dfrac{c_{20}}{n} \right]^2} < \frac{2n^2}{c_{20}^2} \int_{D(n)-1}^{\infty} \frac{dt}{t^2} = \frac{2n^2}{c_{20}^2 (D(n) - 1)}.
\end{aligned} \tag{29b}
$$

Putting (29a) and (29b) into (27) we obtain

$$
1 \leqq \frac{9\pi^2 c_{19}}{2n^2} \cdot \frac{3n^2}{c_{20}^2 [D(n) - 1]} = \frac{27\pi^2}{2} \frac{c_{19}}{c_{20}^2} \frac{1}{D(n) - 1} = \frac{27\pi^2}{2} c_{19}^3 \frac{1}{D(n) - 1}
$$

i.e.

$$
D(n) \leqq 1 + \frac{27\pi^2}{2} c_{19}^3 = c_{22}.
$$

$$
\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{2 c_{22} c_{20}}{n} = \frac{c_{21}}{n}. \qquad \text{Q. e. d.}
$$

For the cases $\left\{ \begin{matrix} \nu = 0 \\ 0 \leqq \delta \leqq \pi/2 \end{matrix} \right\}$, $\left\{ \begin{matrix} \nu = n \\ 0 \leqq \delta \leqq \pi/2 \end{matrix} \right\}$, $\{\pi/2 < \delta \leqq \pi\}$ the proof follows similar lines. Thus the result is established.

3.

THEOREM III. *If for the matrix there exists an L-integrable $s(x)$, which in $[-1, +1]$ lies between two positive bounds a and $b (a < b)$ and is such that*

$$\int_{-1}^{1} l_i(x)s(x)\,dx \geqq 0, \qquad i = 1, 2, \cdots n, \quad n = 1, 2, \cdots,$$

then

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_{23}}{n}, \qquad \nu = 0, 1, \cdots n, \quad n = 1, 2, \cdots.$$

Let[19] μ be the greatest integer not exceeding $\dfrac{n+1}{2}$ i.e. $\mu > n/2, 0 \leqq \eta \leqq \pi/2$ and

$$(30) \qquad f(\theta, n) = \frac{1}{\mu^4}\left[\left(\frac{\sin \mu \dfrac{\theta+\eta}{2}}{\sin \dfrac{\theta+\eta}{2}}\right)^4 + \left(\frac{\sin \mu \dfrac{\theta-\eta}{2}}{\sin \dfrac{\theta-\eta}{2}}\right)^4\right];$$

then, for fixed $\eta, f(\theta, \eta)$ is a cosine polynomial the degree of which does not exceed $(n - 1)$ and for which

$$(31) \qquad |f(\theta, \eta)| \leqq \frac{c_{24}}{\mu^4} \cdot \frac{1}{(\theta - \eta)^4}, \qquad 0 \leqq \theta \leqq \pi.$$

LEMMA I. *If $\pi - 1/2\mu \geqq \eta \geqq 1/2\mu$, then*

$$\frac{c_{25}\eta}{\mu} \leqq J \equiv \int_0^\pi f(\theta, \eta)s(\cos\theta)\sin\theta\,d\theta \leqq \frac{c_{26}\eta}{\mu}.$$

Without loss of generality let $\eta \leqq \pi/2$. We have

$$(32a) \quad J \geqq a\int_\eta^{\eta+\pi/\mu} f(\theta, \eta)\sin\eta\,d\theta \geqq \frac{2a\eta}{\pi\mu^4}\int_\eta^{\eta+\pi/\mu}\left(\frac{\dfrac{2}{\pi}\mu\dfrac{\theta-\eta}{2}}{\dfrac{\theta-\eta}{2}}\right)^4 d\theta = \frac{c_{26}\eta}{\mu}.$$

Further

$$\left|\frac{1}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4 \sin\theta\cdot s(\cos\theta)\,d\theta\right| < \left|\frac{2\cos\eta}{\mu^4}\int_0^\pi \frac{\sin^4\mu\dfrac{\theta+\eta}{2}}{\sin^3\dfrac{\theta+\eta}{2}}\cos\frac{\theta+\eta}{2}\,d\theta\right.$$

[19] Throughout this paragraph the c's are independent of D, η, θ and n, but dependent on a and b.

$$(32\text{b})\qquad -\frac{\sin\eta}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4\cos(\theta+\eta)\,d\theta\,\Bigg|\cdot b$$

$$< c_{27}\left[\frac{1}{\mu^4}\int_{1/\mu}^\infty\frac{dt}{t^3}+\frac{\eta}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4d\theta\right],$$

and

$$\left|\frac{1}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4 s(\cos\theta)\sin\theta\,d\theta\right| < b\,\Bigg|\frac{1}{\mu^4}\sin\eta\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4\cos(\theta-\eta)\,d\theta$$

$$(32\text{c})\qquad +\frac{2\cos\eta}{\mu^4}\int_0^\pi\frac{\sin^4\mu\dfrac{\theta-\eta}{2}}{\sin^3\dfrac{\theta-\eta}{2}}\cos\frac{\theta-\eta}{2}\,d\theta\,\Bigg| < c_{28}\left[\frac{\eta}{\mu^4}\int_0^\pi\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4d\theta\right.$$

$$\left.+\frac{1}{\mu^4}\int_0^{\eta-1/\mu}\frac{d\theta}{(\eta-\theta)^3}+\frac{1}{\mu^4}\int_{\eta-1/\mu}^{\eta+1/\mu}\mu^3\,d\theta+\frac{1}{\mu^4}\int_{\eta+1/\mu}^\pi\frac{1}{(\theta-\eta)^3}\,d\theta\right].$$

Hence by (32b) and (32c)

$$(33)\qquad J\leqq c_{29}\left[\frac{\eta}{\mu^4}\int_0^\pi\left\{\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4+\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4\right\}d\theta+\frac{1}{\mu^2}\right].$$

For the integral on the right, the integrand being even, we have

$$\frac{1}{2}\int_0^{2\pi}f(\theta,\eta)\,d\theta.$$

But

$$(34)\qquad \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\sin\mu\dfrac{\theta+\eta}{2}}{\sin\dfrac{\theta+\eta}{2}}\right)^4d\theta=\mu^2+2((\mu-1)^2+(\mu-2)^2+\cdots+1^2)<c_{30}\mu^3$$

by (22a) and by Parseval's theorem. The same holds for $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\left(\frac{\sin\mu\dfrac{\theta-\eta}{2}}{\sin\dfrac{\theta-\eta}{2}}\right)^4d\theta.$

Hence by $\eta\geqq 1/\mu$, (33) and (34) Lemma I is proved.

LEMMA II. *In the interval*

$$0 \leqq \eta - 1/\mu \leqq \theta \leqq \eta + 1/\mu \leqq \pi$$

we have

$$f(\theta, \eta) \geqq c_{31}.$$

In this interval if $|\alpha| \leqq 1$,

$$f(\theta, \eta) \geqq \frac{1}{\mu^4}\left(\frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2\mu}}\right)^4 \geqq c_{31}.$$

LEMMA III. *Consider the fundamental points* $\cos\theta_\nu^{(n)}$, *for which* $(0 \leqq)\ \eta - 1/\mu \leqq \theta_\nu^{(n)} \leqq \eta + 1/\mu (\leqq \pi)$. *If*

$$\mu_k \equiv \int_0^\pi l_k(\cos\theta)s(\cos\theta)\sin\theta\, d\theta \geqq 0, \qquad k = 1, 2, \cdots n,$$

then

$$S \equiv \sum_{\substack{\nu \\ \eta-1/\mu \leqq \theta_\nu \leqq \eta+1/\mu}} \mu_\nu < c_{32}\eta/\mu.$$

By Lemma II and Lemma I we have

$$c_{31}S < \sum_{\substack{\nu \\ \eta-1/\mu \leqq \theta_\nu^{(n)} \leqq \eta+1/\mu}} f(\theta_\nu^{(n)}, \eta)\mu_\nu \leqq \sum_{\nu=1}^{n} f(\theta_\nu^{(n)}, \eta)\mu_\nu$$

$$= \int_0^\pi f(\theta, \eta)s(\cos\theta)\sin\theta\, d\theta \leqq \frac{c_{26}\eta}{\mu}.$$

Hence

$$S < \frac{c_{26}}{c_{31}}\cdot\frac{\eta}{\mu} = \frac{c_{32}\eta}{\mu}.$$

Now we shall prove our theorem III. Let

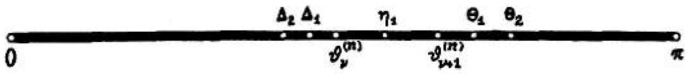
$$\max_{i=0,1,\cdots n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} = \frac{D(n)}{n+1}$$

and we have to prove that $D(n) \leqq c_{23}$. Let $\frac{1}{2}(\theta_{\nu+1}^{(n)} + \theta_\nu^{(n)}) = \eta_1$.

We may suppose without loss of generality that $\eta_1 \leqq \pi/2$ and consider the expression $f(\theta, \eta)$ of (30) for $\eta = \eta_1$. Starting at $\theta_{\nu+1}^{(n)}$ mark off to the right intervals of length $1/\mu$ till the whole of the interval $[\theta_{\nu+1}^{(n)}, \pi]$ is covered; the last interval, the length of which is less than $1/\mu$, we add to the last but one. Similarly, starting at $\theta_\nu^{n)}$ mark off intervals to the left to cover $[0, \theta_\nu^{(n)}]$. Let the end-points (Fig. 2) be $\Theta_1, \Theta_2, \cdots, \Delta_1, \Delta_2, \cdots$ respectively. It is evident that

$$\Theta_r = \eta_1 + \frac{D(n)}{2n+2} + \frac{r}{\mu}, \qquad r = 0, 1, 2, \cdots \tag{35a}$$

(35b) $$\Delta_s = \eta_1 - \frac{D(n)}{2n+2} - \frac{s}{\mu} \qquad s = 0, 1, 2, \cdots$$



Since

$$\sum_{i=0}^{n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \pi,$$

the greatest distance between two consecutive $\theta_\mu^{(n)}$'s is greater than or equal to $\pi/(n+1)$ i.e. $\eta_1 \geqq \pi/(2n+2)$, $D(n) \geqq \pi/2$. Thus by Lemma I

(36) $$\frac{c_{25}\eta_1}{\mu} \leqq \int_0^\pi f(\theta, \eta_1)s(\cos\theta)\sin\theta\, d\theta = \sum_{i=1}^{n} f(\theta_i^{(n)}, \eta_1)\mu_i.$$

In the sum on the right consider the members for which $\theta_i^{(n)}$ lies in $[\Theta_r, \Theta_{r+1}]$. For these we have by (31)

(37) $$|f(\theta_i^{(n)}, \eta_1)| \leqq \frac{c_{24}}{\mu^4}\frac{1}{(\Theta_r - \eta_1)^4}$$

and by Lemma III with $\eta = \dfrac{\Theta_r + \Theta_{r+1}}{2} = \eta_1 + \dfrac{D(n)}{2n+2} + \dfrac{r+\frac{1}{2}}{\mu}$ [20]

(38) $$\sum_{\Theta_r \leqq \theta_i^{(n)} \leqq \Theta_{r+1}} \mu_i < \frac{c_{32}\left[\eta_1 + \dfrac{D(n)}{2n+2} + \dfrac{2r+1}{2\mu}\right]}{\mu}.$$

Hence this part of the sum (36) is, by (37) and (38), less than

(39) $$\frac{c_{24}c_{32}}{\mu^5}\frac{\eta_1 + \dfrac{D(n)}{2n+2} + \dfrac{2r+1}{2\mu}}{\left(\dfrac{D(n)}{2n+2} + \dfrac{r}{\mu}\right)^4} = c_{24}c_{32}\left[\frac{\dfrac{\eta_1}{\mu^5} + \dfrac{1}{2\mu^6}}{\left(\dfrac{D(n)}{2n+2} + \dfrac{r}{\mu}\right)^4} + \frac{1}{\left(\dfrac{D(n)}{2n+2} + \dfrac{r}{\mu}\right)^3}\cdot\frac{1}{\mu^5}\right].$$

This holds evidently also for $[\theta_{\nu+1}^{(n)}, \Theta_1]$ with $r = 0$. Hence the partial sum of (36) summed over all $i \geqq \nu + 1$ is in consequence of $\frac{1}{2}(n+1) \geqq \mu \geqq n/2$ less than

$$c_{33}\left[\frac{\eta_1}{n}\sum_{r=0}^{\infty}\frac{1}{(D(n)+4r)^4} + \frac{1}{n^2}\sum_{r=0}^{\infty}\frac{1}{(D(n)+r)^4} + \frac{1}{n^2}\sum_{r=0}^{\infty}\frac{1}{(D(n)+r)^3}\right]$$

[20] It is clear, that for each such η, $\eta \geqq 1/2\mu$ is satisfied.

and by $\eta_1 \geqq \pi/2n$

$$(40) \qquad < \frac{c_{34}\eta_1}{n}\left[\frac{1}{D(n)^3} + \frac{2}{D(n)^2}\right] < \frac{3c_{34}\eta_1}{nD(n)^2}.$$

The sum for $i \leqq \nu$ may be estimated in the same way. Thus by (36) and (40) we obtain

$$\frac{c_{25}\eta_1}{\mu} \leqq \frac{3c_{34}\eta_1}{nD(n)^2},$$

$$D(n) \leqq c_{23}. \qquad \text{Q. e. d.}$$

THEOREM IV. *If for a matrix we have*

$$\left| \int_{-1}^{1} l_\nu(x)\,dx \right| \leqq c_{11} n^{c_{12}},$$

then

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_{13}\log(n+1)}{n}, \qquad \nu = 0, 1, \cdots n, \quad n = 1, 2, \cdots.$$

PROOF. Let $n \geqq 9$, r be even and $\geqq 4$, the odd integer $m \geqq 5$ so that

$$(41) \qquad \tfrac{1}{2}(m-1)r \leqq n-1;$$

r and m are indefinite for the moment with only the restriction that both tend to infinity as $n \to \infty$. Let

$$\max_{i=0,1,\cdots n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} = 2\delta(n)$$

$$\tfrac{1}{2}(\theta_{\nu+1}^{(n)} + \theta_\nu^{(n)}) = \theta',$$

and

$$(42) \qquad f(\theta) = \frac{1}{m^r}\left[\left(\frac{\sin m\dfrac{\theta+\theta'}{2}}{\sin\dfrac{\theta+\theta'}{2}}\right)^r + \left(\frac{\sin m\dfrac{\theta-\theta'}{2}}{\sin\dfrac{\theta-\theta'}{2}}\right)^r\right].$$

Evidently $f(\theta)$ is a trigonometric polynomial of degree $\frac{1}{2}(m-1)r \leqq n-1$ and as $f(\theta) = f(-\theta)$, it is a pure cosine polynomial. First consider $0 \leqq \theta' \leqq \pi/2$; if $\pi/2 < \theta' \leqq \pi$, the proof is similar. As $\delta(n) \geqq \pi/(2n+2)$ we have $\theta' \geqq \pi/(2n+2)$.

As $m \geqq 5$, the interval $[\theta', \theta' + \pi/m]$ will lie entirely in $[0, \pi]$.

Then, as r is even and $\theta' \leqq \pi/2$,

$$(43) \qquad \int_0^\pi f(\theta)\sin\theta\,d\theta \geqq \frac{1}{m^r}\int_{\theta'}^{\theta'+\pi/m}\left(\frac{\dfrac{2}{\pi}m\dfrac{\theta-\theta'}{2}}{\dfrac{\theta-\theta'}{2}}\right)^r \frac{2}{\pi}\delta(n)\,d\theta = \left(\frac{2}{\pi}\right)^r \frac{2}{m}\delta(n).$$

The degree of $f(\theta)$ being $(n-1)$ at most, we have

$$\sum_{i=1}^{n} f(\theta_i^{(n)})\, l_i(\cos\theta) \equiv f(\theta),$$

hence by (43)

$$
\begin{aligned}
(44)\qquad 2\left(\frac{2}{\pi}\right)^r \frac{\delta(n)}{m} &\leqq \int_0^\pi f(\theta)\sin\theta\, d\theta = \left|\sum_{i=1}^{n} f(\theta_i^{(n)}) \int_0^\pi l_i(\cos\theta)\sin\theta\, d\theta\right| \\
&= \left|\sum_{i=1}^{n} f(\theta_i^{(n)}) \int_{-1}^{1} l_i(x)\,dx\right| \leqq c_{11} n^{c_{12}}\cdot n \max_{i=1,\cdots n} |f(\theta_i^{(n)})|.
\end{aligned}
$$

As $|\frac{1}{2}(\theta-\theta')| \leqq \pi/2$, we have from (42)

$$(45)\qquad |f(\theta_i^{(n)})| \leqq \frac{1}{m^r}\left[\frac{1}{\left[\frac{1}{\pi}(\theta_i^{(n)}-\theta')\right]^r} + \frac{1}{\left[\frac{1}{\pi}\delta(n)\right]^r}\right] \leqq 2\left(\frac{\pi}{m\delta(n)}\right)^r;$$

by putting (45) into (44)

$$2\left(\frac{2}{\pi}\right)^r \frac{\delta(n)}{m} \leqq c_{11} n^{1+c_{12}}\, 2\left(\frac{\pi}{m\delta(n)}\right)^r,$$

thus

$$(46)\qquad \delta(n) \leqq \frac{\pi^2}{m}\left(\frac{2c_{11}}{\pi^2}\, m^2 n^{1+c_{12}}\right)^{\frac{1}{r+1}}.$$

Now let r be the greatest even integer not exceeding $\log n$ and m be the greatest integer less than $(2n-2)/r$; then (41) is satisfied for sufficiently great n and thus by (46) we have

$$\delta(n) \leqq \frac{c_{13}\log(n+1)}{n},$$

which establishes the result.

4.

Here we have to prove that from

$$(47)\qquad |\theta_k(x)|\, l_k(x)^2 \equiv |h_k(x)| \leqq c_{36}$$

$$-1 \leqq x \leqq +1, \qquad k = 1, 2, \cdots n, \qquad n = 1, 2, \cdots,$$

follows

$$\frac{c_{37}}{n} \leqq \theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \leqq \frac{c_{38}}{n}.$$[21]

[21] The upper estimate holds for $\nu = 0, 1, 2 \cdots n$, the lower for $\nu = 1, 2, \cdots (n-1)$.

We obtain the lower estimate as in theorem I, namely

$$\frac{1}{|\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}|} = \left| \frac{h_\nu(\cos\theta_\nu^{(n)}) - h_\nu(\cos\theta_{\nu+1}^{(n)})}{\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}} \right| = \left| \frac{dh_\nu(\cos\theta)}{d\theta} \right|_{\theta=\varphi_1}$$

where according to Rolle's theorem φ_1 lies between $\theta_\nu^{(n)}$ and $\theta_{\nu+1}^{(n)}$. Since $h_\nu(\cos\theta)$ is a trigonometric polynomial of degree $(2n - 1)$, we obtain by Bernstein's theorem

$$\frac{1}{|\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}|} \leqq (2n-1) \max_{-1\leqq x\leqq 1} |h_\nu(x)| \leqq c_{36}(2n-1),$$

hence

$$\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)} \geqq \frac{1}{c_{36}(2n-1)} > \frac{c_{37}}{n}, \qquad \nu = 1, 2, \cdots (n-1). \tag{48}$$

Let us now consider the upper bound. Let

$$\max_{i=0,1,\cdots n} (\theta_{i+1}^{(n)} - \theta_i^{(n)}) = \theta_{\mu+1}^{(n)} - \theta_\mu^{(n)} = \frac{2D(n)}{n+1}, \tag{49a}$$

and

$$\tfrac{1}{2}(\theta_\mu^{(n)} + \theta_{\mu+1}^{(n)}) = \varphi_2. \tag{49b}$$

From

$$\sum_{\nu=0}^{n} (\theta_{\nu+1}^{(n)} - \theta_\nu^{(n)}) = \pi$$

we obtain

$$D(n) \geqq \pi/2. \tag{49c}$$

Without any loss of generality we may suppose

$$0 \leqq \varphi_2 \leqq \pi/2. \tag{49d}$$

Let $\varphi(x)$ be the polynomial (its degree does not exceed $(n - 1)$), for which $\varphi(\cos\theta)$ is identical with the polynomial defined at (42) if we replace θ' by φ_2 and $r = 10$, $m = \left[\frac{n-1}{5}\right]$. Since

$$|\phi(\cos\theta_\nu^{(n)})| < \frac{c_{39}}{n^{10}|\theta_\nu^{(n)} - \varphi_2|^{10}}, \tag{50}$$

we have

$$1 \leqq \phi(\cos\varphi_2) = \sum_{\nu=1}^{n} \phi(\cos\theta_\nu^{(n)})\, l_\nu(\cos\varphi_2) < \frac{c_{39}}{n^{10}} \sum_{\nu=1}^{n} \frac{|l_\nu(\cos\varphi_2)|}{|\theta_\nu^{(n)} - \varphi_2|^{10}}. \tag{51}$$

CASE I. $|l_\nu(\cos\varphi_2)| \leqq n^8 |\theta_\nu^{(n)} - \varphi_2|^8$, $\nu = 1, 2, \cdots n$.

From (51) we have

$$1 \leqq \frac{c_{39}}{n^2} \sum_{\nu=1}^{n} \frac{1}{|\theta_\nu^{(n)} - \varphi_2|^2} \tag{52a}$$

and by (48)

$$|\theta_\nu^{(n)} - \varphi_2| \geqq \frac{D(n) + |\mu - \nu| c_{37}}{n+1} c_{40}. \tag{52b}$$

Finally by putting (52b) into (52a) we obtain

$$1 \leqq c_{41} \sum_{\nu=\mu}^{\infty} \frac{1}{[D(n) + c_{37}(\nu - \mu)]^2} < \frac{c_{42}}{D(n)},$$

$$D(n) \leqq c_{38},$$

which settles case I.

CASE II. A. There is a k such, that

$$|l_k(\cos \varphi_2)| > n^8 (\theta_k^{(n)} - \varphi_2)^8 \tag{53}$$

and $l_k(\cos \theta)$ takes its absolute maximum in $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$. First we require two lemmas.

LEMMA 1. *Let $f(\theta)$ be a cosine polynomial of degree m, the roots of which are all real and distinct*

$$(\psi_0 \equiv) 0 \leqq \psi_1 < \psi_2 < \cdots < \psi_m \leqq \pi (\equiv \psi_{m+1}),$$

taking its absolute maximum in $[\psi_\nu, \psi_{\nu+1}]$; then to every ξ in $[\psi_\nu, \psi_{\nu+1}]$ there exists an interval l such that:

1. *l lies in $[\psi_\nu, \psi_{\nu+1}]$,*
2. *ξ is an endpoint of l,*
3. *The length of l is greater than $1/2m$,*
4. *For every θ lying in l we have*

$$|f(\theta)| > \tfrac{1}{2} |f(\xi)|.$$

PROOF. According to the hypothesis $f(\theta)$ has in $[\psi_\nu, \psi_{\nu+1}]$ the unique extreme $\theta = \varphi_3$; we can suppose this to be a maximum. Suppose first $\xi \leqq \varphi_3$. If $\xi + 1/2m \leqq \varphi_3$; our lemma follows from the fact, that $f(\theta)$ is monotonously increasing in $[\xi, \varphi_3]$.

Suppose now $\xi + 1/2m > \varphi_3$. Then Bernstein's well-known theorem gives

$$|f'(\theta)| \leqq mf(\varphi_3)$$

and from this we have

$$\begin{aligned} f\left(\xi + \frac{1}{2m}\right) &= f(\varphi_3) + \int_{\varphi_3}^{\xi+1/2m} f'(\theta)\, d\theta \\ &> f(\varphi_3) - \frac{1}{2m} \cdot mf(\varphi_3) = \tfrac{1}{2} f(\varphi_3) \geqq \tfrac{1}{2} f(\xi). \end{aligned} \tag{54}$$

As (54) a fortiori holds for $\xi \leqq \theta \leqq \xi + 1/2m$, the lemma is proved for $\xi \leqq \varphi_3$. Similarly for $\xi > \varphi_3$ we consider the interval $[\xi - 1/2m, \xi]$.

Let us now consider the case II A. Since $l_k(\cos\theta)$ takes its absolute maximum in $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$, we obtain from our lemma by putting $f(\theta) = l_k(\cos\theta)$, $\xi = \varphi_2$, and (53), that in $[\varphi_2, \varphi_2 \pm 1/(2n-2)]$[22] and a fortiori in $[\varphi_2, \varphi_2 \pm 1/2n]$

$$| l_k(\cos\theta) | \geqq \tfrac{1}{2} | l_k(\cos\varphi_2) | > \frac{n^8}{2} | \theta_k^{(n)} - \varphi_2 |^8.$$

Thus for $\varphi_2 \pm 1/2n \equiv \varphi_4$[23] we have

$$(55) \qquad | l_k(\cos\varphi_4) | > \frac{n^8}{2} | \theta_k^{(n)} - \varphi_2 |^8.$$

A simple geometrical observation shows that if for the linear function $a(x) = \alpha x + \beta$

$$a(\xi_1) = 1,$$

and, further, ξ_2 and ξ_3 lie on the same side of ξ_1, then

$$(56) \qquad \max(| a(\xi_2) |, | a(\xi_3) |) \geqq \frac{\left| \dfrac{\xi_2 - \xi_3}{2} \right|}{\left| \dfrac{\xi_2 + \xi_3}{2} - \xi_1 \right|} = \frac{\left| \dfrac{\xi_2 + \xi_3}{2} - \xi_3 \right|}{\left| \dfrac{\xi_2 + \xi_3}{2} - \xi_1 \right|}.$$

By applying (56) to $a(x) \equiv \theta_k(x)$ and putting $\xi_1 = \cos\theta_k^{(n)}$, $\xi_2 = \cos\varphi_2$, $\xi_3 = \cos\varphi_4$ we obtain

$$(57) \qquad \max(| \theta_k(\cos\varphi_2) |, | \theta_k(\cos\varphi_4) |) \geqq \frac{\left| \dfrac{\cos\varphi_2 + \cos\varphi_4}{2} - \cos\varphi_4 \right|}{\left| \dfrac{\cos\varphi_2 + \cos\varphi_4}{2} - \cos\theta_k^{(n)} \right|}.$$

Replacing in (57) $\frac{1}{2}(\cos\varphi_2 + \cos\varphi_4)$ by $\cos\varphi_5 (\varphi_2 \leqq \varphi_5 \leqq \varphi_4$ or $\varphi_4 \leqq \varphi_5 \leqq \varphi_2)$ we have

$$(58) \qquad \max(| \theta_k(\cos\varphi_2) |, | \theta_k(\cos\varphi_4) |) \geqq \left| \frac{\cos\varphi_5 - \cos\varphi_4}{\cos\varphi_5 - \cos\theta_k^{(n)}} \right|.\text{[24]}$$

Now we prove

LEMMA 2. *Let* $0 \leqq \lambda_1 < \lambda_2 < \lambda_3 \leqq \pi$, *then*

$$(59) \qquad \left| \frac{\cos\lambda_1 - \cos\lambda_2}{\cos\lambda_1 - \cos\lambda_3} \right| \geqq \frac{4}{\pi^2} \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 - \lambda_3} \right)^2.\text{[25]}$$

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[22] The sign which we must take depends only on the position of φ_2.

[23] From the lemma it follows that φ_4 also lies in $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$.

[24] It is clear, that the numerator can also be written in the form $| \frac{1}{2}(\cos\varphi_2 + \cos\varphi_4) - \cos\varphi_2 |$. We use this form if φ_2 lies between φ_5 and $\theta_k^{(n)}$; if φ_4 lies between φ_5 and $\theta_k^{(n)}$ we use the form (58).

[25] Obviously one and only one of these two cases arises; we suppose in the text the second one. Evidently the same inequality holds for $0 \leqq \lambda_3 < \lambda_2 < \lambda_1 \leqq \pi$.

PROOF. Obviously if $\angle\ 0_1 0 A_\nu = \lambda_\nu$, $|\cos\lambda_\nu - \cos\lambda_\mu|$ is the projection of $A_\mu A_\nu$; then for fixed $\lambda_2 - \lambda_1$ and $\lambda_3 - \lambda_1$ the quotient $\left|\dfrac{\cos\lambda_1 - \cos\lambda_2}{\cos\lambda_1 - \cos\lambda_3}\right|$ takes its minimum at $\lambda_1 = 0$; the value of the minimum is evidently

$$\frac{1 - \cos(\lambda_2 - \lambda_1)}{1 - \cos(\lambda_3 - \lambda_1)} = \left(\frac{\sin\dfrac{\lambda_2 - \lambda_1}{2}}{\sin\dfrac{\lambda_3 - \lambda_1}{2}}\right)^2 \geqq \frac{4}{\pi^2}\left(\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}\right)^2,$$

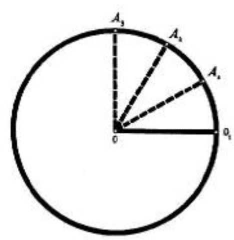

FIG. 3

which proves the lemma.
Applying lemma 2 to (58) we obtain

$$\max(|\theta_k(\cos\varphi_2)|, |\theta_k(\cos\varphi_4)|) \geqq \frac{4}{\pi^2}\left(\frac{\varphi_5 - \varphi_4}{\varphi_5 - \theta_k^{(n)}}\right)^2. \tag{60}$$

Now from the definition of φ_5 and by $|\varphi_2 - \varphi_4| = 1/2n$ we obtain

$$|\varphi_5 - \varphi_4| > \frac{1}{10n} \tag{61a}$$

and

$$|\varphi_5 - \theta_k^{(n)}| < |\varphi_2 - \theta_k^{(n)}|.\ ^{26} \tag{61b}$$

Putting (61a) and (61b) into (60) we obtain

$$\max(|\theta_k(\cos\varphi_2)|, |\theta_k(\cos\varphi_4)|) \geqq \frac{1}{25\pi^2 n^2(\varphi_2 - \theta_k^{(n)})^2}, \tag{62}$$

which with (53) and (55) gives

$$c_{36} \geqq \max(|l_k(\cos\varphi_2)|, |l_k(\cos\varphi_4)|) \geqq c_{43} n^{14}(\varphi_2 - \theta_k^{(n)})^{14}. \tag{63}$$

[26] If φ_2 lies between φ_5 and $\theta_k^{(n)}$ (see footnote 24) then $|\varphi_5 - \varphi_2| > 1/10\,n$ holds instead of (61a) and $|\varphi_5 - \theta_k^{(n)}| \leqq (1 + \pi^{-1})|\varphi_2 - \theta_k^{(n)}|$ instead of (61b), since according to (49c) and $|\varphi_5 - \varphi_2| < |\varphi_4 - \varphi_2| = 1/2\,n$,

$$\left|\frac{\varphi_5 - \theta_k^{(n)}}{\varphi_2 - \theta_k^{(n)}}\right| = \left|\frac{\varphi_5 - \varphi_2}{\varphi_2 - \theta_k^{(n)}}\right| + 1 < 1 + \frac{1}{\pi}.$$

Hence

$$|\varphi_2 - \theta_k^{(n)}| < \frac{c_{44}}{n}, \tag{64}$$

which evidently means that

$$|\theta_{\mu+1}^{(n)} - \theta_\mu^{(n)}| \leqq 2 |\varphi_2 - \theta_k^{(n)}| \leqq \frac{c_{38}}{n} \tag{65}$$

and this settles II A.

CASE II B. There exists a k such that

$$|l_k(\cos \varphi_2)| \geqq n^8(\varphi_2 - \theta_k^{(n)})^8 \tag{66}$$

and $l_k(\cos \theta)$ takes its absolute maximum outside $[\theta_\mu^{(n)}, \theta_{\mu+1}^{(n)}]$, at $\theta = \varphi_6$. From the definition of φ_6 we have

$$|l_k(\cos \varphi_6)| \geqq n^8(\varphi_2 - \theta_k^{(n)})^8. \tag{67}$$

The only property of φ_4 used in the proof of (62) was that its distance from φ_2 lies e.g. between $\pi/8n$ and $\pi/4n$. Thus (62) holds here too, if φ_2 has the same meaning as in case I and instead of φ_4 we take an arbitrary point φ of the interval $[\varphi_2 \pm \pi/8n, \varphi_2 \pm \pi/4n]$.[23] We remark, that φ_6 is farther from φ_2 than φ, and note, that by using (66) and (67) the whole idea of the proof of case II A may be applied here too, if we have proved following lemma.

LEMMA 3. *Let $\psi(x)$ be linear and denote the minimum of max $(|\psi(\cos \varphi_2)|, |\psi(\cos \alpha)|)$ by $M = M(\varphi_2, \alpha, \theta_k^{(n)})$, if $\psi(x)$ runs over the lines, for which $\psi(\cos \theta_k^{(n)}) = 1$. Then M does not decrease, if $\theta_k^{(n)}$ and φ_2 are fixed and $|\alpha - \varphi_2|$ increases $(\alpha \neq \theta_k^{(n)})$.*

If φ_2 and α lie on the same side of $\theta_k^{(n)}$ and α is fixed then the minimum is attained for the straight line connecting the point $(\cos \theta_k^{(n)}, 1)$ with the bisecting point of the distance $(\cos \alpha, 0)$ and $(\cos \varphi_2, 0)$. This evidently proves the lemma for this case and it is also clear, that the minimum is less than 1. If α and φ_2 are situated on opposite sides of $\theta_k^{(n)}$, the minimum is attained if $\psi(x) \equiv 1$ and then its value is 1.

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