Here (7) follows from (*b*); (8) from (i) of the lemma and (*c*); (9) from (5) and (*c*); and (10) from (6) and (*c*).

Finally, making ϵ tend to zero and using property (*d*) again, we have

$$\bar{M}(\xi) \leqslant [0, \xi]_{\bar{f}}.$$

Hence the result of the theorem.

In conclusion, I may point out that Theorem II may be generalized by a weakening of the hypothesis (*d*).

(1) In the first place, continuity of $[\alpha, \beta]_f$ with respect to the pair of variables α, β may be replaced by upper semi-continuity. This generalization requires no change in the proof.

(2) This continuity (or upper semi-continuity) with respect to (α, β) is used only to show that the set A_δ is closed. A slight change in the proof shows that (*d*) may be replaced by

(d') if α is fixed and $0 \leqslant \alpha < a$, then $[\alpha, \beta]_f$ is an upper semi-continuous function of β in the range $\alpha < \beta \leqslant a$.

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A GENERALIZATION OF A THEOREM OF BESICOVITCH

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Let us denote by δ_a the density of the integers which have a divisor between a and $2a$. Besicovitch‡ has proved that $\liminf_{a\to\infty} \delta_a = 0$. I have proved§ that $\lim \delta_a = 0$. I now prove the following more general

THEOREM. *Let ϵ_a be an arbitrary function of a such that $\lim_{a\to\infty} \epsilon_a = 0$, and let d_a be the density of the integers having a divisor between a and $a^{1+\epsilon_a}$. Then $\lim_{a\to\infty} d_a = 0$.*

It can easily be proved that, if ϵ_a does not tend to 0, then $\overline{\lim}\, d_a > 0$.

We may suppose without loss of generality that $a^{\epsilon_a} \to \infty$, for, if not, we can find ϵ_a' such that $\epsilon_a \leqslant \epsilon_a'$, $\epsilon_a' \to 0$, $a^{\epsilon_a'} \to \infty$, and then the theorem for ϵ_a follows from the theorem for ϵ_a'.

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‡ *Math. Annalen*, 110 (1934), 336–341.

§ *Journal London Math. Soc.* 10 (1935), 126–128.

We introduce the following notation:

1. $A_1, A_2, \ldots$ denote the integers which are composed entirely of primes not greater than a^{ϵ_a};

2. $x = \log(1/\epsilon_a)$;

3. $B_1, B_2, \ldots$ denote those integers in the interval $(a^{1-\epsilon_a x}, a^{1+\epsilon_a})$† which are composed entirely of primes greater than a^{ϵ_a};

4. $B_1^*, B_2^*, \ldots$ denote those B's which have not more than $\frac{2}{3}x$ different prime factors;

5. $B_1^+, B_2^+, \ldots$ denote those B's which have more than $\frac{2}{3}x$ but not more than $\frac{4}{3}x$ different prime factors;

6. $B_1^{(r)}, B_2^{(r)}, \ldots$ denote those B's which have exactly r different prime factors, where $r \leqslant \frac{4}{3}x$;

7. c's denote suitable positive constants;

8. $p_1, p_2, \ldots, p_s$ denote the primes in the interval $(a^{\epsilon_a}, a^{1+\epsilon_a})$;

9. $C_1, C_2, \ldots$ denote the integers composed entirely of the primes $p_1, p_2, \ldots, p_s$;

10. N is a sufficiently large number.

We require six lemmas.

LEMMA 1. *The number of integers $m \leqslant N$ which are divisible by an $A > a^{x\epsilon_a}$ is less than $c_1 N/x$.*

Proof. Denote by $A(m)$ the greatest A which divides m. We have (in analogy with Legendre's formula for $n!$)

$$\prod_{i=1}^{N} A(i) < \prod_{p \leqslant a^{\epsilon_a}} p^{N/(p-1)} = \exp\left(N \sum_{p \leqslant a^{\epsilon_a}} \frac{\log p}{p-1}\right) < a^{c_1 N \epsilon_a},$$

since

$$\sum_{p \leqslant y} \frac{\log p}{p-1} < c_1 \log y.$$

Hence, if we denote by U the number of integers $m \leqslant N$ for which $A(m) > a^{x\epsilon_a}$, we have

$$a^{c_1 N \epsilon_a} > a^{U x \epsilon_a};$$

and thus

$$U < c_1 N/x.$$

† We consider the upper bound to be included in the interval but not the lower bound.

LEMMA 2. *The number of integers $m \leqslant N$ divisible by a $C > a^{1/\epsilon_a}$ is less than $c_2 N \epsilon_a$.*

Proof. Denote by $C(m)$ the greatest C which divides m; we have, by Legendre's formula,

$$\prod_{i=1}^{N} C(i) < \prod_{p \leqslant a^{1+\epsilon_a}} p^{N/p-1} = \exp\left(N \sum_{p \leqslant a^{1+\epsilon_a}} \frac{\log p}{p-1}\right) < \exp(c_2 N \log a) = a^{c_2 N}.$$

Hence, if we denote by V the number of integers $m \leqslant N$ for which $C(m) > a^{1/\epsilon_a}$, we have

$$a^{c_2 N} > a^{V/\epsilon_a};$$

and thus

$$V < c_2 N \epsilon_a.$$

LEMMA 3. $$\sum_{i=1}^{\infty} \frac{1}{B_i^*} < \epsilon_a^{\frac{1}{50}}.$$

Proof. First we estimate $\displaystyle\sum_{i=1}^{\infty} \frac{1}{B_i^{(r)}}$.

If $B_i^{(r)} = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $p_1^{\alpha_1} < p_2^{\alpha_2} < \dots < p_r^{\alpha_r}$, then $p_r^{\alpha_r} > (B_i^{(r)})^{1/r}$, and

$$p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{r-1}^{\alpha_{r-1}} < (B_i^{(r)})^{(r-1)/r} < a^{\{(1+\epsilon_a)(r-1)\}/r}. \tag{1}$$

The sum of the reciprocals of the $B_i^{(r)}$'s of which the first $r-1$ prime factors are $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{r-1}^{\alpha_{r-1}}$ is evidently not greater than $\displaystyle\frac{1}{p_1^{\alpha_1} \dots p_{r-1}^{\alpha_{r-1}}} \sum' \frac{1}{p_r^{\alpha_r}}$, where Σ' means that $p_r^{\alpha_r}$ runs through the interval

$$\left(\frac{a^{1-\epsilon_a x}}{p_1^{\alpha_1} \dots p_{r-1}^{\alpha_{r-1}}},\quad \frac{a^{1+\epsilon_a}}{p_1^{\alpha_1} \dots p_{r-1}^{\alpha_{r-1}}}\right).$$

Now it is known that, for $y \geqslant 3$,

$$\sum_{p^\alpha \leqslant y} \frac{1}{p^\alpha} = \log\log y + c_3 + O\left(\frac{1}{\log y}\right).$$

Hence

$$\sum_{u \leqslant p \leqslant uv} \frac{1}{p^\alpha} = \log\log(uv) - \log\log u + O\left(\frac{1}{\log u}\right)$$

$$= \log(\log u + \log v) - \log\log u + O\left(\frac{1}{\log u}\right) < \frac{\log v}{\log u} + O\left(\frac{1}{\log u}\right),$$

since $\log(1+x) \leqslant x$ for $x \geqslant 0$.

Hence, taking

$$u = \frac{a^{1-\epsilon_a x}}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_{r-1}^{\alpha_{r-1}}} \quad \text{and} \quad v = a^{\epsilon_a(1+x)},$$

we have

$$\Sigma' \frac{1}{p_r^{\alpha_r}} < \frac{\epsilon_a(1+x)\log a}{\log\left(a^{1-\epsilon_a x}/p_1^{\alpha_1}\cdots p_{r-1}^{\alpha_{r-1}}\right)} + O\left(\frac{1}{\log\left(a^{1-\epsilon_a x}/p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{r-1}^{\alpha_{r-1}}\right)}\right);$$

and so, from (1),

$$\Sigma' \frac{1}{p_r^{\alpha_r}} < \frac{\epsilon_a(1+x)\log a}{\log a^{1-\epsilon_a x-\{(1+\epsilon_a)(r-1)\}/r}} + O\left(\frac{1}{\log a^{1-\epsilon_a x-\{(1+\epsilon_a)(r-1)\}/r}}\right).$$

Now $(1+x)\,\epsilon_a \log a$ tends to infinity because a^{ϵ_a} tends to infinity, and so we may write

$$\Sigma' \frac{1}{p_r^{\alpha_r}} < \frac{2\epsilon_a(1+x)\log a}{\log a^{1-\epsilon_a x-\{(1+\epsilon_a)(r-1)\}/r}}$$

for $a > a_0$, say. Thus

$$\Sigma' \frac{1}{p_r^{\alpha_r}} < \frac{2\epsilon_a(1+x)}{1-\epsilon_a x-\{(1+\epsilon_a)(r-1)\}/r} = \frac{2r\,\epsilon_a(1+x)}{1-\epsilon_a(rx+r-1)}.$$

Now $r \leqslant \frac{4}{3}x$, and so $\epsilon_a(rx+r-1) \leqslant \epsilon_a 3x^2 < \frac{1}{2}$ for sufficiently large a. Hence

$$\Sigma' \frac{1}{p_r^{\alpha_r}} < 4\epsilon_a r(x+1) < 8\epsilon_a x^2.$$

From this, we have

$$\sum_{i=1}^{\infty} \frac{1}{B_i^{(r)}} < 8\epsilon_a x^2 \sum_{p_1^{\alpha_1}\cdots p_{r-1}^{\alpha_{r-1}} < a^{1+\epsilon_a}} \frac{1}{p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{r-1}^{\alpha_{r-1}}} < 8\epsilon_a x^2 \left\{\left(\sum_{p,\,\alpha}\frac{1}{p^{\alpha}}\right)^{r-1}\Big/(r-1)!\right\}$$

$$< 8\epsilon_a x^2 \frac{(x+1)^{r-1}}{(r-1)!},$$

since

$$\sum_{p,\,\alpha} \frac{1}{p^{\alpha}} = \log\log a^{1+\epsilon_a} - \log\log a^{\epsilon_a} + O\left(\frac{1}{\log a^{\epsilon_a}}\right)$$

$$= \log(1+\epsilon_a) - \log\epsilon_a + O\left(\frac{1}{\log a^{\epsilon_a}}\right) < x+1.$$

Hence

$$\sum_{i=1}^{\infty}\frac{1}{B_i^*} = \sum_{r=1}^{[\frac{2}{3}x]}\sum_{i=1}^{\infty}\frac{1}{B_i^{(r)}} < 8\epsilon_a x^2 \sum_{r=1}^{[\frac{2}{3}x]}\frac{(x+1)^{r-1}}{(r-1)!} < \frac{16}{3}\epsilon_a x^3 \frac{(x+1)^{[\frac{2}{3}x]}}{([\frac{2}{3}x])!},$$

since $(x+1)^{r-1}/(r-1)!$ increases with r for $r < x+1$.

From the inequality $$n! > \frac{n^n}{e^n}$$

we have $$\sum_{i=1}^{\infty}\frac{1}{B_i^*} < 6\epsilon_a x^3 \frac{(x+1)^{\frac{2}{3}x+1}e^{\frac{2}{3}x+1}}{(\frac{2}{3}x)^{\frac{2}{3}x}}.$$

Now

$$(x+1)^x < ex^x,$$

and so

$$\sum_{i=1}^{\infty} \frac{1}{B_i^*} < 6e^2 \epsilon_a x^3(x+1) \frac{x^{\frac{2}{3}x} e^{\frac{2}{3}x}}{(\frac{2}{3}x)^{\frac{2}{3}x}} = 6e^2 \epsilon_a x^3(x+1) e^{\frac{2}{3}x} (\tfrac{3}{2})^{\frac{2}{3}x}.$$

Further†,

$$(\tfrac{3}{2})^{\frac{2}{3}} < e^{\frac{8}{27}}.$$

Hence

$$\sum_{i=1}^{\infty} \frac{1}{B_i^*} < 6e^2 \epsilon_a x^3(x+1) e^{\frac{26}{27}x},$$

but

$$\epsilon_a = e^{-x},$$

and so

$$\sum_{i=1}^{\infty} \frac{1}{B_i^*} < 6e^2 x^3(x+1) \epsilon_a^{\frac{1}{27}} < \epsilon_a^{\frac{1}{30}}$$

for $a > a_0$, which proves Lemma 3.

Lemma 4.

$$\sum_{i=1}^{\infty} \frac{1}{B_i^+} < x^3.$$

Proof. As in Lemma 2, we have

$$\sum_{i=1}^{\infty} \frac{1}{B_i^+} = \sum_{r>\frac{2}{3}x}^{r\leqslant\frac{4}{3}x} \sum_{i=1}^{\infty} \frac{1}{B_i^{(r)}} < 8\epsilon_a x^2 \sum_{r>\frac{2}{3}x}^{r\leqslant\frac{4}{3}x} \frac{(x+1)^{r-1}}{(r-1)!} < 8\epsilon_a x^2 e^{x+1} = 8e\,x^2 < x^3.$$

Lemma 5. *The number W of integers $m \leqslant N$ which are divisible by not more than $\frac{2}{3}x$ of the p_i's is less than $N\epsilon_a^{\frac{1}{30}}$.*

Proof. We split these integers m into two classes. Put in the first class those for which $C(m) > a^{1/\epsilon_a}$. The number of these is, by Lemma 2, less than $c_2 N\epsilon_a$.

For the integers of the second class $C(m) \leqslant a^{1/\epsilon_a}$.

The number Z of integers $m \leqslant N$ for which $C(m) = C_i \leqslant a^{1/\epsilon_a}$ is equal to the number of integers less than or equal to N/C_i not divisible by any p_i. For this number we have, from the sieve of Eratosthenes, the inequality

$$Z < \frac{N}{C_i} \prod_{p_i} \left(1 - \frac{1}{p_i}\right) + 2^s < c_4 \frac{N\epsilon_a}{C_i},$$

since $C_i \leqslant a^{1/\epsilon_a}$ and the number of the p_i's is independent of N. Hence

$$W < c_2 N\epsilon_a + c_4 N\epsilon_a \sum_{C_i \leqslant a^{1/\epsilon_a}}' \frac{1}{C_i},$$

where the dash means that C_i runs through the C's having not more than

† We have $(\frac{3}{2})^{\frac{9}{4}} = (\frac{3}{2})^2 (\frac{3}{2})^{\frac{1}{4}} < \frac{9}{4}\,\frac{6}{5} = 2\frac{7}{10} < e.$

$\frac{2}{3}x$ prime factors. Thus

$$W < c_2 N\epsilon_a + c_4 N\epsilon_a \left(\sum_{p,\,a} \frac{1}{p^a} + \frac{\left(\sum_{p,\,a} \frac{1}{p^a}\right)^2}{2!} + \frac{\left(\sum_{p,\,a} \frac{1}{p^a}\right)^3}{3!} + \ldots + \frac{\left(\sum_{p,\,a} \frac{1}{p^a}\right)^{[\frac{2}{3}x]}}{[\frac{2}{3}x]!} \right),$$

where p runs through $p_1, p_2, \ldots, p_s$.

Finally, exactly as in the proof of Lemma 3,

$$W < c_2 N\epsilon_a + c_4 N\epsilon_a \left((x+1) + \frac{(x+1)^2}{2!} + \ldots + \frac{(x+1)^{[\frac{2}{3}x]}}{[\frac{2}{3}x]!} \right) < N\epsilon_a^{\frac{1}{30}}.$$

LEMMA 6. *The number of integers $m \leqslant N$ divisible by $\frac{4}{3}x$ or more of the p_i's (multiple factors counted multiply) is less than ηN for every $\eta > 0$, if $N > N(\eta)$.*

The proof follows easily by Turán's method†, so that it will be sufficient to sketch it.

The number of integers less than or equal to N divisible by a p_i^2 is less than

$$\sum_{p_i} \frac{N}{p_i^2} < \frac{\eta}{2} N;$$

hence it will be sufficient to consider the m's containing the p_i's to the first power only.

Let $f(m)$ be the number of p_i's contained in m. We prove that

$$\sum_{m=1}^{N} [f(m)-x]^2 < c_5 N x, \tag{2}$$

where c_5 is independent of N and x. Evidently

$$\sum_{m=1}^{N} [f(m)-x]^2 = \sum_{m=1}^{N} f(m)^2 - 2x \sum_{m=1}^{N} f(m) + Nx^2. \tag{3}$$

We can easily show that

$$\sum_{m=1}^{N} f(m)^2 = 2 \sum_{\substack{p_i,\, p_k \\ p_i \neq p_k}} \left[\frac{N}{p_i p_k}\right] + \sum_{p_i} \left[\frac{N}{p_i^2}\right] \leqslant N \left(\sum_{p_i} \frac{1}{p_i}\right)^2 = Nx^2 + O(Nx). \tag{4}$$

Further,
$$\sum_{m=1}^{N} f(m) = \sum_{p_i} \left[\frac{N}{p_i}\right] = Nx + O(N). \tag{5}$$

Substituting from (4) and (5) in (3), we immediately obtain (2).

From (2) we deduce that the number of integers less than or equal to N for which $f(m) > \frac{4}{3}x$ is less than $9c_5 N/x < \frac{1}{2}\eta N$ for sufficiently large x; thus Lemma 6 is proved.

Proof of the theorem. We divide the integers of the interval $(a, a^{1+\epsilon_a})$ into two classes. In the first class are the integers which are divisible by

† *Journal London Math. Soc.*, 9 (1934), 274–276.

an $A > a^{x\epsilon_a}$. By Lemma 1 the number of integers $m \leqslant N$ divisible by an integer of the first class is less than $c_1 N/x$. The second class contains the other integers in the interval in question. Every integer I of the second class is divisible by a B_i, for, if A_I is the largest A contained in I, then I/A_I contains no prime less than or equal to a^{ϵ_a}, since these have been absorbed by I_A; also $I/A_I > a^{1-\epsilon_a x}$, since $A_I < a^{x\epsilon_a}$; hence I/A_I is a B. We divide these B's into three classes. Put in the first the B_i^*'s, in the second the B_i^+'s, and in the third the B's having more than $\frac{4}{3}x$ prime factors.

The number of integers not greater than N divisible by a B^* is less than

$$\sum_{i=1}^{\infty} \frac{N}{B_i^*} < N \epsilon_a^{\frac{1}{30}},$$

by Lemma 3.

We subdivide the integers less than or equal to N divisible by a B_i^+ into two sets, putting in the first those of the form tB_i^+, where $t \leqslant N/B_i^+$ and t has at most $\frac{2}{3}x$ different prime factors among the p_i's. The number of integers in the first set is less than

$$\epsilon_a^{\frac{1}{30}} \sum_{i=1}^{\infty} \frac{N}{B_i^+} < x^3 \epsilon_a^{\frac{1}{30}},$$

by Lemma 4, since, by Lemma 5, the number of t's is less than $(N/B_i^+)\,\epsilon_a^{\frac{1}{30}}$. The second set includes the integers of the form tB_i^+, where t has more than $\frac{2}{3}x$ different prime factors among the p_i's. These integers have more than $\frac{4}{3}x$ prime factors (multiple factors counted multiply) among the p_i's, and so, by Lemma 6, their number is less than ηN.

Similarly the number of integers $m \leqslant N$ divisible by a B of the third class is less than ηN.

Hence the number of integers not greater than N having a divisor in the interval $(a_1, a^{1+\epsilon_a})$ is less than

$$N\left(\frac{c_1}{x} + \epsilon_a^{\frac{1}{30}} + x^3 \epsilon_a^{\frac{1}{30}} + \eta\right);$$

thus their density is less than

$$\frac{c_1}{x} + \epsilon_a^{\frac{1}{30}} + x^3 \epsilon_a^{\frac{1}{30}} + \eta,$$

which is arbitrarily small. This proves the theorem.

By a more precise argument we can prove that the density in question is less than $\epsilon_a^{c_6}$.

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