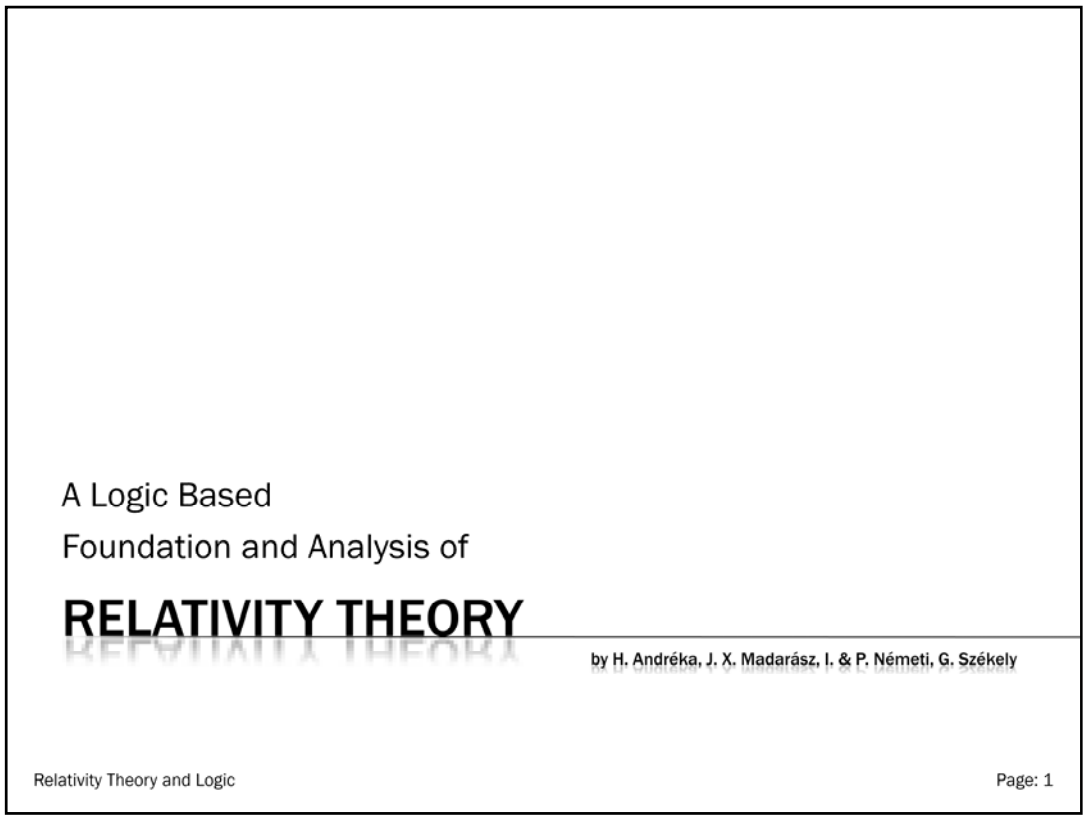


GR and BH theory are extremely exciting, new frontier areas of science. It is an inviting application area for logic and logicians. We claim that logic can be fruitfully applied in this field.

PART I

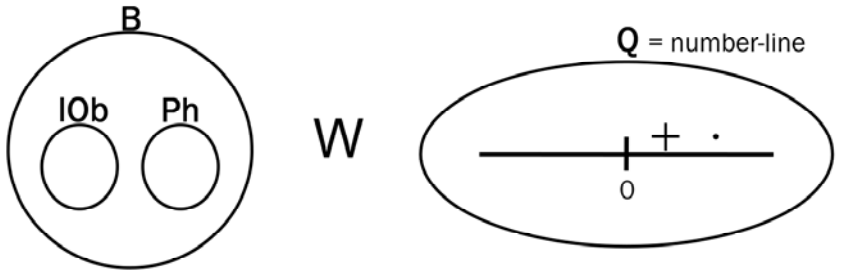
Special Relativity

After the Intro, we begin with special relativity theory.

LANGUAGE FOR SPECREL

$$\langle B, IOb, Ph, Q, +, \cdot, W \rangle$$

Bodies (test particles), Inertial Observers, Photons, Quantities, usual operations on it, Worldview



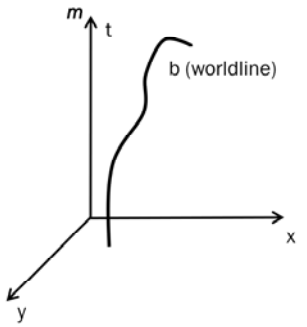
Relativity Theory and Logic Page: 3

Our first example is SR. There are many other incarnations of RT which we investigate e.g. GR, but we have to start somewhere. So let's consider SR.
In our language for SR there are two basic kinds of entities one can talk about, these are test particles (anything that moves) B, and quantities (numbers) Q. We can think of these as the physical and the mathematical universes.
We represent motion by changing place in time, we represent place and time by coordinates. Observer is picturesc word for coordinate system.

LANGUAGE FOR SPECREL

$$W \subseteq IOb \times Q^4 \times B$$

$\mathrm{W(m, t\ x\ y\ z, b)}$ ⇔ body "b" is present at coordinates "t x y z" for observer "m"



$worldline$: $wline_m(b) := \{p \in Q^4 : W(m,p,b)\}$

Axiom 0: Every observer coordinates his world by 4 coordinates, 1 time coord. and 3 space coord. This axiom is built into the language.
Worldview: this is how we illustrate the worldview relation W.
Worldline: body b is present for observer m at these coordinates

AXIOMS FOR SPECREL

- AxField

Usual properties of addition and multiplication on Q :
Q is an ordered Euclidean field.

1. (Q , + , ·) is a field in the sense of abstract algebra (with 0 , −, 1 , / as derived operations)
2. $0 = x^2 + y^2 + z^2 \longrightarrow x = y = z = 0$
3. $\exists y(x = y^2 \; or \; -x = y^2)$

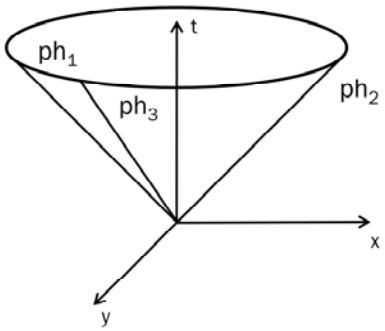
Ordering derived: $x \leq y \overset{d}{\Leftrightarrow} \exists z(y - x = z^2)$

Ordered Euclidean field means that positive members have square roots.
They are called Euclidean fields because of their role in Tarski's FOL axiomatization of Euclidean geometry.
This is a „mathematical” axiom, physicists use it tacitly.

AXIOMS FOR SPECREL

- AxPh

For all inertial observers the speed of light is the same in all directions and is finite.
In any direction it is possible to send out a photon.



Formalization:

$$(\forall m \in IOb)(\exists c \in Q)(\forall p, q \in Q^4)$$

$$\left[\begin{array}{c} |p_s - q_s| = c \cdot |p_t - q_t| \\ \leftrightarrow \\ (\exists ph \in Ph)\ p, q \in wline_m(ph) \end{array}\right]$$

$$where\ p_s = \langle p_2, p_3, p_4 \rangle\ and\ p_t = p_1.$$

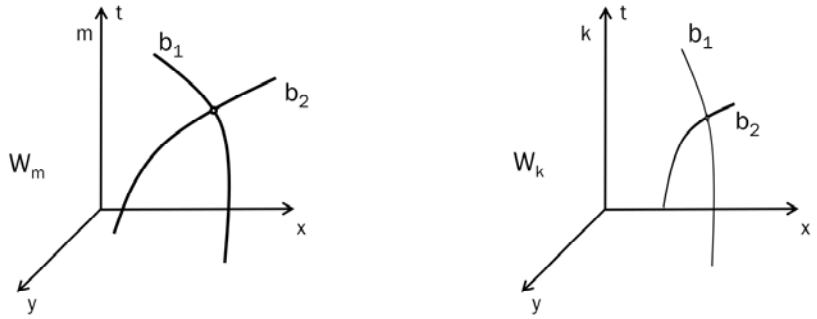
Relativity Theory and Logic Page: 6

This axiom is the outcome of the Michelson&Morley experiment.
Besides M&M this is being tested since then, nowadays it is tested by GPS technology.
Key axiom, with a physical meaning.

AXIOMS FOR SPECREL

- AxEv

All inertial observers coordinatize the same events.



Formalization: $(\forall m,k \in IOb)(\forall txyz)$

$$\exists t'x'y'z' \; \forall b[W(m,t\ x\ y\ z,b) \leftrightarrow W(k,t'x'y'z',b)]$$

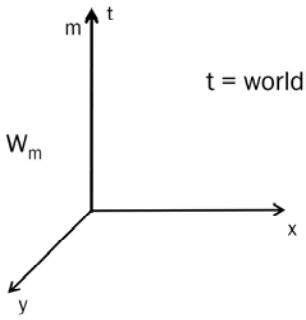
Relativity Theory and Logic Page: 7

These bodies are possible bodies (=test particles).
The same events exist for all observers.
„sees” = coordinatizes
This means that there is an outside reality (or, all observers talk about the same outside reality).

AXIOMS FOR SPECREL

- AxSelf

An inertial observer sees himself as standing still at the origin.



t = worldline of the observer

Formalization: $(\forall m \in IOb)(\forall txyz)$

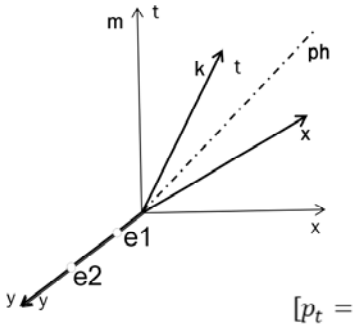
$$W(m,t\ x\ y\ z,m) \leftrightarrow x = y = z = 0$$

The last two axioms are „book-keeping”, simplifying axioms. We could leave them out and nothing would be lost, only the formalizations of the theorems would become more complicated.

AXIOMS FOR SPECREL

- AxSymd

Any two observers agree on the spatial distance between two events, if these two events are simultaneous for both of them, and $|v_m(ph)|=1$.



Formalization: $(\forall m, k \in IOb)(\forall p, q, p', q' \in Q^4)$

$$[p_t = q_t \wedge p'_t = q'_t \wedge ev_m(p) = ev_k(p') \wedge ev_m(q) = ev_k(q')] \rightarrow |p_s - q_s| = |p'_s - q'_s|$$

$ev_m(p) = \{b \in B: W(m,p,b)\}$ is the **event** occurring at *p* in *m*'s worldview

Relativity Theory and Logic Page: 9

Symmetry axiom: all observers use the same units of measurements.

Simplifying axiom.

SpecRel without symmetry axiom already proves all the important predictions of usual special relativity with the only exception that in SpecRel0 different observers might use different units of measurements. The assumption that all observers use the same units of measurements is clearly of a book-keeping nature only and therefore it is expendable. So we could, in principle, regard SpecRel without AxSym as the full theory of SR. However, this would lead to complicated formulation of the important theorems and the complications would go in a completely irrelevant direction. Therefore we add to SpecRel without AxSym the so-called symmetry axiom, a simplifying principle. It is important to note that the symmetry axiom has no deep physical content, it is only conventional, by assuming it we do not use any relevant information.

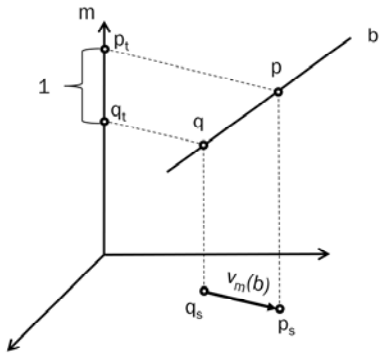
AXIOMS FOR SPECREL

What is speed?

$p = \langle p_1, p_2, p_3, p_4 \rangle$

$p_t = p_1$

$p_s = \langle p_2, p_3, p_4 \rangle$



Relativity Theory and Logic

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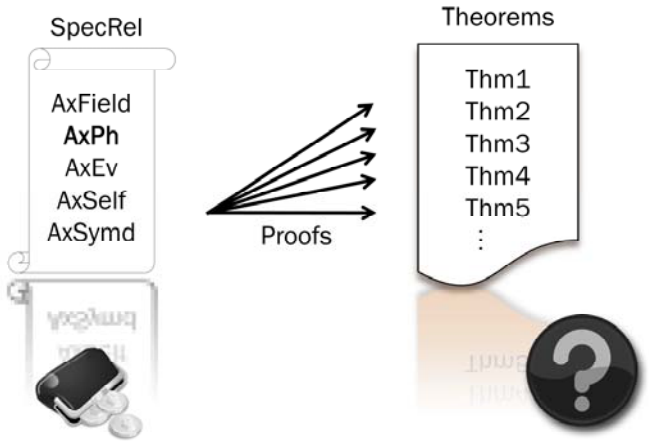
V_m(b) = speed of b relative to m
Space-component of p is denoted by p_s
Time-component of p is denoted by p_t
This is how we formalize speed by using the field-structure of our theory (if the worldline of b is a subset of a straight line)
In the picture you can replace 1 and v_m(b) by t and t times v_m(b).

SPECREL

$$SpecRel = \{AxField, AxPh, AxEv, AxSelf, AxSymd\}$$



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5 axioms, AxPh is the most important one of them
We have paid, what do we get for our price?

SPECREL

- Thm1

$$SpecRel \vdash (\forall m, k \in IOb)(wline_m(k)\ is\ a\ straight\ line)$$

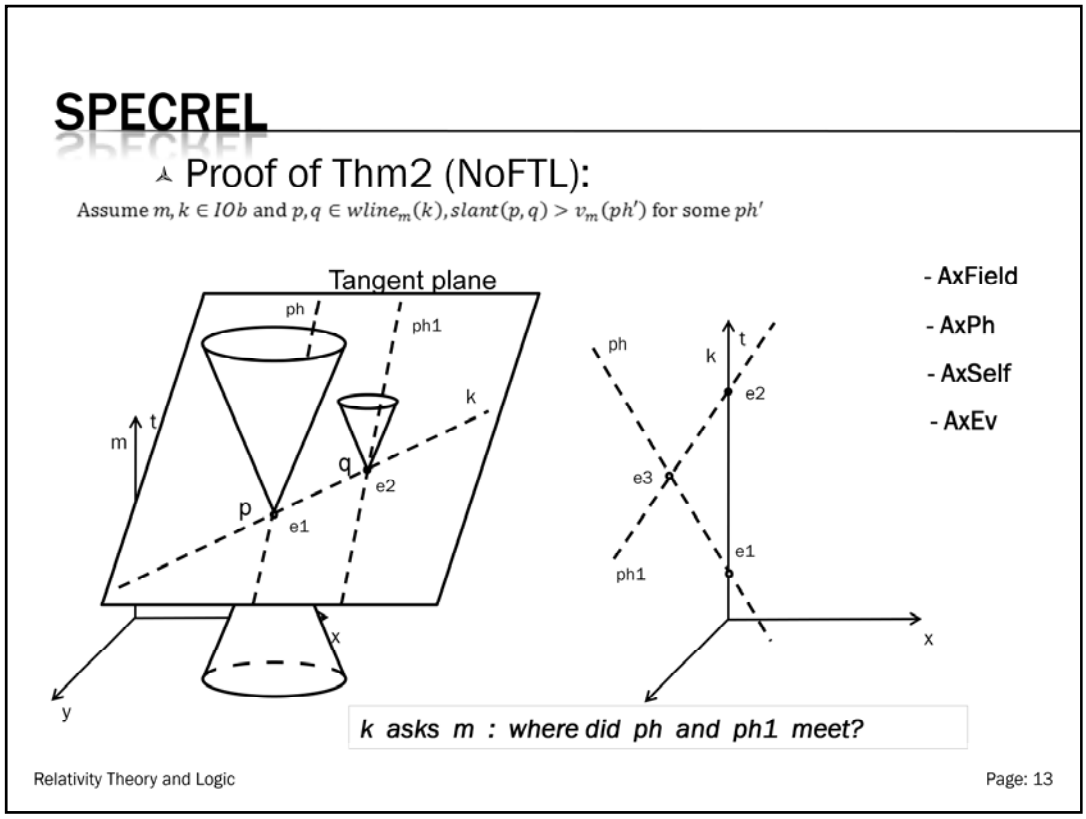
- Thm2

$$SpecRel \vdash NoFTL\ travel$$

$$NoFTL \overset{df}{\Leftrightarrow} (\forall m, k \in IOb)|v_m(k)| < |v_m(ph)| \quad for\ some\ ph \in Ph$$

Relativity Theory and Logic Page: 12

About Thm2:
V_m(k) is the velocity of k as seen by m.
This will be our generic example for “fancy theorem” from “plain axioms”. Analog case will be E=mc^2 in place of NoFTL.
NoFTL is removed from the cost-side and is put on the gain-side.
Bonus: we can tackle “why type” questions, i.e., which axioms to weaken or leave out to get “FTL”.



Proofsketch: (For simplicity, we assume only 2 space-dimensions. Proof goes through for 3 or more space dimensions.)
Assume that in m's worldview, k moves faster than light. Let p,q be distinct, on the worldline of k (in m's worldview). That k moves faster than light means that the worldline of k is outside of the *lightcone emanated from p* (=union of worldlines of photons going through p). The lightcone is a regular cone-shape by **AxPh**. Then there is a plane containing the worldline of k and tangent to this lightcone, by **AxField** (because, AxField ensures that we can solve quadratic equations just as in real numbers.) Let ph be a photon whose worldline is the intersection of the cone and this plane. There is such by **AxPh**. (In the present version of AxPh, this line is populated by a set of photons only, and not necessarily by a single one, but this can be overcome by making the proof slightly more complicated.)
Let's move over to k's worldview. By **AxSelf** and **AxEv,** the events that occur at p and q in m's worldview are situated on the time-axis in k's worldview. By **AxField** again, there is a straight line with the same slope as that of the worldline of ph in k's worldview, and intersecting that. Let this line be the worldline of the photon ph1 (**AxPh)**. Then ph and ph1 meet in k's worldview.
Let's go back to m's worldview. What is the worldline of ph1 like in m's worldview? If this line is not in the tangent plane, then it does not intersect the worldline of ph. If it is in the plane, then it is parallel with ph's worldline by **AxPh** and **AxField** (since this is a tangent plane). So ph and ph1 do not meet in this second case, either. This is a violation of **AxEv.**

SPECREL

- Conceptual analysis
  - Which axioms are needed and why
- Project:
  - Find out the limits of NoFTL
  - How can we weaken the axioms to make NoFTL go away

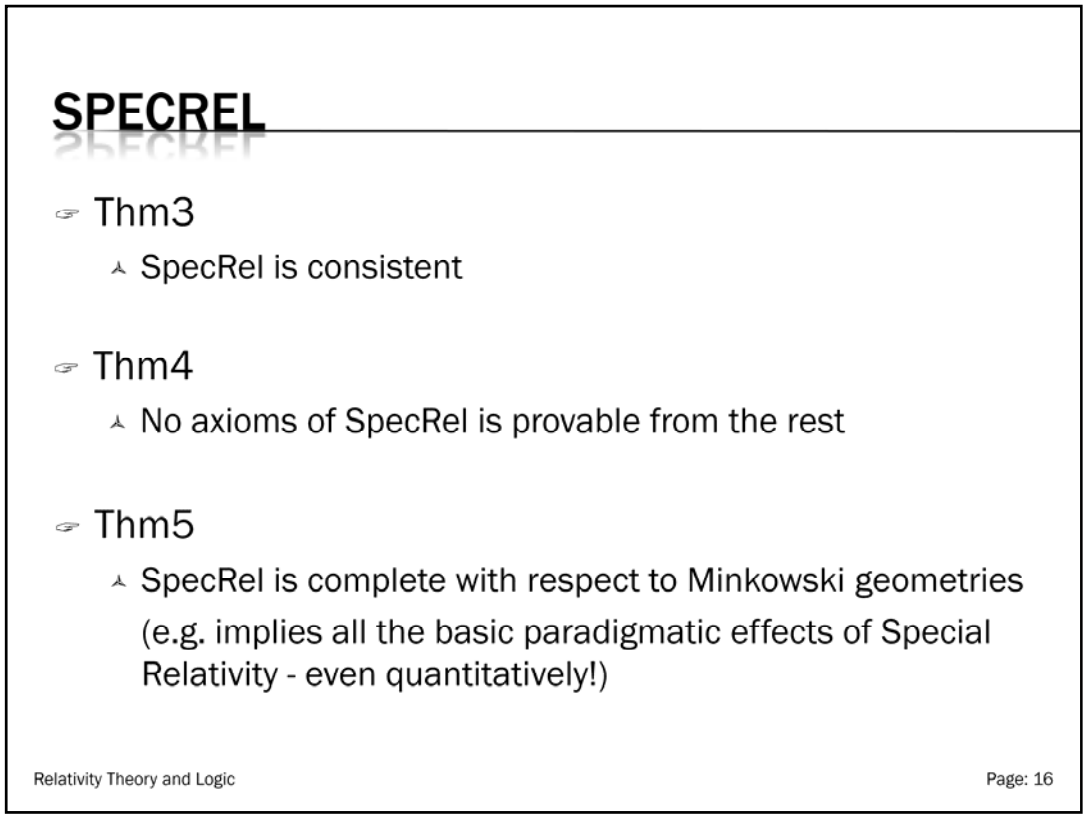
Why type questions
E.g., preparing the road to future theories like Quantum Gravity.

SPECREL

Igor D. Novikov,
September 3 Budapest:

It is an important research today to study spacetimes with more than one time dimensions

Why type questions
E.g., preparing the road to future theories like Quantum Gravity



Theorems 3,4 show what we cannot prove from SpecRel. Them 5 shows what we CAN prove from SpecRel. It is a completeness theorem for SR.
The completeness theorem for SR is a completeness theorem in the usual sense of logic, namely it says that SR is complete wrt its "standard model", or "intended model", i.e. wrt Minkowskian geometry. (See also a later slide.)
This theorem implies the so-called *paradigmatic (i.e., characteristic) effects of SR*, e.g. the fact that "moving clocks run slow", and that "moving clocks get out of synchronism". In fact, we prove this completeness theorem via proving the paradigmatic effects one-by-one, directly from the axioms of SR and then we show that the completeness theorem follows. This illuminates or illustrates why the paradigmatic effects are true and show how to perform a conceptual analysis.

After the next slide come the three paradigmatic effects and their proofs.

SPECREL

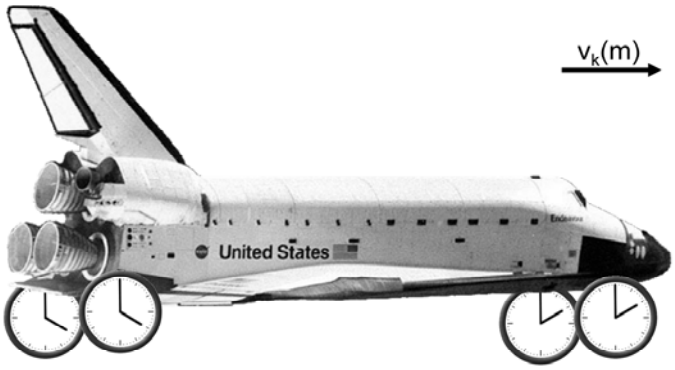
- Thm6
  - SpecRel generates an undecidable theory. Moreover, it enjoys both of Gödel's incompleteness properties
- Thm7
  - SpecRel has a decidable extension, and it also has a hereditarily undecidable extension. Both extensions are physically natural.

Proofidea for Thm6: we can use the worldline of a periodically circling body to select a subset of Q which together with the field-operations satisfy Robinson's arithmetic.

Proofidea for the existence of a decidable extension: We may postulate that the field-reduct is a real-closed field (which has a decidable FOL theory, Tarski's thm), all bodies are observers or photons, there are no two observers with the same worldview, and a few similar extra axioms.

RELATIVISTIC EFFECTS

Thm8



- Moving clocks get out of synchronism.
- Captain claims that the two clocks show the same time.

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This is the first and most important paradigmatic effect.

The picture shows a spaceship as seen by an observer moving relative to the ship. Captain claims that the two clocks show the same time.

The paradigmatic effects of SR are about comparing the worldviews of inertial observers moving relative to each other. Next slide shows formal statement of the theorem.

This effect has deep philosophical consequences. E.g. there is no such thing as absolute present. Mixes time and space.

Most of the fancy, exciting things in Relativity and Cosmology are based on this Thm!

Contradicts common sense. Like in models of set theory: external properties and internal properties (size of a set) may differ.

Paves the road for proving completeness theorem.

MOVING CLOCKS GET OUT OF SYNCHRONISM

☞ Thm8 (formalization of clock asynchronism)

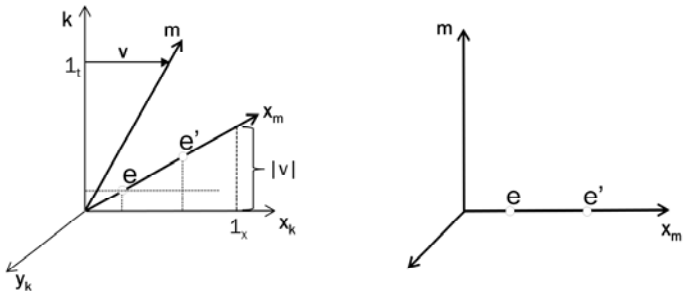
Assume SpecRel. Assume $m,k \in IOb$ and events e, e' are simultaneous for m,

$i.e.\ \ loc_m(e)_t = loc_m(e')_t$

(1) Assume e, e' are separated in the direction of motion of m in k's worldview,

$i.e.\ \ loc_k(e)_s - loc_k(e')_s \parallel v_k(m)$

$Then\ \ |loc_k(e)_t - loc_k(e')_t| = |loc_k(e)_s - loc_k(e')_s| \cdot |v_k(m)|$

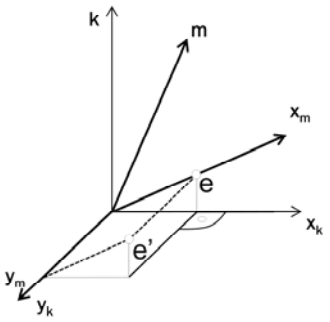


This is the formal statement of the (first part of the) effect. Only the formal statement has to be taken seriously (consult this if there is any confusion).

Events in the direction of motion do get out of synchronism.

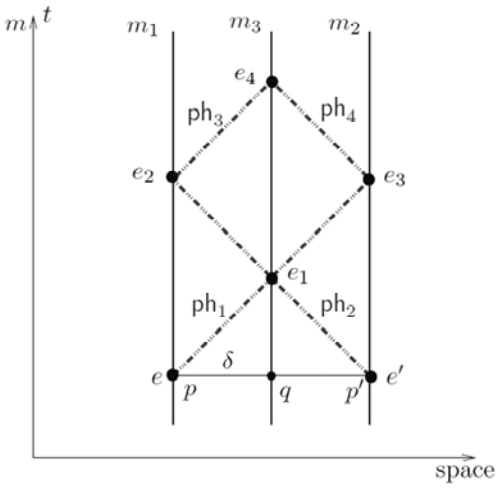
MOVING CLOCKS GET OUT OF SYNCHRONISM

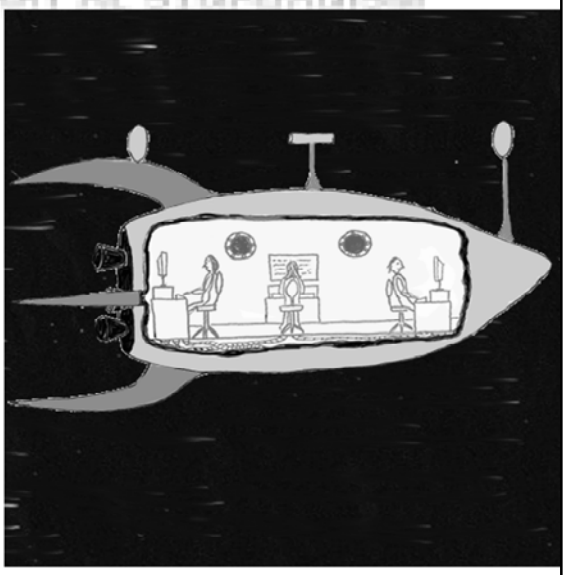
(2) *e*, *e'* are simultaneous for *k*, too
⇔ *e*, *e'* are separated orthogonally to $v_k(m)$ in *k*'s worldview
$i.e.\ \ loc_k(e)_s - loc_k(e')_s \perp v_k(m)$



Events separated orthogonally to the motion do not get out of synchronism

MOVING CLOCKS GET OUT OF SYNCRONISM





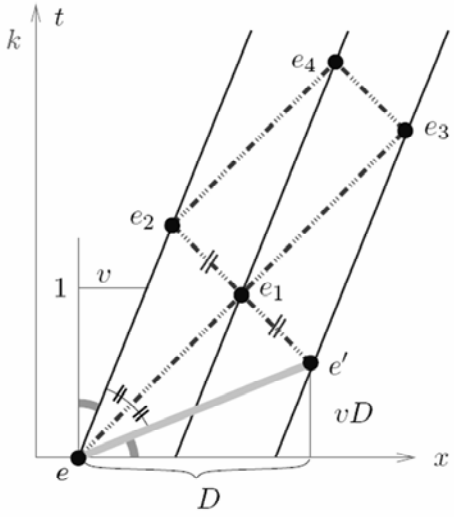
Thought-experiment for proving relativity of simultaneity.

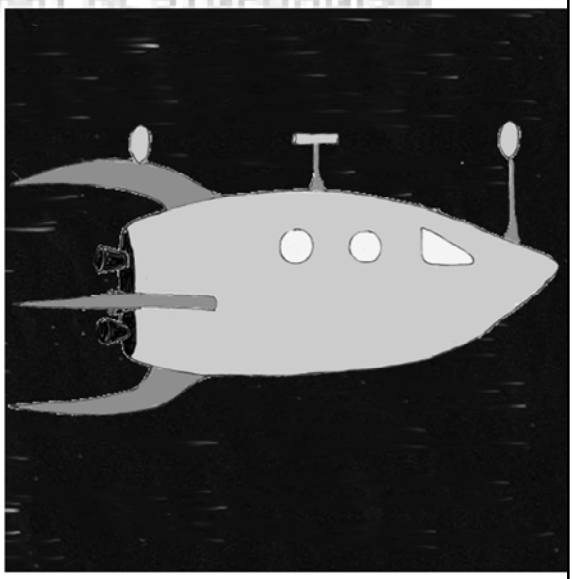
Experiment for establishing “simultaneity for m”.

This is the worldview of the ship. The captain is sitting in the middle of the ship and sends photon-signals to two mirrors, one at the nose and one at the rear of the ship. These two signals bounce on the mirrors and arrive back to the captain at the same time. This is how the captain knows that he is sitting exactly in the middle of the ship. By the photon-axiom the forward going and backward going photons travel with the same speed, thus the two bouncing events happen at the same time in this ship.

On the lelft-hand side there is the space-time diagram for the situation.

MOVING CLOCKS GET OUT OF SYNCRONISM

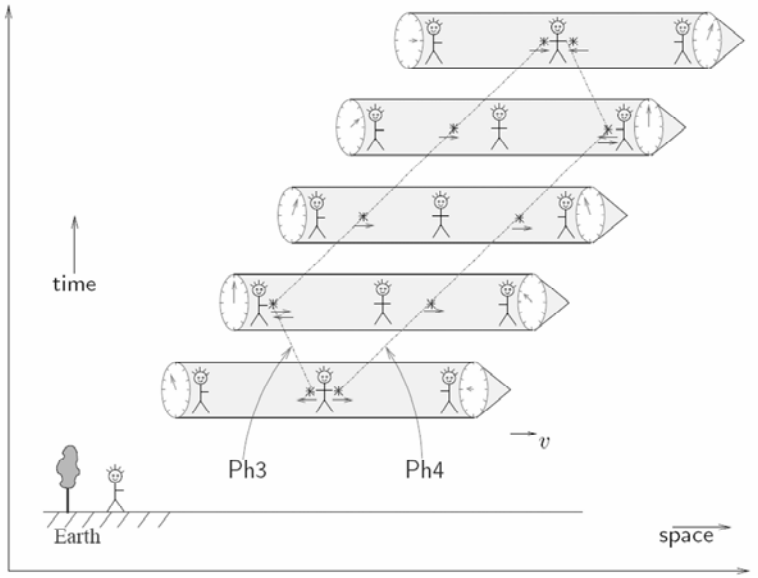




Thought-experiment for proving relativity of simultaneity.

This is the worldview of an observer moving relative to the ship, lets call him Earthling. So this ship is moving in this worldview. We are following it with our camera, thats why it stays in the middle of the picture. Now by the photon-axiom, the Earthling sees the forward and backward moving photons move with the same speed. Since the ship moves in this worldview, the forward going photons race with the ship, thus the photons move slowly relative to the ship. By the same token, the backward going photons move very fast relative to the ship. By the event axiom, they meet where the captain sits, hence in the middle of the ship in this worldview, too. Therefore in this worldview the nose-bouncing event has to take place later than the rear-bouncing event.

MOVING CLOCKS GET OUT OF SYNCRONISM



Relativity Theory and Logic

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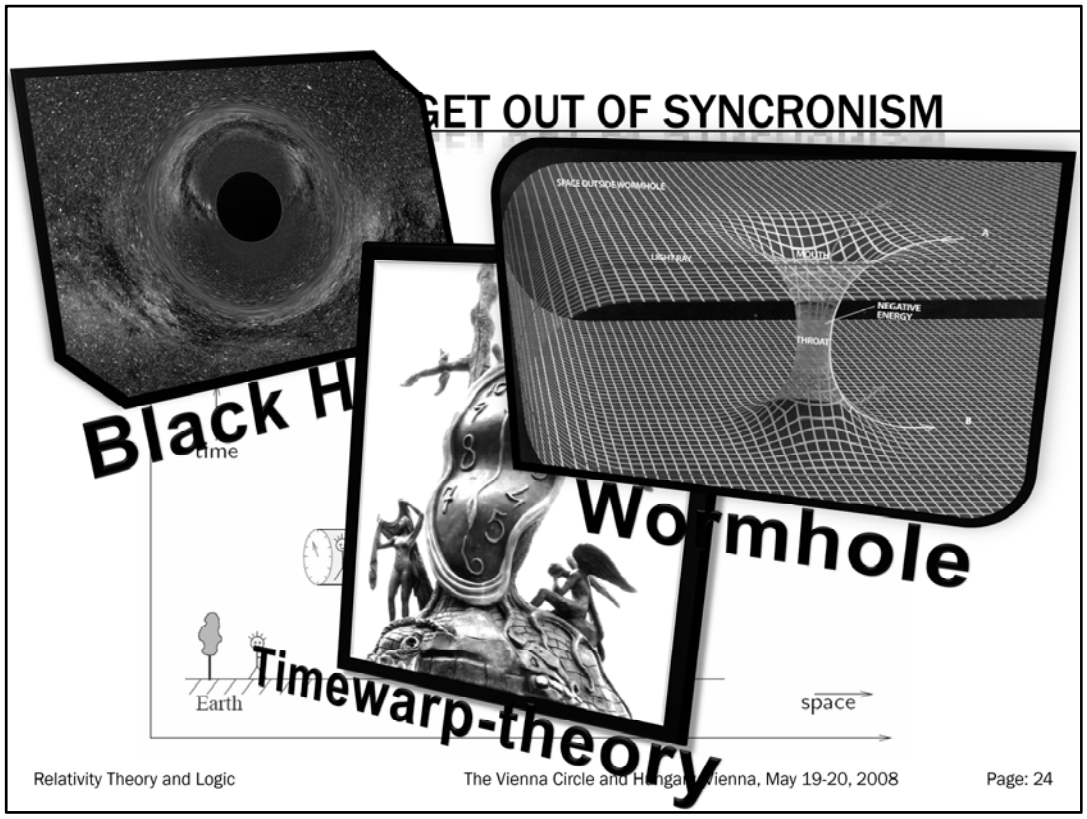
As a consolation, “present” becomes a defined, “testable” concept. (half-simultaneity)

This leads up to black-hole, wormhole, timewarp-theory (via Einstein`s equivalence principle).

Shows that time and space get mixed.



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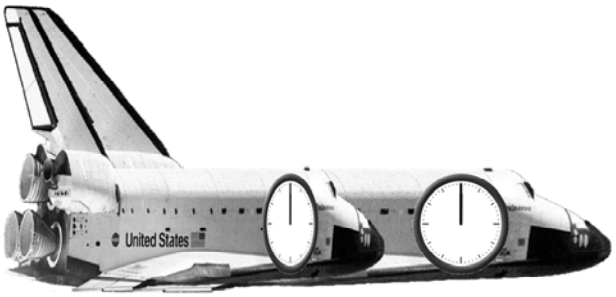
RELATIVISTIC EFFECTS

Thm9



- Moving clocks tick slowly
- Moving spaceships shrink

Relativity Theory and Logic Page: 25

These are the other two paradigmatic effects of special relativity theory. They can be stated and proved analogously to the previous one, as will be shown in the next slides. The three effects together prove the theorem coming afterwards, which is the key in proving the completeness theorem.

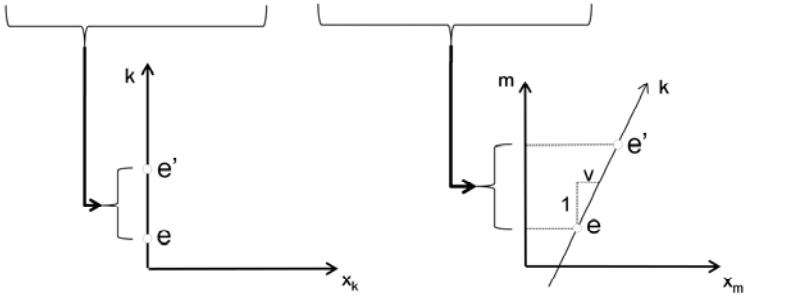
MOVING CLOCKS TICK SLOWLY

- Thm9 (formalization of time-dilation)

  Assume SpecRel. Let *m,k* ϵ *IOb* and events e, e' are on *k*'s lifeline.

  i.e. $loc_k(e)_s = loc_k(e')_s$

  Then $|loc_k(e)_t - loc_k(e')_t| = |loc_m(e)_t - loc_m(e')_t| \cdot \sqrt{1 - |v_m(k)|^2}$



Relativity Theory and Logic

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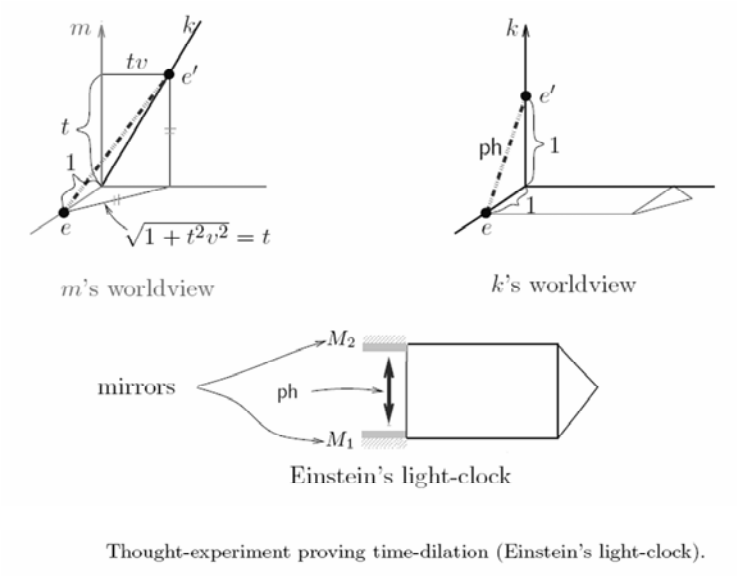
K measures less time between e and e' than m.

In other words, k's clocks tick slowly as m observes them. This is called relativistic time dilation.

MOVING CLOCKS TICK SLOWLY



Thought-experiment proving time-dilation (Einstein's light-clock).

Proof of relativistic time dilation (via thought experiment)

MOVING SPACESHIPS SHRINK

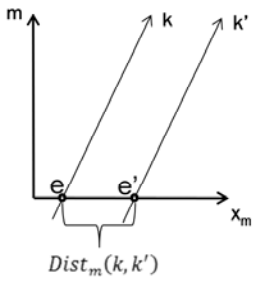
- Thm10 (formalization of spaceship shrinking)

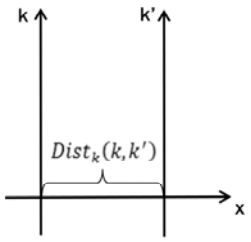
  Assume SpecRel. Let *m,k,k'* ϵ *IOb* and assume $v_k(k') = 0$.

  $$Dist_m(k,k') \coloneqq |loc_m(e) - loc_m(e')|$$

  $$where\ \ loc_k(e)_s = loc_k(e')_s = 0\ \ and\ loc_m(e)_t - loc_m(e')_t$$

  $$Dist_m(k,k') = \sqrt{1 - |v_m(k)|^2} \cdot Dist_k(k,k')$$



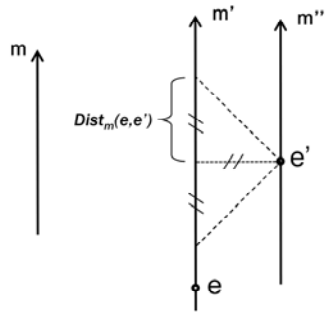


According to m, k's ship is shorter than what k claims.

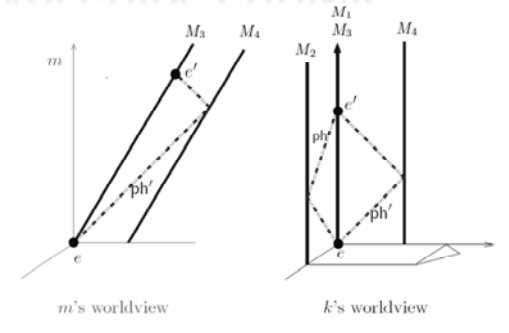
This is called relativistic length contraction.

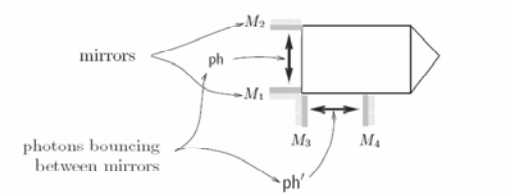
MOVING SPACESHIPS SHRINK

- Experiment for measuring distance (for m) by radar:



MOVING SPACESHIPS SHRINK

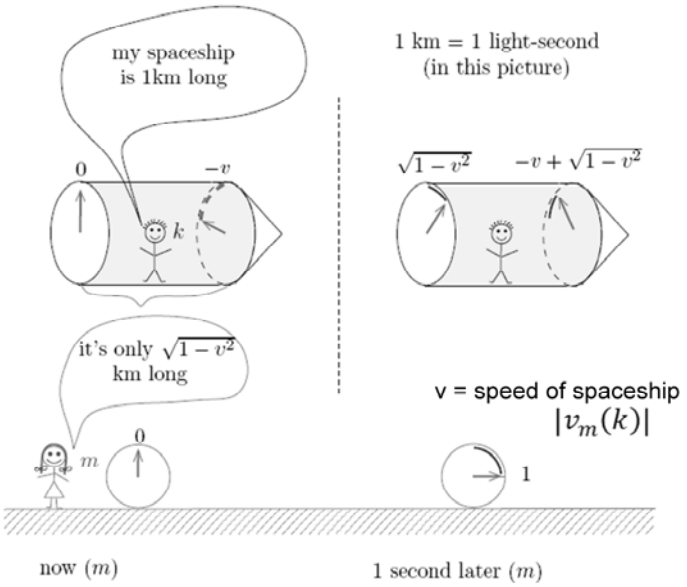




Thought-experiment proving length-contraction

Proof of relativistic length contraction (via thought experiment)

RELATIVISTIC EFFECTS



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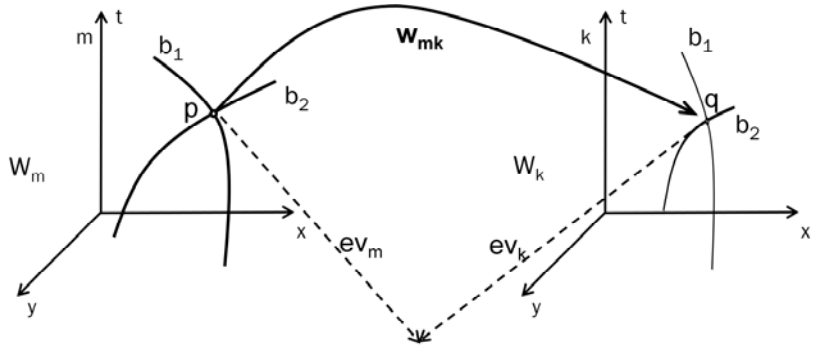
v = speed of spaceship

Quantitatively, too.

WORLDVIEW TRANSFORMATION



$$w_{mk} := \{\langle p, q\rangle : ev_m(p) = ev_k(q)\}$$

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comparison between two maps of the same city

The worldview transformation w_{mk} connects spacetime locations where m and k, respectively, “see” the same events.

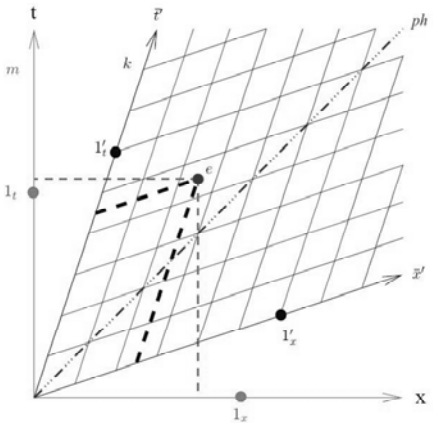
w_{mk} = ev_m circle ev_k^{-1},

w_{mk} is a binary relation between coordinate systems

SPECREL

- Thm11

$SpecRel \vdash$ The worldview transformations w_{mk} are Lorentz transformations (composed perhaps with a translation).



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K's coordinate system drawn into m's coordinate system. The worldview of one observer contains all the information of spacetime.

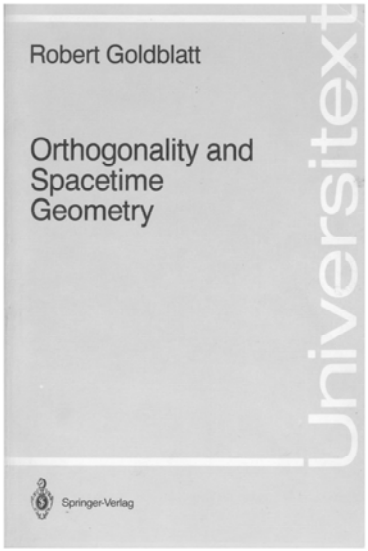
Lorentz transformations are affine maps which are like on the picture in one plane and are identity in directions orthogonal to this plane.

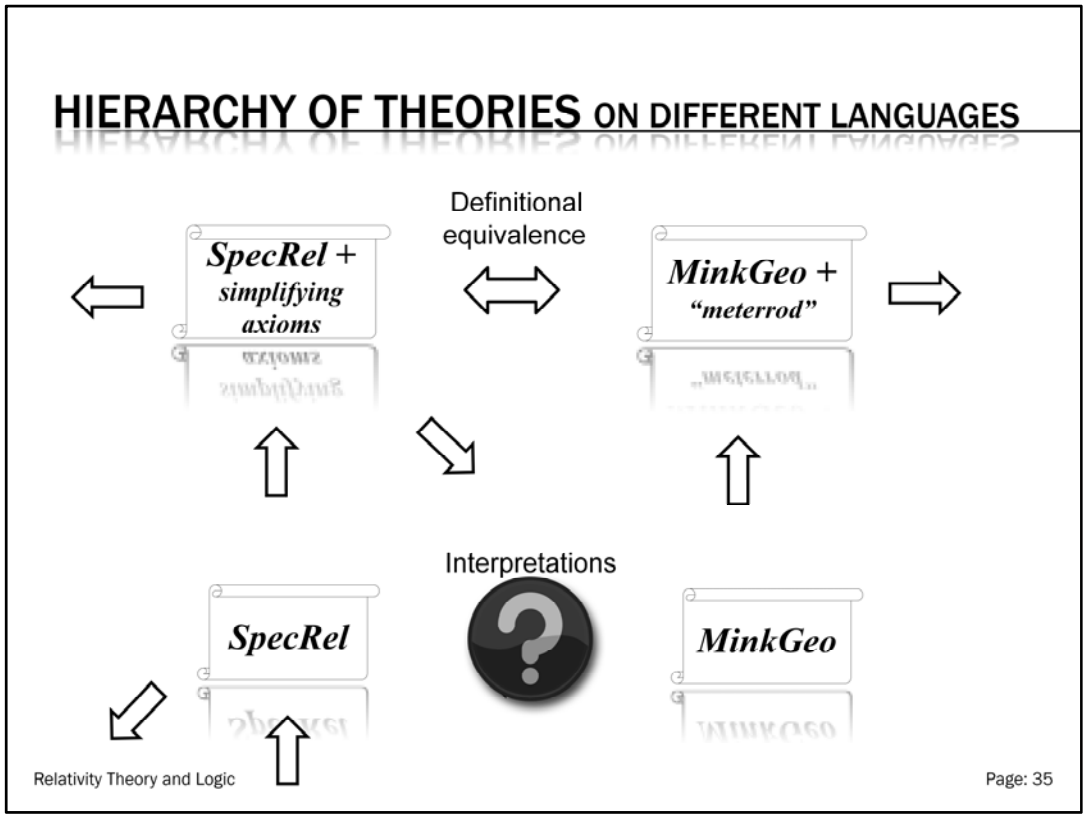
Lorentz transformations composed with translations are exactly the bijections preserving the so-called Minkowski distance. Thm11 is the key step in the proof of the completeness thm for SpecRel. Thm11 is a corollary of the three paradigmatic effects.

OTHER FORMALISATIONS OF SPECREL

Minkowskian Geometry: Top-down approach to SR, observer-free.

Robert Goldblatt:
complete FOL axiom
system MinkGeo





What is the connection between SpecRel and MinkGeo?

There are interpretations in both directions between them.

SpecRel and MinkGeo are FOL theories on languages with different vocabularies. The vocabulary of MinkGeo contains basic symbols like points, lines, betweenness relation between points, and orthogonality of lines. These are notions geometric in nature. An interpretation of SpecRel in MinkGeo consists of interpreting the basic notions of SpecRel with these geometric basic notions, and then proving in MinkGeo the axioms of SpecRel written up with these interpretations (i.e., definitions) in place of the original "atomic" basic symbols. An interpretation is "analysing further the basic notions of SpecRel". An interpretation of SpecRel in MinkGeo gives us a recipe for how to set up the coordinate systems (observers) so that the axioms of SpecRel be true, and such an interpretation also tells us how our notion of numbers (the quantities with addition, multiplication) comes from geometry.

In definability theory, the strongest relation between two theories is definitional equivalence. When two theories are definitionally equivalent, they are the same theory in different linguistic representation. Are SpecRel and MinkGeo definitionally equivalent? Can't be because SpecRel is undecidable and MinkGeo is decidable (when the field-structure is appropriate).

How to measure the difference between SpecRel and MinkGeo? Add simplifying axioms to SpecRel, add two constants to MinkGeo, and then the two theories become definitionally equivalent.

OTHER FORMALISATIONS OF SPECREL

Minkowskian Geometry
James Ax: Signals
Alfred Robb: causality
Patrick Suppes: worldview transformations
…

Connections between theories. Dynamics of theories.
Interpretations between them. Theory morphisms. Definitional equivalence.

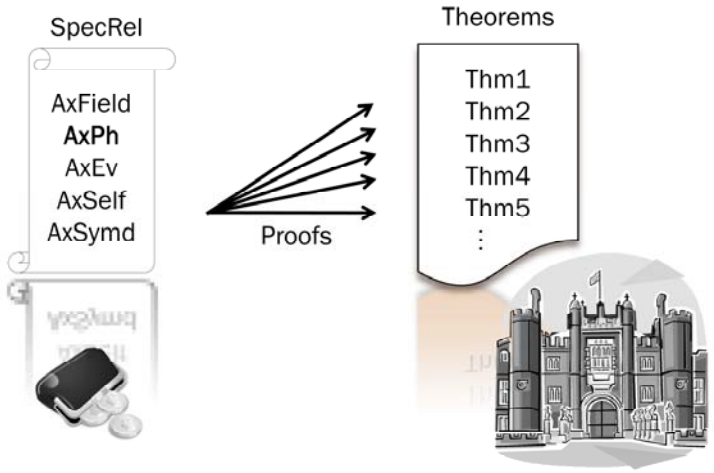
Tamás Füzessy, Judit Madarász and Gergely Székely began joint work in this

Definability theory of logic! (Tarski, Makkai)

Contribution of relativity to logic: definability theory with new objects definable (and not only with new relations definable). J. Madarász' dissertation.

SPECREL

$$SpecRel = \{AxField, AxPh, AxEv, AxSelf, AxSymd\}$$



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5 axioms, AxPh is the most important one of them.
We got lots of theorems.

SPECREL

Conceptual analysis of SR goes on ... on our homepage

☞ New theory is coming:

We have to interrupt studying this theory in this prezentation since we want to go on towards general relativity theory,
Comes Part II: theory of accelerated observers in special relativity.